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Mixed LQG and H_∞ coherent feedback control for linear quantum systems

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ABSTRACT
The purpose of this paper is to study the mixed linear quadratic Gaussian (LQG) and H_∞ optimal control problem for linear quantum stochastic systems, where the controller itself is also a quantum system, often referred to as 'coherent feedback controller'. A lower bound of the LQG control is proved. Then two different methods, rank-constrained linear matrix inequality method and genetic algorithm are for controller design. A passive system (cavity) and a non-passive one (degenerate parametric amplifier) demonstrate the effectiveness of these two proposed algorithms.

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1. Introduction

With the rapid development of quantum technology in recent years, more and more researchers are paying attention to quantum control systems, which are an important part in quantum information science. On the other hand, it is found that many methodologies in classical (namely non-quantum) control theory can be extended into the quantum regime (Bouten, Handel, & James, 2007; Doherty & Jacobs, 1999; Doherty, Habib, Jacobs, Mabuchi, & Tan, 2000; Hamerly & Mabuchi, 2013; James, Nurdin, & Petersen, 2008; Petersen, 2013; Wang & James, 2015; Zhang, Lee, Huang, & Zhang, 2012). Meanwhile, quantum control has its special features absent in the classical case (see e.g. Wiseman & Milburn, 2010; Wang, Nurdin, Zhang, & James, 2013; Zhang & James, 2011; Zhang & James, 2012). For example, a controller in a quantum feedback control system may be classical, quantum or even mixed quantum/classical (James et al., 2008). Generally speaking, in the physics literature, the feedback control problem in which the designed controller is also a fully quantum system is often named as 'coherent feedback control'.

Optimal control, as a vital concept in classical control theory (Zhou, Doyle, & Glover, 1996), has been widely studied. H_2 and H_∞ control are the two foremost control methods in classical control theory, which aim to minimise cost functions with specific forms from exogenous inputs (disturbances or noises) to pertinent performance outputs. When the disturbances and measurement noises are Gaussian stochastic processes with known power spectral densities, and the objective is a quadratic performance criterion, then the problem of minimising this quadratic cost function of linear systems is named as LQG control problem, which has been proved to be equivalent to an H_2 optimal control problem (Zhou et al., 1996). On the other hand, H_∞ control problem mainly concerns the robustness of a system to parameter uncertainty or external disturbance signals, and a controller to be designed should make the closed-loop system stable, meanwhile minimising the influence of disturbances or system uncertainties on the system performance in terms of the H_∞ norm of a certain transfer function. Furthermore, the mixed LQG (or H_2) and H_∞ control problem for classical systems has been studied widely during the last three decades. When the control system is subject to both white noises and signals with bounded power, not only optimal performance (measured in H_2 norm) but also robustness specifications (in terms of an H_∞ constraint) should be taken into account, which is one of the main motivations for considering the mixed control problem (Zhou, Glover, Bodenheimer, & Doyle, 1994); see also Campos-Delgado and Zhou (2003), Doyle, Zhou, Glover, and Bodenheimer (1994), Khargonekar and Rotea (1991), Neumann and Araujo (2004), Qiu, Shi, Yao, Xu, and Xu (2015), Zhou et al. (1996), Zhou et al. (1994), and the references therein.

Very recently, researchers have turned to consider the optimal control problem of quantum systems. For instance, H_∞ control of linear quantum stochastic systems is investigated in James et al. (2008), three different types of controllers are designed. Nurdin, James, and Petersen (2009) proposes a method for quantum LQG control, for which the designed controller is also a fully quantum system. In Zhang and James (2011), direct

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coupling and indirect coupling for quantum linear systems have been discussed. It is shown in Zhang et al. (2012) that phase shifters and ideal squeezers can be used in feedback loop for better control performance. Nevertheless, all of the above papers mainly focus on the vacuum inputs, while the authors in Hamerly and Mabuchi (2013) concern not only the vacuum case, but also the thermal input. They also discussed how to design both classical and non-classical controllers for LQG control problem. Besides, because of nonlinear and non-convex constraints in the coherent quantum controller synthesis, Harno and Petersen (2015) uses a differential evolution algorithm to construct an optimal linear coherent quantum controller. Notwithstanding the above research, to our best knowledge, there is little research on the mixed LQG and H_∞ coherent control problem for linear quantum systems, except Bian, Zhang, and Lee (2015).

Similar to the classical case, in mixed LQG and H_∞ quantum coherent control, LQG and H_∞ performances are not independent. Moreover, because the controller to be designed is another quantum mechanical system, it has to satisfy additional constraints, which are called 'physical realisability conditions' (James et al., 2008; Zhang and James, 2012). For more details, please refer to Section 3.

The contribution of the paper is threefold. First, a *mixed* LQG and H_∞ coherent feedback control problem have been studied, while most of the present literatures (except the conference paper Bian et al. (2015), by one of the authors) only focus on LQG or H_∞ control problem separately. For a typical quantum optical system, there exist quantum white noise as well as finite energy signals (like lasers), while quantum white noise can be dealt with LQG control, finite energy disturbance can better handled by H_∞ control. As a result, it is important to study the mixed control problem. Second, we extend Theorem 4.1 in Zhang et al. (2012), and prove a general result for the lower bound of LQG index. Finally, we propose a genetic algorithm (GA)-based method to design a coherent controller for this mixed problem. In contrast to the numerical algorithm proposed in the earlier conference paper (Bian et al., 2015) by one of the authors, the new algorithm is much simpler and is able to produce better results, as clearly demonstrated by numerical studies.

The organisation of the paper is as follows. In Section 2, quantum linear systems are briefly discussed. Section 3 formulates the mixed LQG and H_∞ coherent feedback control problem. Two algorithms, rank-constrained linear matrix inequality (LMI) method and GA, are proposed in Section 4. Section 5 presents numerical studies to demonstrate the proposed algorithms. Section 6 concludes the paper.

Notation Let $i = \sqrt{-1}$ be the imaginary unit. F denotes a real skew symmetric 2×2 matrix $F = [0\,1; -1\,0]$. Then define a real antisymmetric matrix Θ with components Θ_{jk} is *canonical*, which means $\Theta = \mathrm{diag}(F, F, \ldots, F)$. Given a column vector of operators $x = [x_1 \ldots x_m]^T$ where m is a positive integer, define $x^\# = [x_1^* \ \ldots \ x_m^*]^T$, where the asterisk * indicates Hilbert space adjoint or complex conjugation. Furthermore, define the doubled-up column vector to be $\breve{x} = [x^T \ (x^\#)^T]^T$, and the matrix case can be defined analogously. Given two matrices $U, V \in \mathbb{C}^{r\times k}$, a doubled-up matrix $\Delta(U, V)$ is defined as $\Delta(U, V) := [U \ V; V^\# \ U^\#]$. Let I_N be an identity matrix of dimension N, and define $J_N = \mathrm{diag}(I_N, -I_N)$, where the 'diag' notation indicates a block diagonal matrix assembled from the given entries. Then for a matrix $X \in \mathbb{C}^{2N\times 2M}$, define $X^\flat := J_M X^\dagger J_N$. Finally, the symbol $[\ ,\ ]$ is defined for commutator $[A, B] := AB - BA$.

2. Linear quantum systems

2.1 Open linear quantum systems

An open linear quantum system G consists of N quantum harmonic oscillators $a = [a_1 \ldots a_N]^T$ interacting with N_w-channel quantum fields. Here a_j is the *annihilation operator* of the jth quantum harmonic oscillator and a_j^* is the *creation operator*, they satisfy canonical commutation relations (CCR): $[a_j, a_k^*] = \delta_{jk}$, and $[a_j, a_k] = [a_j^*, a_k^*] = 0$ $(j, k = 1, \ldots, N)$. Such a linear quantum system can be specified by a triple of physical parameters (S, L, H) (Hudson & Parthasarathy, 1984).

In this triple, S is a unitary scattering matrix of dimension N_w. L is a vector of coupling operators defined by

$$L = C_- a + C_+ a^\#, \tag{1}$$

where C_- and $C_+ \in \mathbb{C}^{N_w\times N}$. H is the Hamiltonian describing the self-energy of the system, satisfying

$$H = \frac{1}{2}\breve{a}^\dagger \Delta(\Omega_-, \Omega_+)\breve{a}, \tag{2}$$

where Ω_- and $\Omega_+ \in \mathbb{C}^{N\times N}$ with $\Omega_- = \Omega_-^\dagger$ and $\Omega_+ = \Omega_+^T$.

The *annihilation–creation representation* for linear quantum stochastic systems can be written as the following quantum stochastic differential equations (QSDEs):

$$\begin{aligned} d\breve{a}(t) &= \breve{A}\breve{a}(t)dt + \breve{B}d\breve{b}_{in}(t), \quad \breve{a}(0) = \breve{a}_0 \\ d\breve{y}(t) &= \breve{C}\breve{a}(t)dt + \breve{D}d\breve{b}_{in}(t), \end{aligned} \tag{3}$$

where the correspondences between system matrices $(\breve{A}, \breve{B}, \breve{C}, \breve{D})$ and parameters (S, L, H) are as follows:

$$\begin{aligned} \breve{A} &= -\frac{1}{2}\breve{C}^{\flat}\breve{C} - iJ_N\Delta(\Omega_-, \Omega_+), \quad \breve{B} = -\breve{C}^{\flat}\Delta(S, 0), \\ \breve{C} &= \Delta(C_-, C_+), \quad \breve{D} = \Delta(S, 0). \end{aligned} \tag{4}$$

2.2 Quadrature representation of linear quantum systems

In addition to annihilation–creation representation, there is an alternative form, *amplitude–phase quadrature representation*, where all the operators are observable (self-adjoint operators) and all corresponding matrices are real, so this form is more convenient for standard matrix analysis software packages and programs (Bian et al., 2015; Zhang et al., 2012).

First, define a unitary matrix

$$\Lambda_n = \frac{1}{\sqrt{2}}\begin{bmatrix} I & I \\ -iI & iI \end{bmatrix}_{n\times n} \tag{5}$$

and denote $q_j = (a_j + a_j^*)/\sqrt{2}$ as the *real quadrature*, and $p_j = (-ia_j + ia_j^*)/\sqrt{2}$ as the *imaginary* or *phase quadrature*. It is easy to show these two quadratures also satisfy the CCR $[q_j, p_k] = i\delta_{jk}$ and $[q_j, q_k] = [p_j, p_k] = 0$ $(j, k = 1, \ldots, N)$.

By defining a coordinate transform

$$x := \Lambda_n \breve{a}, \quad w := \Lambda_{n_w}\breve{b}_{in}, \quad y := \Lambda_{n_y}\breve{y}, \tag{6}$$

we could get

$$\begin{aligned} dx(t) &= Ax(t)dt + Bdw(t), \quad x(0) = x_0 \\ dy(t) &= Cx(t)dt + Ddw(t), \end{aligned} \tag{7}$$

where $n = 2N$, $n_w = 2N_w$, $n_y = 2N_y$ are positive even integers, and $x(t) = [q_1(t) \ldots q_N(t)\, p_1(t) \ldots p_N(t)]^T$ is the vector of system variables, $w(t) = [w_1(t) \; \ldots \; w_{n_w}(t)]^T$ is the vector of input signals, including control input signals, noises and disturbances, $y(t) = [y_1(t) \; \ldots \; y_{n_y}(t)]^T$ is the vector of outputs. A, B, C and D are real matrices in $\mathbb{R}^{n\times n}$, $\mathbb{R}^{n\times n_w}$, $\mathbb{R}^{n_y\times n}$ and $\mathbb{R}^{n_y\times n_w}$, respectively. The correspondences between these coefficient matrices of two different representations are

$$\begin{aligned} A &= \Lambda_n\breve{A}\Lambda_n^{\dagger}, \quad B = \Lambda_n\breve{B}\Lambda_{n_w}^{\dagger}, \\ C &= \Lambda_{n_y}\breve{C}\Lambda_n^{\dagger}, \quad D = \Lambda_{n_y}\breve{D}\Lambda_{n_w}^{\dagger}. \end{aligned} \tag{8}$$

Remark 2.1: For simplicity in calculation, we usually do a simple linear transformation to obtain $x(t) = [x_1(t) \ldots x_n(t)]^T = [q_1(t)\, p_1(t) \ldots q_N(t)\, p_N(t)]^T$, and similarly in $w(t)$, $y(t)$ and corresponding matrices (Zhang et al., 2012). In the rest of this paper, we only focus on the quadrature form after the linear transformation.

Assumption 2.1: *Without loss of generality, we give some assumptions for quantum systems (Bian et al., 2015; Nurdin et al., 2009):*

1. *The initial system variable $x(0) = x_0$ is Gaussian.*
2. *The vector of inputs $w(t)$ could be decomposed as $dw(t) = \beta_w(t)dt + d\tilde{w}(t)$, where $\beta_w(t)$ is a self-adjoint adapted process, $\tilde{w}(t)$ is the noise part of $w(t)$, and satisfies $d\tilde{w}(t)d\tilde{w}^T(t) = F_{\tilde{w}}dt$, where $F_{\tilde{w}}$ is a non-negative Hermitian matrix. In quantum optics, $\tilde{w}(t)$ is quantum white noise, and $\beta_w(t)$ is the signal, which in many cases is L_2 integrable.*
3. *The components of $\beta_w(t)$ commute with those of $d\tilde{w}(t)$ and also those of $x(t)$ for all $t \geq 0$.*

2.3 Physical realisability conditions of linear QSDEs

The QSDEs (7) may not necessarily represent the dynamics of a meaningful physical system, because quantum mechanics demands physical quantum systems to evolve in a unitary manner. This implies the preservation of canonical commutation relations $x(t)x^T(t) - (x(t)x^T(t))^T = i\Theta$ for all $t \geq 0$, and also another constraint related to the output signal. These constraints are formulated as physically realisability of quantum linear systems in James et al. (2008).

A linear non-commutative stochastic system of quadrature form (7) is physically realisable if and only if

$$iA\Theta + i\Theta A^T + BT_{\tilde{w}}B^T = 0\,, \tag{9a}$$

$$B\begin{bmatrix} I_{n_y\times n_y} \\ 0_{(n_w-n_y)\times n_y} \end{bmatrix} = \Theta C^T \mathrm{diag}_{N_y}(F)\,, \tag{9b}$$

$$D = \begin{bmatrix} I_{n_y\times n_y} & 0_{n_y\times(n_w-n_y)} \end{bmatrix}, \tag{9c}$$

where the first equation determines the Hamiltonian and coupling operators, and the others relate to the required form of the output equation.

2.4 Direct coupling

There are also some additional components and relations in quantum systems, such as direct coupling, phase shifter, ideal squeezer, etc. Interested readers could refer to, for example, Zhang and James (2011), Zhang and James (2012) and Zhang et al. (2012). Depending on the need of this paper, we just briefly introduce the *direct coupling*.

In quantum mechanics, two independent systems G_1 and G_2 may interact by exchanging energy directly. This energy exchange can be described by an interaction Hamiltonian H_{int}. In this case, it is said that these two systems are directly coupled. When they are expressed in annihilation-creation operator form, such as

$$d\breve{a}_1(t) = \breve{A}_1\breve{a}_1(t)dt + \breve{B}_{12}\breve{a}_2(t)dt,$$
$$d\breve{a}_2(t) = \breve{A}_2\breve{a}_2(t)dt + \breve{B}_{21}\breve{a}_1(t)dt,$$

where the subscript 1 means that corresponding terms belong to the system G_1, and similar for subscript 2. B_{12} and B_{21} denote the direct coupling between two systems, and satisfy the relations as follows:

$$B_{12} = -\Delta(K_-, K_+)^\flat,$$
$$B_{21} = -B_{12}^\flat = \Delta(K_-, K_+),$$

where K_- and K_+ are arbitrary constant matrices of appropriate dimensions.

Definition 2.1: For a quantum linear system in the annihilation–creation operator form which is defined by parameters $(C_-, C_+, \Omega_-, \Omega_+, K_-, K_+)$, there will have the following classifications:

(1) If all 'plus' parameters (i.e. C_+, Ω_+ and K_+) are equal to 0, the system is called a passive system;
(2) Otherwise, it is called a non-passive system.

Examples for these two different systems are given in Section 5.

3. Synthesis of mixed LQG and H_∞ coherent feedback control problem

In this section, we first formulate the QSDEs for the closed-loop system, in which both plant and controller are quantum systems, as well as the specific physical realisability conditions. Then H_∞ and LQG control problems are discussed.

3.1 Composite plant -controller system

Consider the closed-loop system as shown in Figure 1. The quantum plant P is described by QSDEs in

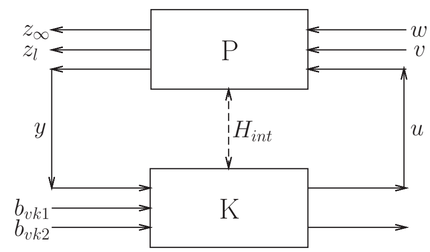

Figure 1. Schematic of the closed-loop plant -controller system.

quadrature form (Bian et al. (2015))

$$\begin{aligned} dx(t) &= Ax(t)dt + B_0dv(t) + B_1dw(t) + B_2du(t),\\ dy(t) &= C_2x(t)dt + D_{20}dv(t) + D_{21}dw(t),\\ dz_\infty(t) &= C_1x(t)dt + D_{12}du(t),\\ z_l(t) &= C_zx(t) + D_z\beta_u(t), \end{aligned} \tag{10}$$

where A, B_0, B_1, B_2, C_2, D_{20}, D_{21}, C_1, D_{12}, C_z and D_z are real matrices in $\mathbb{R}^{n\times n}$, $\mathbb{R}^{n\times n_v}$, $\mathbb{R}^{n\times n_w}$, $\mathbb{R}^{n\times n_u}$, $\mathbb{R}^{n_y\times n}$, $\mathbb{R}^{n_y\times n_v}$, $\mathbb{R}^{n_y\times n_w}$, $\mathbb{R}^{n_\infty\times n}$, $\mathbb{R}^{n_\infty\times n_u}$, $\mathbb{R}^{n_l\times n}$ and $\mathbb{R}^{n_l\times n_u}$, respectively, and n, n_v, n_w, n_u, n_y, n_∞ and n_l are positive integers. $x(t) = [x_1(t) \ldots x_n(t)]^T$ is the vector of self-adjoint possibly non-commutative system variables; $u(t) = [u_1(t) \ \ldots \ u_{n_u}(t)]^T$ is the controlled input; $v(t) = [v_1(t) \ \ldots \ v_{n_v}(t)]^T$ and $w(t) = [w_1(t) \ \ldots \ w_{n_w}(t)]^T$ are other inputs. $z_\infty(t) = [z_{\infty_1}(t) \ \ldots \ z_{\infty_{n_\infty}}(t)]^T$ and $z_l(t) = [z_{l_1}(t) \ \ldots \ z_{l_{n_l}}(t)]^T$ are controlled outputs which are referred to as H_∞ and LQG performance, respectively.

The purpose is to design a coherent feedback controller K to minimise the LQG norm and the H_∞ norm of closed-loop system simultaneously, and K has the following form:

$$\begin{aligned} d\xi(t) &= A_k\xi(t)dt + B_{k1}db_{vk1}(t) + B_{k2}db_{vk2}(t) + B_{k3}dy(t),\\ du(t) &= C_k\xi(t)dt + db_{vk1}(t), \end{aligned} \tag{11}$$

where $\xi(t) = [\xi_1(t) \ \ldots \ \xi_{n_k}(t)]^T$ is a vector of self-adjoint variables, and matrices A_k, B_{k1}, B_{k2}, B_{k3} and C_k have appropriate dimensions.

Assumption 3.1: *Similarly, with Assumption 2.1, we give additional assumptions for the specific plant and controller which we consider:*

(1) *The inputs $w(t)$ and $u(t)$ also have the decompositions: $dw(t) = \beta_w(t)dt + d\tilde{w}(t)$, $du(t) = \beta_u(t)dt + d\tilde{u}(t)$, where the meanings of β_* and $\tilde{*}$ are similar as those in Assumption 2.1.*
(2) *The controller state variable $\xi(t)$ has the same order as the plant state variable $x(t)$.*

(3) *$v(t)$, $\tilde{\omega}(t)$, $b_{vk1}(t)$ and $b_{vk2}(t)$ are independent quantum noises in vacuum state.*
(4) *The initial plant state and controller state satisfy relations: $x(0)x^T(0) - (x(0)x^T(0))^T = i\Theta$, $\xi(0)\xi^T(0) - (\xi(0)\xi^T(0))^T = i\Theta_k$, $x(0)\xi^T(0) - (\xi(0)x^T(0))^T = 0$.*

By the identification $\beta_u(t) \equiv C_k\xi(t)$ and $\tilde{u}(t) \equiv b_{vk1}(t)$, the closed-loop system is obtained as

$$\begin{aligned} d\eta(t) &= M\eta(t)dt + Nd\tilde{\omega}_{cl}(t) + H\beta_w(t)dt, \\ dz_\infty(t) &= \Gamma\eta(t)dt + \Pi d\tilde{\omega}_{cl}(t), \\ z_l(t) &= \Psi\eta(t), \end{aligned} \tag{12}$$

where $\eta(t) = [x^T(t)\ \xi^T(t)]^T$ denotes the state of the closed-loop system, $\beta_w(t)$ is the disturbance, while $\tilde{\omega}_{cl}(t) = [v^T(t)\ \tilde{\omega}^T(t)\ b_{vk1}^T(t)\ b_{vk2}^T(t)]^T$ contains all white noises, and coefficient matrices are shown as follows:

$$\begin{aligned} M &= \begin{bmatrix} A & B_2C_k \\ B_{k3}C_2 & A_k \end{bmatrix}, \\ N &= \begin{bmatrix} B_0 & B_1 & B_2 & 0 \\ B_{k3}D_{20} & B_{k3}D_{21} & B_{k1} & B_{k2} \end{bmatrix}, \\ H &= \begin{bmatrix} B_1 \\ B_{k3}D_{21} \end{bmatrix}, \quad \Gamma = \begin{bmatrix} C_1 & D_{12}C_k \end{bmatrix}, \\ \Pi &= \begin{bmatrix} 0 & 0 & D_{12} & 0 \end{bmatrix}, \quad \Psi = \begin{bmatrix} C_z & D_zC_k \end{bmatrix}. \end{aligned}$$

3.2 Physical realisability conditions

For the plant P introduced in the previous section, we want to design a controller K which is also a quantum system. Hence from James et al. (2008) and Zhang et al. (2012), Equation (11) should also satisfy the following physical realisability conditions:

$$\begin{aligned} A_k\Theta_k + \Theta_kA_k^T \quad &+ B_{k1}\mathrm{diag}_{n_{vk1}/2}(F)B_{k1}^T \\ &+ B_{k2}\mathrm{diag}_{n_{vk2}/2}(F)B_{k2}^T \\ &+ B_{k3}\mathrm{diag}_{n_{vk3}/2}(F)B_{k3}^T = 0, \end{aligned} \tag{13a}$$

$$B_{k1} = \Theta_kC_k^T\mathrm{diag}_{n_u/2}(F). \tag{13b}$$

3.3 LQG control problem

For the closed-loop system (12), we associate a quadratic performance index

$$J(t_f) = \int_0^{t_f} \langle z_l^T(t)z_l(t)\rangle dt, \tag{14}$$

where the notation $\langle\cdot\rangle$ is standard and refers to as quantum expectation (Merzbacher, 1998).

Remark 3.1: In classical control, $\int_0^\infty (x(t)^TPx(t) + u(t)^TQu(t))dt$ is the standard form for LQG performance index, where x is the system variable and u is the control input. However, things are more complicated in the quantum regime. By Equation (11), we can see that $u(t)$ is a function of both $\xi(t)$ (the controller variable) and $b_{vk1}(t)$ (input quantum white noise). If we use $u(t)$ in Equation (11) directly, then there will be quantum white noise in the LQG performance index, which yields an unbounded LQG control performance. On the other hand, by Equation (14), the LQG performance index is a function of $x(t)$ (the system variable) and $\xi(t)$ (the controller variable). This is the appropriate counterpart of the classical case.

Generally, we always focus on the infinite horizon case $t_f \to \infty$. Therefore, as in Nurdin et al. (2009), assume that M is asymptotically stable, by standard analysis methods, we have the infinite-horizon LQG performance index as

$$J_\infty = \lim_{t_f\to\infty} \frac{1}{t_f}\int_0^{t_f} \langle z_l^T(t)z_l(t)\rangle dt = \mathrm{Tr}(\Psi P\Psi^T), \tag{15}$$

where P is the unique symmetric positive definite solution of the Lyapunov equation

$$MP + PM^T + \frac{1}{2}NN^T = 0. \tag{16}$$

Problem 3.1: The LQG coherent feedback control problem is to find a quantum controller K of Equations (11) that minimises the LQG performance index $J_\infty = \mathrm{Tr}(\Psi P\Psi^T)$. Here P is the unique solution of Equation (16), and coefficient matrices of controller satisfy constraints (13).

When considering minimising LQG performance index, first, we want to know the minimum. But for general case, it is too complicated to get the theoretical result, so we choose the orders of plant and controller to be 2. In this case, because $C_z^TC_z$ and $D_z^TD_z$ are both 2-by-2 positive semi-definite real matrices, we denote

$$C_z^TC_z = \begin{bmatrix} c_1 & c_2 \\ c_2 & c_3 \end{bmatrix}, D_z^TD_z = \begin{bmatrix} d_1 & d_2 \\ d_2 & d_3 \end{bmatrix}, \quad C_k = \begin{bmatrix} c_{k1} & c_{k2} \\ c_{k3} & c_{k4} \end{bmatrix},$$

where all parameters in these matrices are real scalars.

In analogy to Theorem 4.1 in Zhang et al. (2012), we have the following result.

Theorem 3.1 (The lower bound of LQG index): *Assume that both the plant and the controller defined in Section 3.1* *are in the ground state, then LQG performance index*

$$J_\infty \geq \frac{c_1 + c_3}{2} + d_2(c_{k1}c_{k3} + c_{k2}c_{k4}),$$

where c_* and d_* come from the matrices above.

Proof: Since $z_l = C_z x + D_z \beta_u = C_z x + D_z C_k \xi$, we could easily get

$$\begin{aligned}\langle z_l^T z_l\rangle &= \langle (C_z x + D_z C_k \xi)^T (C_z x + D_z C_k \xi)\rangle \\ &= \langle x^T C_z^T C_z x\rangle + \langle \xi^T C_k^T D_z^T D_z C_k \xi\rangle \\ &\quad + \langle x^T C_z^T D_z C_k \xi\rangle + \langle \xi^T C_k^T D_z^T C_z x\rangle,\end{aligned} \tag{17}$$

where

$$x = \begin{bmatrix} q \\ p \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}\begin{bmatrix} a \\ a^* \end{bmatrix}, \xi = \begin{bmatrix} q_k \\ p_k \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}\begin{bmatrix} a_k \\ a_k^* \end{bmatrix}.$$

Then we have

$$\begin{aligned}&\left\langle x^T C_z^T C_z x\right\rangle \\ &= \frac{1}{2}\left\langle [a\ a^*]\begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}\begin{bmatrix} c_1 & c_2 \\ c_2 & c_3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}\begin{bmatrix} a \\ a^* \end{bmatrix}\right\rangle \\ &= \frac{1}{2}\left\langle [a\ a^*]\begin{bmatrix} c_1 - c_3 - 2ic_2 & c_1 + c_3 \\ c_1 + c_3 & c_1 - c_3 + 2ic_2 \end{bmatrix}\begin{bmatrix} a \\ a^* \end{bmatrix}\right\rangle \\ &= \frac{1}{2}\langle (c_1 + c_3)a^* a + (c_1 + c_3)aa^* \\ &\quad + (c_1 - c_3 - 2ic_2)aa + (c_1 - c_3 + 2ic_2)a^* a^*]\rangle \\ &= \left\langle (c_1 + c_3)a^* a + \frac{c_1 + c_3}{2}\right\rangle,\end{aligned} \tag{18}$$

where the last equality follows from our assumption that the plant is in the ground state, and $[a, a^*] = 1 \Rightarrow aa^* = 1 + a^* a$. The second term of Equation (17) becomes

$$\begin{aligned}&\langle \xi^T C_k^T D_z^T D_z C_k \xi\rangle \\ &= \left\langle [q_k\ p_k]\begin{bmatrix} c_{k1} & c_{k3} \\ c_{k2} & c_{k4} \end{bmatrix}\begin{bmatrix} d_1 & d_2 \\ d_2 & d_3 \end{bmatrix}\begin{bmatrix} c_{k1} & c_{k2} \\ c_{k3} & c_{k4} \end{bmatrix}\begin{bmatrix} q_k \\ p_k \end{bmatrix}\right\rangle \\ &= \left\langle [q_k\ p_k]\begin{bmatrix} e_1 & e_2 \\ e_2 & e_3 \end{bmatrix}\begin{bmatrix} q_k \\ p_k \end{bmatrix}\right\rangle \\ &= \left\langle e_1 q_k^2 + e_3 p_k^2 + e_2(q_k p_k + p_k q_k)\right\rangle,\end{aligned} \tag{19}$$

where $e_1 = d_1 c_{k1}^2 + d_3 c_{k3}^2 + 2d_2 c_{k1} c_{k3}$, $e_3 = d_1 c_{k2}^2 + d_3 c_{k4}^2 + 2d_2 c_{k2} c_{k4}$, $e_2 = d_1 c_{k1} c_{k2} + d_3 c_{k3} c_{k4} + d_2(c_{k1} c_{k4} + c_{k2} c_{k3})$.

While $q_k = \frac{a_k + a_k^*}{\sqrt{2}}$ and $p_k = \frac{-ia_k + ia_k^*}{\sqrt{2}}$, we get

$$\begin{aligned} q_k^2 &= \frac{1}{2}\left[a_k^2 + (a_k^*)^2 + 2a_k^* a_k + 1\right], \\ p_k^2 &= -\frac{1}{2}\left[a_k^2 + (a_k^*)^2 - 2a_k^* a_k - 1\right], \\ q_k p_k + p_k q_k &= -i\left[a_k^2 - (a_k^*)^2\right],\end{aligned}$$

and

$$\begin{aligned}&\left\langle \xi^T C_k^T D_z^T D_z C_k \xi\right\rangle \\ &= \left\langle \frac{e_1}{2}\left[a_k^2 + (a_k^*)^2 + 2a_k^* a_k + 1\right]\right. \\ &\quad \left. - \frac{e_3}{2}\left[a_k^2 + (a_k^*)^2 - 2a_k^* a_k - 1\right] - e_2 i\left[a_k^2 - (a_k^*)^2\right]\right\rangle.\end{aligned} \tag{20}$$

Since both the plant and the controller are in the ground state, all terms containing a, a^*, a_k and a_k^* are 0; and the plant state x commutes with the controller state ξ, so the third and fourth terms of Equation (17) are also 0. By substituting (18) and (20) into (17), we obtain the result of $\langle z_l^T z_l\rangle$:

$$\begin{aligned}\langle z_l^T z_l\rangle &= \frac{c_1 + c_3}{2} + \frac{e_1 + e_3}{2} \\ &= \frac{d_1(c_{k1}^2 + c_{k2}^2) + d_3\left(c_{k3}^2 + c_{k4}^2\right) + 2d_2\left(c_{k1}c_{k3} + c_{k2}c_{k4}\right)}{2} \\ &\quad + \frac{c_1 + c_3}{2}.\end{aligned} \tag{21}$$

Consequently, all square terms are not less than 0, so $J_\infty \geq \frac{c_1 + c_3}{2} + d_2(c_{k1}c_{k3} + c_{k2}c_{k4})$. The proof is completed. ■

Remark 3.2: Sometimes for simplicity, we could choose the coefficient matrix D_z satisfying $d_2 = 0$, then the bound of LQG index becomes $J_\infty \geq \frac{c_1 + c_3}{2}$, which is a constant, independent with the designed controller. This is consistent with the result in Zhang et al. (2012).

Meanwhile, it is easy to see that physical realisability conditions (13) of the coherent controller K are polynomial equality constraints, so they are difficult to solve numerically using general existing optimisation algorithms. Hence sometimes we reformulate Problem 3.1 into a rank-constrained LMI feasibility problem, by letting the LQG performance index $J_\infty < \gamma_l$ for a prespecified constant $\gamma_l > 0$. This is given by the following result.

Lemma 3.1 (**Relaxed LQG problem;** Nurdin et al., 2009): *Given Θ_k and $\gamma_l > 0$, if there exist symmetric matrix $P_L = P^{-1}$, Q and coefficient matrices of controller such* *that physical realisability constraints (13) and the following inequality constraints*

$$\begin{bmatrix} M^T P_L + P_L M & P_L N \\ N^T P_L & -I \end{bmatrix} < 0,$$
$$\begin{bmatrix} P_L & \Psi^T \\ \Psi & Q \end{bmatrix} > 0,$$
$$\mathrm{Tr}(Q) < \gamma_l \quad (22)$$

hold, then the LQG coherent feedback control problem admits a coherent feedback controller K of the form (11).

3.4 H_∞ control problem

For linear systems, the H_∞ norm can be expressed as follows:

$$\|T\|_\infty = \sup_{\omega\in\mathbb{R}} \sigma_{\max}[T(j\omega)] = \sup_{\omega\in\mathbb{R}} \sqrt{\lambda_{\max}(T^*(j\omega)T(j\omega))} \quad (23)$$

where $\sigma_{\max}$ is the maximum singular value of a matrix, and $\lambda_{\max}$ is the maximum eigenvalue of a matrix.

Since we consider the H_∞ control problem for the closed-loop system (12), and only β_w part contains exogenous signals while the others are all white noises, we interpret $\beta_w \to z_\infty$ as the robustness channel for measuring H_∞ performance, and our objective to be minimised is

$$\begin{aligned}\|G_{\beta_w\to z_\infty}\|_\infty &= \|D_{cl} + C_{cl}(sI - A_{cl})^{-1}B_{cl}\|_\infty \\ &= \|\Gamma(sI - M)^{-1}H\|_\infty\end{aligned} \quad (24)$$

Problem 3.2: The H_∞ coherent feedback control problem is to find a quantum controller K of form (11) that minimises the H_∞ performance index $\|G_{\beta_w\to z_\infty}\|_\infty$, while coefficient matrices of controller A_k, B_{k1}, B_{k2}, B_{k3} and C_k satisfy constraints (13) simultaneously.

Similarly to the LQG case, we proceed to relax Problem 3.2 into a rank-constrained LMI feasibility problem, i.e. let $\|G_{\beta_w\to z_\infty}\|_\infty < \gamma_\infty$ for a pre-specified constant $\gamma_\infty > 0$, then we get the following lemma.

Lemma 3.2 (**Relaxed H_∞ problem;** Zhang and James (2011))**:** *Given Θ_k and $\gamma_\infty > 0$, if there exist A_k, B_{k1}, B_{k2}, B_{k3}, C_k and a symmetric matrix P_H such that physical realisability constraints (13) and the following inequality constraints*

$$\begin{bmatrix} M^T P_H + P_H M & P_H H & \Gamma^T \\ H^T P_H & -\gamma_\infty I & 0 \\ \Gamma & 0 & -\gamma_\infty I \end{bmatrix} < 0,$$
$$P_H > 0 \quad (25)$$

hold, then the H_∞ coherent feedback control problem admits a coherent feedback controller K of the form (11).

Meanwhile, we also want to know the lower bound of H_∞ performance index. It is in general difficult to derive the minimum value of H_∞ index analytically, here we just present a simple example. We begin with the following remark.

Remark 3.3: By referring to James et al. (2008), there exists an H_∞ controller of form (11) for the quantum system (10), if and only if the following pair of algebraic Riccati equations

$$\begin{aligned}&(A - B_2E_1^{-1}D_{12}^TC_1)^TX + X\left(A - B_2E_1^{-1}D_{12}^TC_1\right) \\ &\quad + X\left(B_1B_1^T - \gamma_\infty^2 B_2E_1^{-1}B_2^T\right)X \\ &\quad + \gamma_\infty^{-2}C_1^T\left(I - D_{12}E_1^{-1}D_{12}^T\right)C_1 = 0\end{aligned} \quad (26)$$

and

$$\begin{aligned}&(A - B_1D_{21}^TE_2^{-1}C_2)Y + Y\left(A - B_1D_{21}^TE_2^{-1}C_2\right)^T \\ &\quad + Y\left(\gamma_\infty^{-2}C_1^TC_1 - C_2^TE_2^{-1}C_2\right)Y \\ &\quad + B_1\left(I - D_{21}^TE_2^{-1}D_{21}\right)B_1^T = 0\end{aligned} \quad (27)$$

have positive definite solutions X and Y, where $D_{12}^TD_{12} = E_1 > 0$, $D_{21}D_{21}^T = E_2 > 0$.

We consider a simple example. The system equations are described as

$$\begin{aligned}dx(t) &= -\frac{1}{2}\begin{bmatrix} 0.89 & 0 \\ 0 & 0.91 \end{bmatrix}x(t)dt - \sqrt{0.5}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}dv(t) \\ &\quad - \sqrt{0.2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}dw(t) - \sqrt{0.2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}du(t), \\ dy(t) &= \sqrt{0.5}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ x(t)dt &+ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}dv(t) + \delta\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}dw(t), \\ dz_\infty(t) &= \sqrt{0.2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}x(t)dt + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}du(t),\end{aligned}$$

where δ is a very small positive real number.

There has no problem to calculate the first Riccati equation (26). For the second one (27), denote $Y =$

$\begin{bmatrix} y_1 & y_2 \\ y_2 & y_3 \end{bmatrix}$, we get

$$\begin{bmatrix} \left(\frac{0.2}{\gamma_\infty^2} - \frac{0.5}{\delta^2}\right)(y_1^2 + y_2^2) - \left(0.89 - \frac{2\sqrt{0.1}}{\delta}\right) y_1 & \left[\left(\frac{0.2}{\gamma_\infty^2} - \frac{0.5}{\delta^2}\right)(y_1 + y_3) - \left(0.9 - \frac{2\sqrt{0.1}}{\delta}\right)\right] y_2 \\ \left[\left(\frac{0.2}{\gamma_\infty^2} - \frac{0.5}{\delta^2}\right)(y_1 + y_3)\left(0.9 - \frac{2\sqrt{0.1}}{\delta}\right)\right] y_2 & \left(\frac{0.2}{\gamma_\infty^2} - \frac{0.5}{\delta^2}\right)(y_2^2 + y_3^2) - \left(0.91 - \frac{2\sqrt{0.1}}{\delta}\right) y_3 \end{bmatrix} = 0. \tag{28}$$

Notice that, since δ is very small, $0.89 - \frac{2\sqrt{0.1}}{\delta}$, $0.9 - \frac{2\sqrt{0.1}}{\delta}$ and $0.91 - \frac{2\sqrt{0.1}}{\delta}$ are negative.

From the (1,2) term, we make a classification: $y_2 = 0$ or $y_2 \neq 0$.

(1) $y_2 = 0$: Since (1,1) and (2,2) terms are 0, we get

$$y_1 = 0 \quad or \quad y_1 = \frac{0.89 - \frac{2\sqrt{0.1}}{\delta}}{\frac{0.2}{\gamma_\infty^2} - \frac{0.5}{\delta^2}},$$

$$y_3 = 0 \quad or \quad y_3 = \frac{0.91 - \frac{2\sqrt{0.1}}{\delta}}{\frac{0.2}{\gamma_\infty^2} - \frac{0.5}{\delta^2}}.$$

(2) $y_2 \neq 0$: From the (1,2) term, we get

$$y_1 + y_3 = \frac{0.9 - \frac{2\sqrt{0.1}}{\delta}}{\frac{0.2}{\gamma_\infty^2} - \frac{0.5}{\delta^2}}.$$

After doing the calculation that the (1,1) term minus the (2,2) term, and substituting $y_1 + y_3$ into it, we get

$$y_1 + y_3 = 0.$$

This contradicts the above equation.

Consequently, if Equation (28) has positive definite solution Y, it must satisfy $\frac{0.2}{\gamma_\infty^2} - \frac{0.5}{\delta^2} < 0$, implying the condition $\gamma_\infty > \sqrt{0.4}\delta$.

3.5 Mixed LQG and H_∞ control problem

After the above derivations, we find that when we consider H_∞ control, we intend to design a controller K to minimise $\|\Gamma(sI - M)^{-1}H\|_\infty$, which depends on matrices M, H and Γ, but these three matrices only depend on controller matrices A_k, B_{k3} and C_k. Then we use physical realisability constraints to design other matrices B_{k1} and B_{k2} to guarantee the controller is also a quantum system, but these will affect the LQG index, which depends on M, N and Ψ, so further depends on all matrices of the controller. That is, the LQG problem and the H_∞ problem are not independent.

According to the above analysis, we state the mixed LQG and H_∞ coherent feedback control problem for linear quantum systems.

Problem 3.3: The mixed LQG and H_∞ coherent feedback control problem is to find a quantum controller K of form (11) that minimises LQG and H_∞ performance indices simultaneously, while its coefficient matrices satisfy the physical realisability constraints (13).

Lemma 3.3 (**Relaxed mixed problem;** Bian et al., 2015**):** *Given Θ_k, $\gamma_l > 0$ and $\gamma_\infty > 0$, if there exist A_k, B_{k1}, B_{k2}, B_{k3}, C_k, Q, and symmetric matrices $P_L = P^{-1}$, P_H such that physical realisability constraints (13) and inequality constraints (22) and (25) hold, where P is the solution of Equation (16), then the mixed LQG and H_∞ coherent feedback control problem admits a coherent feedback controller K of the form (11).*

4. Algorithms for mixed LQG and H_∞ coherent feedback control problem

In this section, the coherent feedback controllers for mixed LQG and H_∞ problems are constructed by using two different methods, rank-constrained LMI method and GA.

4.1 Rank-constrained LMI method

In Lemma 3.3, for the mixed problem, obviously constraints (22) and (25) are nonlinear matrix inequalities, and physical realisability conditions (13) are non-convex constraints. Therefore, it is difficult to obtain the optimal solution by existing optimisation algorithms. By referring to Bian et al. (2015), Nurdin et al. (2009) and Scherer, Gahinet, and Chilali (1997), we could translate these non-convex and nonlinear constraints to linear ones.

First, we redefine the original plant (10) to a *modified plant* as follows:

$$\begin{aligned} dx(t) &= Ax(t)dt + B_w d\tilde{w}_{cl}(t) + B_1\beta_w(t)dt \\ &\quad + B_2\beta_u(t)dt, \\ dy'(t) &= [b_{vk1}^T(t)\; b_{vk2}^T(t)\; y^T(t)]^T \\ &= Cx(t)dt + D_w d\tilde{w}_{cl}(t) + D\beta_w(t)dt, \\ dz_\infty(t) &= C_1x(t)dt + D_\infty d\tilde{w}_{cl}(t) + D_{12}\beta_u(t)dt, \\ z_l(t) &= C_z(t) + D_z\beta_u(t), \end{aligned} \tag{29}$$

where $B_w = [B_0\, B_1\, B_2\, 0]$, $C = [0\, 0\, C_2^T]^T$, $D = [0\, 0\, D_{12}^T]^T$, $D_\infty = [0\, 0\, D_{12}\, 0]$ and $D_w = \begin{bmatrix} 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ D_{20} & D_{21} & 0 & 0 \end{bmatrix}$. Correspondingly, the *modified controller* equations become the following form:

$$\begin{aligned} d\xi(t) &= A_k\xi(t)dt + B_{wk}dy'(t), \\ \beta_u(t) &= C_k\xi(t) \end{aligned} \tag{30}$$

with $B_{wk} = [B_{k1}\, B_{k2}\, B_{k3}]$, and the closed-loop system still has the same form as (12).

Assumption 4.1: *For simplicity, we assume $P_H = P_L = P^{-1}$.*

We proceed to introduce matrix variables Ξ, Σ, X, Y, $Q \in \mathbb{R}^{n\times n}$, where X, Y and Q are symmetric. Then define the change of controller variables as follows:

$$\begin{aligned} \hat{A} &:= \Xi A_k\Sigma^T + \Xi B_{wk}CX + YB_2C_k\Sigma^T + YAX, \\ \hat{B} &:= \Xi B_{wk}, \\ \hat{C} &:= C_k\Sigma^T, \end{aligned} \tag{31}$$

where $\Sigma\Xi^T = I - XY$.

By using (31), LQG inequality constrains (22) can be transformed to (32). Similarly, H_∞ inequality constraints (25) become (33). It is obvious that the following matrix inequalities are linear, so they can be easily solved by MATLAB:

$$\begin{bmatrix} AX + XA^T + B_2\hat{C} + (B_2\hat{C})^T & \hat{A}^T + A & B_w \\ \hat{A} + A^T & A^TY + YA + \hat{B}C + (\hat{B}C)^T & YB_w + \hat{B}D_w \\ B_w^T & (YB_w + \hat{B}D_w)^T & -I \end{bmatrix} < 0,$$

$$\begin{bmatrix} X & I & (C_zX + D_z\hat{C})^T \\ I & Y & C_z^T \\ (C_zX + D_z\hat{C}) & C_z & Q \end{bmatrix} > 0,$$

$$\mathrm{Tr}(Q) < \gamma_l. \tag{32}$$

$$\begin{bmatrix} AX + XA^T + B_2\hat{C} + (B_2\hat{C})^T & \hat{A}^T + A & * & * \\ \hat{A} + A^T & A^TY + YA + \hat{B}C + (\hat{B}C)^T & * & * \\ B_1^T & (YB_1 + \hat{B}D)^T & -\gamma_\infty I & * \\ C_1X + D_{12}\hat{C} & C_1 & 0 & -\gamma_\infty I \end{bmatrix} < 0. \tag{33}$$

From (31), we can obtain $C_k = \hat{C}\Sigma^{-T}$, $B_{wk} = \Xi^{-1}\hat{B}$, and $A_k = \Xi^{-1}(\hat{A} - \Xi B_{wk}CX - YB_2C_k\Sigma^T - YAX)\Sigma^{-T}$. After substituting A_k, B_{wk} and C_k into (13) and introducing new variables $\tilde{\Xi} = \Xi J_{N_\xi}$, $\tilde{A}_k = \Xi A_k$, $\tilde{B}_{ki} = \Xi B_{ki}$, $i = 1, 2, 3$, physical realisability constraints (13) become

$$\begin{aligned} &(-\hat{A}\Sigma^{-T} + (\tilde{B}_{k3}C_2 + YA)X\Sigma^{-T} + YB_2C_k)\tilde{\Xi}^T \\ &+\tilde{\Xi}(\hat{A}\Sigma^{-T} - (\tilde{B}_{k3}C_2 + YA)X\Sigma^{-T} - YB_2C_k)^T \\ &\qquad + \textstyle\sum_{i=1}^3 \tilde{B}_{ki}J_{n_{vki}/2}\tilde{B}_{ki}^T = 0, \end{aligned} \tag{34a}$$

$$\tilde{B}_{k1} - \tilde{\Xi}C_k^TJ_{n_{vk1}/2} = 0. \tag{34b}$$

We get the following result for the mixed LQG and H_∞ coherent feedback control problem.

Lemma 4.1: *Given Θ_k, $\gamma_l > 0$ and $\gamma_\infty > 0$, if there exist matrices $\hat{A}$, $\tilde{B}_{k1}$, $\tilde{B}_{k2}$, $\tilde{B}_{k3}$, $\hat{C}$, X, Y, $\tilde{\Xi}$, Σ, Ξ, C_k such that the LMIs (32), (33) and equality constraints (34) hold, then the mixed LQG and H_∞ coherent feedback control problem admits a coherent feedback controller K of the form (11).*

Algorithm 4.1 (Rank-constrained LMI method; Bian et al., 2015): First, introduce 13 basic matrix variables: $M_1 = \hat{A}$, $M_2 = \tilde{B}_{k1}$, $M_3 = \tilde{B}_{k2}$, $M_4 = \tilde{B}_{k3}$, $M_5 = \hat{C}$, $M_6 = X$, $M_7 = Y$, $M_8 = \tilde{\Xi}$, $M_9 = \Sigma$, $M_{10} = \Xi$, $M_{11} = C_k$, $M_{12} = \check{A} = \hat{A}\Sigma^{-T}$, $M_{13} = \check{X} = X\Sigma^{-T}$. And define 18 matrix lifting variables: $W_i = \tilde{B}_{ki}J_{N_{vki}}$ $(i = 1, 2, 3)$, $W_4 = YB_2$, $W_5 = \tilde{B}_{k3}C_2 + YA$, $W_6 = \tilde{\Xi}C_k^T$, $W_7 = \tilde{\Xi}\check{X}^T$, $W_8 = \check{A}\tilde{\Xi}^T$, $W_9 = YX$, $W_{10} = W_4W_6^T$, $W_{11} = W_5W_7^T$, $W_{12} = W_1\tilde{B}_{k1}^T$, $W_{13} = W_2\tilde{B}_{k2}^T$, $W_{14} = W_3\tilde{B}_{k3}^T$, $W_{15} = \Xi\Sigma^T = I - YX$, $W_{16} = \check{A}\Sigma^T = \hat{A}$, $W_{17} = \check{X}\Sigma^T = X$, $W_{18} = C_k\Sigma^T = \hat{C}$.

By defining

$$\begin{aligned} V &= [I\, Z_{m_1,1}^T \ldots Z_{m_{13},1}^T\, Z_{w_1,1}^T \ldots Z_{w_{18},1}^T]^T \\ &= [I\, M_1^T \ldots M_{13}^T\, W_1^T \ldots W_{18}^T]^T, \end{aligned} \tag{35}$$

we could let Z be a $32n \times 32n$ symmetric matrix with $Z = VV^T$. It is obvious that $Z_{m_i,w_i} = Z_{m_i,1}(Z_{w_i,1})^T$.

Meanwhile, because of relations between these 31 variables, we require the matrix Z to satisfy the following constraints:

$$\begin{aligned} Z &\geq 0, \\ Z_{0,0} - I_{n\times n} = 0, &\quad Z_{w_7,1} - Z_{m_8,m_{13}} = 0, \\ Z_{1,x_6} - Z_{m_6,1} = 0, &\quad Z_{w_8,1} - Z_{m_{12},m_8} = 0 \\ Z_{1,x_7} - Z_{m_7,1} = 0, &\quad Z_{w_9,1} - Z_{m_7,m_6} = 0, \\ Z_{w_1,1} - Z_{m_2,1}J_{n_{vk1}/2} = 0, &\quad Z_{w_{10},1} - Z_{w_4,w_6} = 0, \\ Z_{w_2,1} - Z_{m_3,1}J_{n_{vk2}/2} = 0, &\quad Z_{w_{11},1} - Z_{w_5,w_7} = 0, \\ Z_{w_3,1} - Z_{m_4,1}J_{n_{vk3}/2} = 0, &\quad Z_{w_{12},1} - Z_{w_1,m_2} = 0, \\ Z_{w_4,1} - Z_{m_7,1}B_2 = 0, &\quad Z_{w_{13},1} - Z_{w_2,m_3} = 0, \\ Z_{w_5,1} - Z_{m_4,1}C_2 - Z_{m_7,1}A = 0, &\quad Z_{w_{14},1} - Z_{w_3,m_4} = 0, \\ Z_{w_6,1} - Z_{m_8,m_{11}} = 0, &\quad Z_{w_{15},1} - Z_{m_{10},m_9} = 0, \\ Z_{w_{16},1} - Z_{m_{12},m_9} = 0, &\quad Z_{w_{17},1} - Z_{m_{13},m_9} = 0, \\ Z_{w_{18},1} - Z_{m_{11},m_9} = 0, &\quad Z_{w_{15},1} - I + Z_{w_9,1} = 0, \\ Z_{m_1,1} - Z_{w_{16},1} = 0, &\quad Z_{m_6,1} - Z_{w_{17},1} = 0, \\ Z_{m_8,1} - Z_{m_{10},1}J_{n_\xi/2} = 0, &\quad Z_{m_5,1} - Z_{w_{18},1} = 0, \end{aligned} \tag{36}$$

and moreover, Z satisfies a rank n constraint, i.e. $rank(Z) \leq n$.

Then, we use $Z_{m_1,1}$, $[Z_{m_2,1}\ Z_{m_3,1}\ Z_{m_4,1}]$, $Z_{m_5,1}$, $Z_{m_6,1}$, $Z_{m_7,1}$ to replace $\hat{A}$, $\hat{B}$, $\hat{C}$, X, Y in LMI constraints (32) and (33), and convert physical realisability conditions (34) to

$$\begin{aligned} &-Z_{w_8,1} + Z_{w_8,1}^T + Z_{w_{11},1} - Z_{w_{11},1}^T + Z_{w_{10},1} - Z_{w_{10},1}^T \\ &\quad + Z_{w_{12},1} + Z_{w_{13},1} + Z_{w_{14},1} = 0, \end{aligned} \tag{37a}$$

$$Z_{m_2,1} - Z_{w_6,1}J_{N_{vk1}} = 0. \tag{37b}$$

We have transformed the mixed problem to a rank-constrained problem, which could be solved by using Toolbox: Yalmip (Lofberg, 2004), SeDuMi and LMIRank (Orsi, Helmke, & Moore, 2006).

Remark 4.1: The above LMI-based approach solves a suboptimal control problem for the mixed LQG/H_∞ coherent feedback control. Once a feasible solution is found by implementing Algorithm 4.1, we then know that the LQG index is bounded by γ_l from above, and *simultaneously*, the H_∞ index is bounded by γ_∞ from above.

4.2 Genetic algorithm

GA is a search heuristic that mimics the process of natural selection in the field of artificial intelligence. This heuristic (sometimes called metaheuristic) is routinely used to generate useful solutions to optimisation and search problems. GA belongs to the larger class of evolutionary algorithms, which get solutions using techniques inspired by natural evolution, such as inheritance, mutation, selection and crossover, etc. GA is a useful method for controller design (see e.g. Campos-Delgado & Zhou, 2003; Michalewicz, Janikow, & Krawczyk, 1992; Neumann & Araujo, 2004; Pereira & Araujo, 2004). In the field of quantum control, GA methods are applied to design quantum coherent feedback controllers (see e.g. Harno & Petersen, 2015; Zhang et al., 2012).

We briefly introduce the procedures of GA as follows.

Algorithm 4.2 (Genetic algorithm):

Step 1 : Initialisation for the population (the first generation), by using random functions, and binary strings denote controller parameters we want to design.

Step 2 : Transform binary strings to decimal numbers, and calculate the results of these parameters.

Step 3 : After obtaining coefficient matrices of the controller, we restrict one of the LQG or H_∞ indices in an interval, then minimise the other index (the fitness function in our problem). Since the lower bounds of these two indices can be calculated a priori, see Sections 3.3 and 3.4, the above-mentioned interval can always be found. By the above procedure, we get the best individual and corresponding performance index in this generation.

Step 4 : Perform the selection operation, for yielding new individuals.

Step 5 : Perform the cross-over operation, for yielding new individuals.

Step 6 : Perform the mutation operation, for yielding new individuals.

Step 7 : Back to Step 2, recalculate all parameters and corresponding best fitness function result for new generation.

Step 8 : At the end of iterations, compare all best results of every generation, and get the optimal solution.

Remark 4.2: Algorithm 4.2 does not minimise both LQG and H_∞ performance indices simultaneously. More specifically, as can be seen in Step 3, one of the indices is first fixed, then the other one is minimised. This procedure is repeated as can be seen from Step 7. Therefore, Algorithm 4.2 is an iterative minimisation algorithm.

In our problem, because the coherent feedback controller K to be designed is a quantum system, it can be described by the (S, L, H) language introduced in Section 2.1. With this, physical realisability conditions are naturally satisfied. As a result, we apply the GA to find K by minimising the LQG and H_∞ performance indices directly.

5. Numerical simulations and comparisons

In this section, we provide two examples to illustrate the methods proposed in the previous section.

5.1 Numerical simulations

Example 1: This example is taken from Section VII of James et al. (2008). The plant is an optical cavity resonantly coupled to three optical channels.

The dynamics of this optical cavity system can be described by following equations:

$$\begin{aligned} dx(t) &= -\frac{\gamma}{2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}x(t)\,dt - \sqrt{\kappa_1}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}dv(t)\\ &\quad - \sqrt{\kappa_2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}dw(t) - \sqrt{\kappa_3}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}du(t),\\ dy(t) &= \sqrt{\kappa_2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}x(t)dt + \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}dw(t),\\ dz_\infty(t) &= \sqrt{\kappa_3}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}x(t)dt + \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}du(t),\\ z_l(t) &= \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}x(t) + \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\beta_u(t) \end{aligned} \tag{38}$$

with parameters $\gamma = \kappa_1 + \kappa_2 + \kappa_3$, $\kappa_1 = 2.6$, $\kappa_2 = \kappa_3 = 0.2$. In this example, $v(t)$ is quantum white noise, while $w(t)$ is a sum of quantum white noise and L_2 disturbance (See Assumption 2.1 for details). Therefore, there are two types of noises in this system. LQG control is used to suppress the influence of quantum white noise, while H_∞ control is used to attenuate the L_2 disturbance.

Example 2: In this example, we choose a DPA as our plant. For more details about DPA, one may refer to Leonhardt (2003). The QSDEs of DPA are

$$\begin{aligned} dx(t) &= -\frac{1}{2}\begin{bmatrix}\gamma-\epsilon & 0\\ 0 & \gamma+\epsilon\end{bmatrix}x(t)\,dt - \sqrt{\kappa_3}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}dv(t)\\ &\quad - \sqrt{\kappa_1}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}dw(t) - \sqrt{\kappa_2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}du(t),\\ dy(t) &= \sqrt{\kappa_3}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}x(t)dt + \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}dv(t),\\ dz_\infty(t) &= \sqrt{\kappa_2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}x(t)dt + \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}du(t),\\ z_l(t) &= \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}x(t) + \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\beta_u(t) \end{aligned} \tag{39}$$

with parameters $\gamma = \kappa_1 + \kappa_2 + \kappa_3$, $\kappa_1 = \kappa_2 = 0.2$, $\kappa_3 = 0.5$, $\varepsilon = 0.01$.

Table 1. Optimisation results only for LQG index.

Plant	Controller	J_∞ (LQG index)
Cavity	Passive controller	1.0005
		1.0000
	Non-passive controller	1.0006
		1.0003
DPA	Passive controller	1.0003
		1.0000
	Non-passive controller	1.0002
		1.0000

Table 2. Optimisation results only for H_∞ index.

Plant	Controller	$\|G_{\beta_w \to z_\infty}\|_\infty$ (H_∞ index)
Cavity	Passive controller	0.0134
		0.0050
	Non-passive controller	0.0196
		0.0075
DPA	Passive controller	0.0070
		0.0044
	Non-passive controller	0.0057
		0.0045

According to Theorem 3.1, it is easy to find that lower bounds of the LQG index for both two examples are 1. First, we only focus on the LQG performance index, and design two different types of controllers to minimise it by using GA. The results are shown in Table 1. For each case, we list two values obtained.

Remark 5.1: J_∞ in Table 1 is the LQG performance index defined in Equation (15). In Theorem 3.1, a lower bound for J_∞ is proposed. This lower bound is obtained when both the plant and the controller are in the ground state, as stated in Theorem 3.1. In Table 1, there are two systems, namely the optical cavity and DPA. For both of them, the lower bound in Theorem 3.1 satisfies $d_2 = 0$ and $c_1 = c_3 = 1$. Therefore, $J_\infty \geq 1$. From Table 1, we can see that our GA finds controllers that yield the LQG performance which is almost optimal. And in this case, as guaranteed by Theorem 3.1, both the plant and the controller are almost in the ground state.

Second, similarly to the LQG case, we only focus on the H_∞ index and design controllers to minimise the objective, getting the following Table 2. For each case, we list two values obtained.

Remark 5.2: Table 2 is for H_∞ performance index. For the cavity case, actually it can be proved analytically that the H_∞ performance index can be made arbitrarily close to zero. On the other hand, by Remark 3.3, H_∞ index has a lower bound $\sqrt{0.4\delta}$. However, for the DPA studied in this example, $\delta = 0$, that is, the lower bound for H_∞ index is also zero. The simulation results in Table 2 confirmed this observation.

From the above results we could see, if we only consider one performance index, either LQG index or

Table 3. Optimisation results by rank-constrained LMI method.

Plant	Constraints		Results	
	γ_∞	γ_l	$\lVert G_{\beta_w \to z_\infty} \rVert_\infty$ (H_∞ index)	J_∞ (LQG index)
Cavity ($\gamma = \kappa_1 + \kappa_2 + \kappa_3, \kappa_1 = 2.6, \kappa_2 = \kappa_3 = 0.2.$)	0.1	2.5	0.039900	1.014555
	0.1	N/A	0.058805	1.039487
	N/A	2.5	0.134558	1.000577
	N/A	3	0.423970	1.379587
	2.8	3	0.444119	1.270835
DPA ($\gamma = \kappa_1 + \kappa_2 + \kappa_3, \kappa_1 = \kappa_2 = 0.2, \kappa_3 = 0.5, \varepsilon = 0.01.$)	0.3	2.5	0.172385	1.175277
	0.5	3	0.447274	1.080976
	N/A	3	0.468007	1.149859
	1	5	0.647468	1.374547

H_∞ index, there are no significant differences between passive controllers and non-passive controllers, both of which can lead to a performance index close to the minimum.

Then we proceed to use these two methods to do simulations for the mixed problem, to see whether we could succeed to make two performance indices close to the minima simultaneously, and find which method is better. The results are shown in Tables 3 and 4, respectively.

5.2 Comparisons of results

After getting numerical results shown in Tables 3 and 4, and doing comparisons with other literatures in coherent optimal control for linear quantum systems, we state the advantages of Algorithm 4.2:

(1) Instead of single LQG or H_∞ optimal control for linear quantum systems, Algorithm 4.2 deals with the *mixed* LQG and H_∞ problem.
(2) Algorithm 4.1 relaxes two performance indices by introducing γ_l and γ_∞. When they are small, it will be quite difficult to solve the problem by Algorithm 4.1. But Algorithm 4.2 is able to minimise the two performance indices directly.
(3) The solution of the differential evolution algorithm in Harno and Petersen (2015) involves a complex algebraic Riccati equation, but all parameters of our Algorithm 4.2 are real. It might be easier to be solved by current computer software such as MATLAB.
(4) The numerical results show that there seems to be a trend between these two indices, that sometimes one increases, while another decreases.
(5) For a passive system (e.g. cavity), both the passive controller and the non-passive controller could let LQG and H_∞ indices go to the minima simultaneously (Table 4).
(6) For a non-passive system (e.g. DPA), neither the passive controller nor the non-passive controller can let these two indices go to the minima simultaneously, but when a direct coupling is added between the plant and the controller, we could use GA to design a passive controller to minimise these two indices simultaneously, which is not achieved using rank-constrained LMI method (Table 4).

Table 4. Optimisation results by genetic algorithm.

Plant	Controller	Results	
		$\lVert G_{\beta_w \to z_\infty} \rVert_\infty$ (H_∞ index)	J_∞ (LQG index)
Cavity ($\gamma = \kappa_1 + \kappa_2 + \kappa_3, \kappa_1 = 2.6, \kappa_2 = \kappa_3 = 0.2.$)	Passive controller	0.003574	1.008917
		0.078977	1.000619
		0.146066	1.000009
	Non-passive controller	0.071270	1.009303
		0.089383	1.002099
		0.123066	1.000283
DPA ($\gamma = \kappa_1 + \kappa_2 + \kappa_3, \kappa_1 = \kappa_2 = 0.2, \kappa_3 = 0.5, \varepsilon = 0.01.$)	Passive controller	0.428312	1.004787
		0.449534	1.000124
	Non-passive controller	0.364979	1.009691
		0.387734	1.007164
	Passive controller + direct coupling	0.039183	1.000079
		0.042960	1.000002

(7) Actually, rank-constrained LMI method could not be used to design specific passive controllers, or non-passive controllers, while this can be easily achieved using GA, by setting all 'plus' terms equal to 0.
(8) Finally, from numerical simulations, GA often provides better results than the rank-constrained LMI method.

6. Conclusion

In this paper, we have studied the mixed LQG and H_∞ coherent feedback control problem. Two algorithms, rank-constrained LMI method and a GA-based method, have been proposed. Two examples are used to illustrate the effectiveness of these two methods, and also verify the superiority of GA by numerical results.

Disclosure statement

No potential conflict of interest was reported by the authors.

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