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On the control of flying qubits ☆

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The control of flying quantum bits (qubits) carried by traveling quantum fields is crucial for coherent information transmission in quantum networks. In this paper, we develop a general framework for modeling the control of flying qubits based on the quantum stochastic differential equation (QSDE) that describes the input-output process actuated by a standing quantum system. Under the continuous time-ordered photon-number basis, the infinite-dimensional QSDE is reduced to a low-dimensional deterministic differential equation for the non-unitary state evolution of the standing quantum system, and the outgoing flying-qubit states can be expressed in the form of randomly occurring quantum jumps. This makes it possible, as demonstrated by examples of flying-qubit generation and transformation, to analyze general cases when the number of excitations is not reserved. The proposed framework lays the foundation for the design of flying-qubit control systems, with which advanced control techniques can be incorporated for practical applications.

1. Introduction

Quantum computation has been evidently demonstrated to be surpassing its classical counterparts (Arute et al., 2019). Further into the future, the precise control of networked quantum systems is highly demanding (DiVincenzo, 1995) for large-scale and distributed quantum information processing (QIP) applications (Duan, Lukin, Cirac, & Zoller, 2001). In the past decades, the control of "standing" components (namely the nodes, e.g., atoms or resonators Gu, Kockum, Miranowicz, Liu, & Nori, 2017) for on-site QIP had been extensively studied (D'Alessandro, 2008; Dong & Petersen, 2010; Glaser et al., 2015; Jacobs, 2014; Wiseman & Milburn, 2010). However, the control of "flying" components for information transmission, as are often called flying qubits, received much less attention (Lucamarini, Di Giuseppe, Vitali, & Tombesi, 2011).

In quantum networks, the transfer of quantum information between distant nodes is physically realized by releasing and catching photons that encode flying qubits (Cirac, Zoller, Kimble, & Mabuchi, 1997; Duan et al., 2001; Gheri, Ellinger, Pellizzari, & Zoller, 1998), which range from optical regime (e.g., from quantum dots or NV-centers Eisaman, Fan, Migdall, & Polyakov, 2011; Kuhn, Hennrich, & Rempe, 2002; Kurtsiefer, Mayer, Zarda, & Weinfurter, 2002) down to the microwave regime (e.g., from superconducting artificial atoms Houck et al., 2007). In laboratories, such processes must be actuated by a standing quantum system (e.g., an atom or a resonator Yao, Liu, & Sham, 2005) that is accessible by control electronics, and the incoming and outgoing flying qubits form its quantum inputs or outputs. The control of flying qubits is thus formulated as the control of such quantum input-output processes through the standing quantum system's coherent driving (Fischer, Trivedi, Ramasesh, Siddiqi, & Vučković, 2018; Pechal et al., 2013) or incoherent tunable coupling to the waveguide (Pierre, Svensson, Sathyamoorthy, Johansson, & Delsing, 2014). Alternatively, the control can be indirectly realized by nonlocal spectral filtering using ancilla qubits (Averchenko, Sych, Marquardt, & Leuchs, 2020) or coherent quantum feedback loops (Dong, Cui, Zhang, & Fu, 2016; Zhang, 2020).

Roughly speaking, flying-qubit control problems can be categorized into the following three classes (see Fig. 1 for illustration).

The first class is the on-demand *generation* of flying qubits for the purpose of coding and sending quantum information. To match remote network nodes (Averchenko, Sych, Schunk, Vogl, Marquardt, & Leuchs, 2017; Forn-Diaz, Warren, Chang, Vadiraj, & Wilson, 2017; Pechal et al., 2013)), the generated flying qubits should be stable (i.e., all generated flying qubits are indistinguishable) and the frequency and shape are flexibly tunable (Averchenko et al., 2020, 2017; Forn-Diaz et al., 2017; Keller, Lange, Hayasaka, Lange, & Walther, 2004; Pechal et al.,

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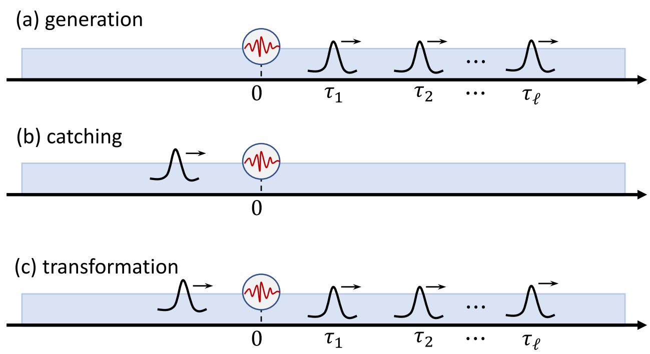

Fig. 1. Schematics of flying-qubit control processes actuated by a standing quantum system: (a) the generation of flying qubits; (b) the catching of flying qubits; (c) the transformation of flying qubits.

2013; Peng, Graaf, Tsai, & Astafiev, 2016). The early studies can be dated back to the quantum interface (Yao et al., 2005) for quantum information transfer from a standing qubit to a flying qubit, which physically led to the generation of shaped single-photon pulses with a spin qubit coupled to a fiber through a ring cavity. Such cavity-assisted schemes have low repetition rates that are limited by the narrow bandwidth of the cavity. An effective approach to overcome this problem is to directly couple the standing system to the waveguide, which has been drawing more and more attention (Kiilerich & Mølmer, 2019; Shi, Chang, & Cirac, 2015; Trivedi, Fischer, Xu, Fan, & Vuckovic, 2018).

The second class is the *catching* of incoming flying qubits for the purpose of receiving and storing quantum information. This is usually done by the interaction with a standing system that is initialized at its ground state (Pan, Zhang, Cui, & James, 2015; Stobinska, Alber, & Leuchs, 2009). As an inverse process of the flying-qubit generation, the standing system receives flying qubits as quantum inputs and yields vacuum quantum outputs. Theoretical analysis shows that only properly shaped flying qubits can be completely caught, e.g., a standing qubit with fixed coupling to a waveguide is only able to catch exponentially rising (Aljunid, Maslennikov, Wang, Dao, Scarani, & Kurtsiefer, 2013; Dao, Aljunid, Maslennikov, & Kurtsiefer, 2012; Pan et al., 2015; Pan, Zhang, & James, 2016; Yamamoto & James, 2014) or time-reversal symmetrized (Srinivasan et al., 2014) single-photon pulses. For more complicated systems, the involved coherent or incoherent controls on the standing system need to be carefully optimized so as to perfectly match the incoming qubit (Nurdin, James, & Yamamoto, 2016; Sounas, 2020; Srinivasan et al., 2014; Wenner et al., 2014).

The third class is the *transformation* of flying qubits for the purpose of information transformation. In this scenario, the standing quantum system yields non-empty quantum outputs in response to incoming flying qubits as a non-empty quantum input. For examples, an incoming flying qubit can be coherently routed into the superposition of two paths (Xia, Jelezko, & Twamley, 2018), or be reshaped into time-reversed waveform to match remote quantum nodes (Leong, Seidler, Steiner, Cere, & Kurtsiefer, 2016; Srivathsan, Gulati, Cere, Chng, & Kurtsiefer, 2014). The engineering of transformation processes can be taken as a generalization of the generation and catching problems.

In these flying-qubit control problems, we are mainly concerned with quantum input–output processes of flying qubits steered by classical time-dependent control functions. The classical control functions may be either coherent (e.g., a semi-classical driving field) or incoherent (e.g., tunable waveguide coupling) in the sense that the control preserves the unitarity of the standing system's evolution or not. The modeling of the underlying control dynamics is very complicated. In particular, when flying qubits are generated at high repetition rates, the single-mode approximation adopted in cavity QED (quantum electrodynamics) systems (Gu et al., 2017; Scully & Zubairy, 1997) is not viable any more because broadband continuous modes are involved (Li, Zhou, & Sun, 2014). Regarding this, the underlying physical processes must be described by waveguide QED instead of cavity QED.

To facilitate the control of single flying qubits, the standing quantum system is often chosen to be nonlinear (e.g., a finite-level system or an anharmonic oscillator) (Trivedi et al., 2018). In contrast to linear quantum systems (Zhang, 2014, 2016; Zhang & Dong, 2022; Zhang & James, 2011) that can be conveniently described by transfer functions, analytical solutions to cases with nonlinear standing quantum systems are usually unavailable.

Moreoever, most existing results (Shen & Fan, 2009; Trivedi et al., 2018; Yao et al., 2005) are heavily dependent on the conservation of energy (or the excitation number) that enables the analysis within a finite-dimensional subspace of quantum states. This symmetry is usually not preserved under time-dependent coherent driving controls, which force the analysis into infinite-dimensional Hilbert spaces. In such cases, approximations have to be introduced, e.g., the coarse discretization of time in the time-resolved scattering of quantum fields by a coherently driven atom (Fischer et al., 2018); and the adiabatic elimination in the shaping of single photons (Hurst & Kok, 2018).

Regarding these difficulties, it is natural to introduce the Quantum Stochastic Differential Equation (QSDE) (Gardiner & Collett, 1985; Hudson & Parthasarathy, 1984) for the modeling of flying-qubit input–output processes. In the literature, the QSDE has been successful in special cases that the standing system is linear or the number of excitations is preserved, including the analysis and filtering of flying-qubit responses (Dong, Zhang, & Amini, 2019; Gough, James, & Nurdin, 2014; Song, Zhang, & Xi, 2016; Zhang, 2014, 2016, 2020, 2021; Zhang & Dong, 2022; Zhang & James, 2011), as well as flying-qubit control applications to the filter-based generation of non-classical quantum fields with linear quantum systems (Gough & Zhang, 2015) and to the catching of single flying qubits using a two-level quantum system with tunable waveguide coupling (Nurdin et al., 2016).

In a preliminary version of this paper (Li, Zhang, & Wu, 2020), we extended the QSDE approach to general flying-qubit generation problems that allows for nonlinear standing quantum systems and coherent driving controls that do not preserve the number of excitations. It is shown in Section 2 that, under the temporal photon-number state basis, the seemingly complicated continuous-mode stochastic different equation can be reduced to a much simpler non-unitary Schrödinger equation, which is fully consistent with the results obtained in Fischer et al. (2018). In this paper, we will delve into more general problems including the catching and transformation of flying qubits, forming a unified framework for the analysis and design of flying-qubit control systems. The major contributions made in this paper are as follows. First, based on the temporal photon-number state basis presented in Section 2, we formalize the discussions in the preliminary version (Li et al., 2020) by introducing Theorem 3.1 and Corollary 3.1. After that we generalize the proposed QSDE formulation (see Theorem 3.1 and Corollary 3.1) to flying-qubit catching and transformation problems by introducing an auxiliary system to generate the required flying-qubit inputs. Then, in the analysis of flying-qubit generation problems, we use the derived formulas to reproduce and generalize an existing result of generating arbitrary-shape single photons with an incoherently driven two-level system (see Proposition 4.1), as well as the analytic expressions of flying-qubit emissions under rectangular coherent driving pulses. In the catching problems, we apply the formulas to reproduce the perfect catching condition (see Proposition 4.2). Finally, in the transformation problems, we apply the formulas to the calculation of outgoing flying-qubit states induced by stimulated emission and to the optimal reshaping of incoming flying qubits.

The remainder of this paper is organized as follows. In Section 2, necessary preliminaries will be provided for the state representation of flying qubits and their joint system with the standing quantum system. In Section 3, the non-unitary Schrödinger equation is derived for calculating multi-photon wave packets under coherent or incoherent controls. In Section 4, we demonstrate the theory with applications to flying-qubit generation, catching and transformation problems, respectively, with a two-level standing quantum system. Finally, concluding remarks are made in Section 5.

2. The quantum state representation of flying qubits

Consider flying qubits carried by single photons in quantized electromagnetic fields traveling in waveguides. The field generally contains a continuum of bosonic modes with mode frequency $\omega \in (-\infty, \infty)$. Each annihilation and creation operators of these continuous modes are defined on the entire Hilbert state space $\mathcal{H}_F$ with the identity operator on the rest modes being omitted, and they satisfy the singular commutation relations $[b(\omega), b^\dagger(\omega')] = \delta(\omega-\omega')$.

In practice, we often represent the field with the inverse Fourier transform of $b(\omega)$ (Gardiner & Collett, 1985)

$$b_\tau = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega\tau} b(\omega)\mathrm{d}\omega, \tag{1}$$

which is also defined on $\mathcal{H}_F$. Such temporal field operator b_τ describes the physical process of annihilating a photon at time τ, and satisfies the singular commutation relation $[b_\tau, b^\dagger_{\tau'}] = \delta(\tau-\tau')$. Further, we can use it to construct the Ito-type quantum Wiener increment

$$\mathrm{d}B_\tau = \int_\tau^{\tau+\mathrm{d}\tau} b_s \mathrm{d}s, \quad \mathrm{d}\tau > 0, \tag{2}$$

that obeys Ito's rule

$$\begin{aligned} &\mathrm{d}B_\tau \mathrm{d}B_s^\dagger = \delta_{\tau,s}\mathrm{d}\tau, \\ &\mathrm{d}B_\tau^\dagger \mathrm{d}B_s = \mathrm{d}B_\tau \mathrm{d}B_s = \mathrm{d}B_\tau^\dagger \mathrm{d}B_s^\dagger = 0. \end{aligned} \tag{3}$$

Using the Ito-type increment $\mathrm{d}B_\tau^\dagger$, we can define the state of an arbitrary single flying qubit as follows:

$$|1_\xi\rangle_F = \int_{-\infty}^{\infty} \xi^\tau \mathrm{d}B_\tau^\dagger |\Omega\rangle_F, \tag{4}$$

with $\int_{-\infty}^{\infty} |\xi^\tau|^2 \mathrm{d}\tau = 1$, where $|\Omega\rangle_F \in \mathcal{H}_F$ is the vacuum state of the quantum field that includes all continuous modes and $|\xi^\tau|^2\mathrm{d}\tau$ indicates the probability of observing the photon between τ and $\tau + d\tau$. Similarly, we can define the ℓ-photon state (Fischer et al., 2018),

$$|\ell_\xi\rangle_F = \int_{-\infty}^{\infty}\int_{-\infty}^{\tau_\ell}\cdots\int_{-\infty}^{\tau_2} \xi^{\tau_1,\ldots,\tau_\ell} \mathrm{d}B_{\tau_1}^\dagger \cdots \mathrm{d}B_{\tau_\ell}^\dagger |\Omega\rangle_F, \tag{5}$$

with

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\tau_\ell}\cdots\int_{-\infty}^{\tau_2} |\xi^{\tau_1,\ldots,\tau_\ell}|^2 \mathrm{d}\tau_1\cdots\mathrm{d}\tau_\ell = 1. \tag{6}$$

The most general flying-qubit state can thus be expanded as the superposition of multi-photon states:

$$|\xi\rangle_F = \sum_{\ell=0}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\tau_\ell}\cdots\int_{-\infty}^{\tau_2} \xi^{\tau_1,\ldots,\tau_\ell} \mathrm{d}B_{\tau_1}^\dagger \cdots \mathrm{d}B_{\tau_\ell}^\dagger |\Omega\rangle_F, \tag{7}$$

with

$$\sum_{\ell=0}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\tau_\ell}\cdots\int_{-\infty}^{\tau_2} |\xi^{\tau_1,\ldots,\tau_\ell}|^2 \mathrm{d}\tau_1\cdots\mathrm{d}\tau_\ell = 1. \tag{8}$$

Under this representation, each component $\xi^{\tau_1,\ldots,\tau_\ell}$ is unnormalized, and the integral

$$P_\ell = \int_{-\infty}^{\infty}\int_{-\infty}^{\tau_\ell}\cdots\int_{-\infty}^{\tau_2} |\xi^{\tau_1,\ldots,\tau_\ell}|^2 \mathrm{d}\tau_1\cdots\mathrm{d}\tau_\ell, \tag{9}$$

indicates the probability of observing ℓ photons in the state $|\xi\rangle_F$.

3. Flying qubit dynamics via classical controls

Throughout this paper, we always assume that the classical controls are imposed on the standing system from the initial time $t_0 = 0$, and the standing system is coupled to a chiral waveguide in which all flying qubits propagate rightwards (Peng et al., 2016). Let $\mathcal{H}_S$ be the Hilbert state space of the standing quantum system. To analyze the time evolution of the joint system, we first expand the time-dependent joint state of the standing system and the flying qubits in $\mathcal{H}_S \otimes \mathcal{H}_F$ as follows:

$$\begin{aligned} |\Psi(t)\rangle = &\sum_{\ell=0}^{\infty}\int_0^t\int_0^{\tau_\ell}\cdots\int_0^{\tau_2} |\psi^{\tau_1,\ldots,\tau_\ell}(t)\rangle \\ &\otimes \mathrm{d}B_{\tau_1}^\dagger\cdots\mathrm{d}B_{\tau_\ell}^\dagger|\Omega\rangle_F, \end{aligned} \tag{10}$$

where $|\psi^{\tau_1,\ldots,\tau_\ell}(t)\rangle \in \mathcal{H}_S$ is the standing system's correlated state at time t when ℓ photons are found at times $\tau_1,\ldots,\tau_\ell$ in the waveguide. Under this representation, the differential $\langle\psi^{\tau_1,\ldots,\tau_\ell}(t)|\psi^{\tau_1,\ldots,\tau_\ell}(t)\rangle\mathrm{d}\tau_1\mathrm{d}\tau_2\cdots\mathrm{d}\tau_\ell$ indicates the probability of sequentially generating ℓ photons in time intervals $[\tau_1, \tau_1 + \mathrm{d}\tau_1), [\tau_2, \tau_1+\mathrm{d}\tau_2), \ldots, [\tau_\ell, \tau_1+\mathrm{d}\tau_\ell)$ by the standing system. The modified lower and upper bounds of the first integral indicate that, physically, flying qubits can only be generated between time 0 and the present time t.

Under coherent driving controls, the above expansion usually involves an infinite number of terms. However, when the standing quantum system only exchanges energy with the waveguide (as its heat bath), the number of excitations will be preserved during the controlled evolution. If the joint system has M excitations in total, the standing system can only stay at its eigenstate $|M-\ell\rangle$ when the field contains ℓ photons, i.e., $|\psi^{\tau_1,\ldots,\tau_\ell}\rangle = \eta_\ell^{\tau_1,\ldots,\tau_\ell}|M-\ell\rangle$. For example, when $M = 1$ and the standing system is a two-level atom, the joint state can be written as follows:

$$|\Psi(t)\rangle = \eta_0(t)|1\rangle\otimes|\Omega\rangle_F + \int_0^t |0\rangle\otimes\eta_1^\tau(t)\mathrm{d}B_\tau^\dagger|\Omega\rangle_F. \tag{11}$$

When $M = 2$ and the system is a three-level atom, the joint state can be written as follows

$$\begin{aligned} |\Psi(t)\rangle = &\eta_0(t)|2\rangle\otimes|\Omega\rangle_F + \int_0^t |1\rangle\otimes\eta_1^\tau(t)\mathrm{d}B_\tau^\dagger|\Omega\rangle_F \\ &+ \int_0^t\int_0^{\tau_2} |0\rangle\otimes\eta_2^{\tau_1,\tau_2}(t)\mathrm{d}B_{\tau_2}^\dagger\mathrm{d}B_{\tau_1}^\dagger|\Omega\rangle_F. \end{aligned} \tag{12}$$

In the following, we will introduce the QSDE to model the flying qubit control processes with vacuum or non-vacuum quantum inputs, respectively.

3.1. Standing system with vacuum quantum inputs

We start from the simple case that the standing system receives vacuum quantum input, i.e., the joint system is initially at $|\Psi(0)\rangle = |\psi_0\rangle\otimes|\Omega\rangle_F$ with $|\psi_0\rangle$ being the initial state of the standing system. The controlled dynamics of the joint system can be described by the following QSDE (Gardiner & Zoller, 2004, Chapter 11):

$$\mathrm{d}U(t) = [(-iH(t) - \frac{1}{2}L^\dagger(t)L(t))\mathrm{d}t + L(t)\mathrm{d}B_t^\dagger - L^\dagger(t)\mathrm{d}B_t]U(t), \tag{13}$$

where $U(t)$ is the unitary propagator of the joint system whose initial value is the identity operator $\mathbb{I}$. The Hamiltonian $H(t)$ represents the total energy of the standing quantum system, including the internal energy and the coherently injected energy through classical driving fields. The operator $L(t)$ represents the coupling of the standing system with the flying qubits in the waveguide.

It is straightforward to derive from Eq. (13) the stochastic Schrödinger equation for $|\Psi(t)\rangle = U(t)|\Psi(0)\rangle$:

$$\mathrm{d}|\Psi(t)\rangle = \left[-iH_{\mathrm{eff}}(t)\mathrm{d}t + L(t)\mathrm{d}B_t^\dagger - L^\dagger(t)\mathrm{d}B_t\right]|\Psi(t)\rangle, \tag{14}$$

where $H_{\mathrm{eff}}(t) = H(t) - \frac{i}{2}L^\dagger(t)L(t)$ is the effective *non-Hermitian* Hamiltonian. According to the definition of the increments $\mathrm{d}B(t)$ and $\mathrm{d}B^\dagger(t)$ that point to the future, we have $[U(t), \mathrm{d}B(t)] = 0$, which implies that

$$\mathrm{d}B_t|\Psi(t)\rangle = \mathrm{d}B_tU(t)|\Psi(0)\rangle = U(t)|\psi_0\rangle \otimes \mathrm{d}B_t|\Omega\rangle_F = 0 \tag{15}$$

when the waveguide is initially empty. Hence, Eq. (14) can be simplified as (Gough, James, & Nurdin, 2013; Nurdin et al., 2016)

$$\mathrm{d}|\Psi(t)\rangle = \left[-iH_{\mathrm{eff}}(t)\mathrm{d}t + L(t)\mathrm{d}B_t^\dagger\right]|\Psi(t)\rangle. \tag{16}$$

Now we present the main Theorem for the analysis of flying-qubit control dynamics.

Theorem 3.1. *Let $V(t)$ be the propagator of the differential equation*

$$\dot{V}(t) = -iH_{\mathrm{eff}}(t)V(t), \quad V(0) = \mathbb{I}. \tag{17}$$

Then, the correlated states satisfy

$$|\psi^{\tau_1,\ldots,\tau_\ell}(t)\rangle = V(t)\tilde{L}(\tau_\ell)\cdots\tilde{L}(\tau_1)|\psi_0\rangle, \tag{18}$$

for $\ell = 0, 1, \ldots$, where $\tilde{L}(\tau_k) = V^{-1}(\tau_k)L(\tau_k)V(\tau_k)$.

Proof. Differentiate both sides of Eq. (10), we have

$$\begin{aligned}\mathrm{d}|\Psi(t)\rangle = &\sum_{\ell=0}^{\infty}\int_0^t\int_0^{\tau_\ell}\cdots\int_0^{\tau_2}|\dot{\psi}^{\tau_1,\ldots,\tau_\ell}(t)\rangle\mathrm{d}t \\ &\otimes \mathrm{d}B_{\tau_1}^\dagger\cdots\mathrm{d}B_{\tau_\ell}^\dagger|\Omega\rangle_F \\ &+ \mathrm{d}B_t^\dagger\sum_{\ell=1}^{\infty}\int_0^t\int_0^{\tau_{\ell-1}}\cdots\int_0^{\tau_2}|\psi^{\tau_1,\ldots,\tau_{\ell-1},t}(t)\rangle \\ &\otimes \mathrm{d}B_{\tau_1}^\dagger\cdots\mathrm{d}B_{\tau_{\ell-1}}^\dagger|\Omega\rangle_F,\end{aligned} \tag{19}$$

where the dot "·" represents the partial derivative with respect to time variable t. Replacing Eqs. (10) and (19) into Eq. (16), we can obtain

$$|\dot{\psi}^{\tau_1,\ldots,\tau_\ell}(t)\rangle = -iH_{\mathrm{eff}}(t)|\psi^{\tau_1,\ldots,\tau_\ell}(t)\rangle, \tag{20}$$

by comparing the terms with $\mathrm{d}t$, where $\ell \geq 0$. The comparison of terms with $\mathrm{d}B_t^\dagger$ gives the boundary condition

$$|\psi^{\tau_1,\ldots,\tau_{\ell-1},t}(t)\rangle = L(t)|\psi^{\tau_1,\ldots,\tau_{\ell-1}}(t)\rangle. \tag{21}$$

The transition from $|\psi^{\tau_1,\ldots,\tau_{\ell-1}}(t)\rangle$ to $|\psi^{\tau_1,\ldots,\tau_{\ell-1},t}(t)\rangle$ implies that the number of photons in the field is increased by 1, where the additional photon is emitted at time t from the standing system.

Observing that all functions $|\psi^{\tau_1,\ldots,\tau_\ell}(t)\rangle$ share the same differential Eq. (20) with respect to time t, we denote by $V(t)$ their common non-unitary propagator steered by the effective Hamiltonian $H_{\mathrm{eff}}(t)$, as in Eq. (17). Consequently, let

$$G(s, s') = V(s)V^{-1}(s'), \tag{22}$$

be the state transition operator from time s' to s. From Eqs. (17) and (20) we have $|\psi(t)\rangle = G(t, 0)|\psi_0\rangle$ and

$$|\psi^{\tau_1,\ldots,\tau_\ell}(t)\rangle = G(t, \tau)|\psi^{\tau_1,\ldots,\tau_\ell}(\tau)\rangle, \tag{23}$$

for any $0 \leq \tau_1 \leq \cdots \leq \tau_\ell \leq \tau \leq t$. Repeatedly using Eqs. (23) and (21), we have

$$\begin{aligned}|\psi^{\tau_1,\ldots,\tau_\ell}(t)\rangle &= G(t, \tau_\ell)|\psi^{\tau_1,\ldots,\tau_\ell}(\tau_\ell)\rangle \\ &= G(t, \tau_\ell)L(\tau_\ell)|\psi^{\tau_1,\ldots,\tau_{\ell-1}}(\tau_\ell)\rangle \\ &\vdots \\ &= G(t, \tau_\ell)L(\tau_\ell)G(\tau_\ell, \tau_{\ell-1})L(\tau_{\ell-1}) \\ &\quad G(\tau_{\ell-1}, \tau_{\ell-2})\cdots L(\tau_1)G(\tau_1, 0)|\psi_0\rangle.\end{aligned} \tag{24}$$

The compact form (18) is obtained from Eq. (24) by applying Eq. (22). □

Remark 3.1. Theorem 3.1 characterizes quantum trajectories that are well-known in the theory of open quantum systems (Wiseman & Milburn, 2010), along which single photons are sequentially released at random time instants $\tau_1, \ldots, \tau_\ell$. These quantum jumps intervene in the non-unitary evolution (20) of $|\psi^{\tau_1,\ldots,\tau_\ell}(t)\rangle$ by $L(\tau_1), \ldots, L(\tau_\ell)$, forming stochastic and discontinuous evolution trajectories that can be numerically calculated to analyze and design the flying qubit generation processes.

Theorem 3.1 provides the entire solution for the transient process of generating flying qubits, which is reduced to Eq. (17) on the space of the standing quantum system. To calculate the steady-state of the outgoing flying qubits, we have the following corollary.

Corollary 3.1. *If the standing system asymptotically decays to its ground state $|0\rangle$, then the outgoing flying qubit state can be decomposed into the superposition of multi-photon states represented by Eq. (7), where*

$$\xi^{\tau_1,\ldots,\tau_\ell} = \langle 0|V(\infty)\tilde{L}(\tau_\ell)\cdots\tilde{L}(\tau_1)|\psi_0\rangle, \tag{25}$$

for $\ell = 0, 1, 2, \ldots$.

Proof. The steady-state of the outgoing flying qubits can be obtained from the asymptotic limit $t \to \infty$ of the joint state $|\Psi(t)\rangle$. When the standing system decays to its ground state $|0\rangle$, only ground-state components in all $|\psi^{\tau_1,\ldots,\tau_\ell}(t)\rangle$ may survive. Hence, the asymptotic state of $|\Psi(t)\rangle$ can be decomposed as

$$\begin{aligned}|\Psi(\infty)\rangle = |0\rangle \otimes &\sum_{\ell=0}^{\infty}\int_0^\infty\int_0^{\tau_\ell}\cdots\int_0^{\tau_2}\eta^{\tau_1,\ldots,\tau_\ell} \\ &\mathrm{d}B_{\tau_1}^\dagger\cdots\mathrm{d}B_{\tau_\ell}^\dagger|\Omega\rangle_F,\end{aligned} \tag{26}$$

from which we see that $\eta^{\tau_1,\ldots,\tau_\ell} = \langle 0|\psi^{\tau_1,\ldots,\tau_\ell}(\infty)\rangle$ is nothing but the wavepacket of the outgoing ℓ-photon component. It is easy to derive Eq. (25) from Eq. (18). □

Remark 3.2. The assumption made in Corollary 3.1, i.e., the standing system decays to its ground state, always holds when the system is a two-level atom that is constantly coupled to the waveguide and is driven by a finite-duration driving pulse. Under this circumstance, the standing system will spontaneously decay to its ground state after the control pulse is turned off, and eventually is disentangled from the flying qubits. It is possible that, under elaborately designed coherent and incoherent

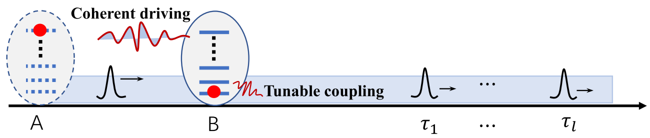

Fig. 2. Schematics of a flying-qubit control system B with quantum inputs generated from an auxiliary system A.

controls, or with multi-level atoms that have non-unique ground states (e.g., a Λ-type atom), the standing system remains entangled with the flying qubits. This has many intriguing applications in quantum networks, e.g., distributing quantum entanglements between remote nodes in the quantum network.

3.2. Standing system with non-vacuum quantum inputs

The above analysis is well suited for the generation of flying qubits, but it is not directly applicable to the catching or transformation problems where the quantum input of incoming flying qubits is not initially empty. As schematically shown in Fig. 2, we can introduce an auxiliary system A that is cascaded to the standing system B, where the auxiliary system is engineered to generate a flying qubit at the same state of the incoming flying qubit (Gough & Zhang, 2015). In this way, the analysis can be equivalently done with the joint AB system with vacuum quantum inputs, to which Theorem 3.1 and Corollary 3.1 become applicable.

Let $H_A(t)$ and $H_B(t)$ be the Hamiltonians of the A-system and B-system, and $L_A(t)$ and $L_B(t)$ be their coupling operators to the waveguide, respectively. According to the (S, L, H) formalism presented in Gough and James (2009), the series product of systems A and B has the following equivalent Hamiltonian

$$H(t) = H_A(t)\otimes \mathbb{I}_B + \mathbb{I}_A \otimes H_B(t) + \frac{1}{2i}\left[L_A(t)\otimes L_B^\dagger(t) - L_A^\dagger(t)\otimes L_B(t)\right], \tag{27}$$

and the equivalent coupling operator

$$L(t) = L_A(t)\otimes \mathbb{I}_B + \mathbb{I}_A \otimes L_B(t). \tag{28}$$

The series product representation makes it possible to apply Theorem 3.1 to analyze flying-qubit catching or transformation problems, because the joint AB system receives vacuum quantum inputs via the subsystem A. Note that the auxiliary system A needs to be carefully designed so as to yield the incoming flying qubits to be caught or transformed.

4. Examples

This section will illustrate how the proposed model is applied to the control of flying qubits. For simplicity, the standing system is chosen as a two-level qubit, whose controlled Hamiltonian is

$$H(t) = \epsilon(t)\sigma_+\sigma_- + \frac{u(t)}{2}\sigma_+ + \frac{u^*(t)}{2}\sigma_-, \tag{29}$$

$$L(t) = \sqrt{2\gamma(t)}\sigma_-, \tag{30}$$

where $\sigma_\pm$ are the standard Pauli raising and lowering operators. The functions $\epsilon(t)$ and $u(t)$ are the detuning frequency and the envelope of the coherent driving field on the qubit. The tunable coupling strength $\gamma(t)$ plays the role of incoherent classical controls. In the following, we discuss the flying-qubit generation and transformation problems, which correspond to the cases with vacuum and non-vacuum quantum inputs, respectively.

4.1. The generation of flying qubits

We will first apply Theorem 3.1 to reproduce a well-known result of generating arbitrary single-photon wavepackets via the incoherent control of tunable waveguide coupling function (Gough et al., 2013; Gough, James, Nurdin, & Combes, 2012; Gough & Zhang, 2015), with a slight generalization in that a time-variant phase is allowed in the wavepacket function.

Proposition 4.1. *Given a differentiable single-photon wavepacket $\xi^\tau = \nu(\tau)e^{-i\phi(\tau)}$, where $\nu(\tau)$ and $\phi(\tau)$ are the amplitude and phase functions of ξ^τ. Then, a flying qubit at state $|1_\xi\rangle$ can be generated without coherent driving $u(t)$ when the standing qubit is initially prepared at the excited state $|1\rangle$ and*

$$\gamma(\tau) = \frac{\nu^2(\tau)}{2\int_\tau^\infty \nu^2(s)\mathrm{d}s}, \quad \epsilon(\tau) = \frac{\mathrm{d}\phi(\tau)}{\mathrm{d}\tau}. \tag{31}$$

Proof. According to Theorem 3.1 and Corollary 3.1, it is straightforward to calculate the wavepacket of outgoing single-photon state as follows:

$$\xi^\tau = \sqrt{2\gamma(\tau)}\exp\left[-\int_0^\tau \gamma(s)\mathrm{d}s\right]\exp\left[-i\int_0^\tau \epsilon(s)\mathrm{d}s\right]. \tag{32}$$

Comparing with the shape function $\xi^\tau = \nu(\tau)e^{i\phi(\tau)}$, we have $\int_0^t \epsilon(\tau)\mathrm{d}\tau = \phi(t)$, which gives the phase condition in Eq. (31). The amplitude condition can be obtained from

$$\nu(\tau) = \sqrt{2\gamma(\tau)}\exp\left[-\int_0^\tau \gamma(s)\mathrm{d}s\right], \quad 0 < \tau < t, \tag{33}$$

in combination with the normalization condition of $\nu(\tau)$. □

The proof of Proposition 4.1 is relatively simple because the controls $\gamma(t)$ and $\epsilon(t)$ preserve the number of excitations in the joint system. Next, we consider the case with a coherent driving pulse that does not preserve the number of excitations. Assume that the driving control posed along the x-axis is resonant with the qubit (i.e., $\epsilon(t) \equiv 0$) and the coupling strength $\gamma(t) \equiv \gamma$. Under a rectangular driving control pulse on $0 < t \le T < \infty$, the effective Hamiltonian can be written as

$$H_{\rm eff}(t) = \begin{cases} \frac{\Omega}{2}\sigma_x - i\frac{\gamma}{2}\sigma_+\sigma_-, & 0 < t \le T \\ -i\frac{\gamma}{2}\sigma_+\sigma_-, & t > T \end{cases} \tag{34}$$

where $\sigma_x = \sigma_+ + \sigma_-$ and Ω is the power of the driving pulse.

Depending on the magnitude of the driving power Ω, we discuss the outgoing flying-qubit states in three regimes, namely the strong-driving regime ($\Omega > \gamma$), the balanced-driving regime ($\Omega = \gamma$) and the weak-driving regime ($\Omega < \gamma$). According to Theorem 3.1 and Corollary 3.1, we can obtain, for the example of strong-driving case, that

$$\xi = \langle 0|V(\infty)|0\rangle = e^{-\frac{\gamma T}{2}}\cos\left(\frac{\omega T}{2} - \phi\right), \tag{35}$$

and the single-photon wavepacket

$$\xi^\tau = \begin{cases} -\frac{i\sqrt{2\gamma}e^{-\frac{\gamma T}{2}}}{\cos^2\varphi}\sin\frac{\omega\tau}{2}\cos\left(\frac{\omega(T-\tau)}{2} - \varphi\right), & \tau \le T \\ -\frac{i\sqrt{2\gamma}e^{-\frac{\gamma T}{2}}}{\cos\varphi}\sin\frac{\omega T}{2}e^{-\gamma(\tau-T)}, & \tau > T \end{cases} \tag{36}$$

where $\varphi = \arcsin \frac{\gamma}{\Omega}$ and $\omega = \sqrt{\Omega^2 - \gamma^2}$. The above results can be inductively generalized to arbitrary ℓ-photon wavepackets

$$\xi^{\tau_1,\ldots,\tau_\ell} = \begin{cases} \frac{(-i\sqrt{2\gamma})^\ell e^{-\frac{\gamma T}{2}}}{\cos^{\ell+1}\varphi} \sin\frac{\omega\delta_1}{2} \sin\frac{\omega\delta_2}{2} \cdots \\ \quad \sin\frac{\omega\delta_\ell}{2} \cos\left(\frac{\omega(T-\tau_\ell)}{2} - \varphi\right), & \tau_\ell \le T \\ \frac{(-i\sqrt{2\gamma})^\ell e^{-\frac{\gamma T}{2}}}{\cos^{\ell}\varphi} \sin\frac{\omega\delta_1}{2} \sin\frac{\omega\delta_2}{2} \cdots \\ \quad \sin\frac{\omega\delta_{\ell-1}}{2} \sin\frac{\omega(T-\tau_{\ell-1})}{2} e^{-\gamma(\tau_\ell - T)}, & \tau_\ell > T \end{cases} \tag{37}$$

where $\delta_k = \tau_k - \tau_{k-1}$ with $\tau_0 = 0$ and $k = 1, 2, \ldots, \ell$, are the intervals between successively released single photons.

In the balanced-driving case, we can derive in the same way that

$$\xi = e^{-\frac{\gamma T}{2}} \left(1 + \frac{\gamma T}{2}\right), \tag{38}$$

and general ℓ-photon wavepackets

$$\xi^{\tau_1,\ldots,\tau_\ell} = \begin{cases} (-i\sqrt{2\gamma})^\ell e^{-\frac{\gamma T}{2}} \frac{\gamma^\ell}{2^\ell} \delta_1\delta_2 \cdots \\ \quad \delta_\ell \left(1 + \frac{\gamma(T-\tau_\ell)}{2}\right), & \tau_\ell \le T \\ (-i\sqrt{2\gamma})^\ell e^{-\frac{\gamma T}{2}} \frac{\gamma^\ell}{2^\ell} \delta_1\delta_2 \cdots \\ \quad \delta_{\ell-1} e^{-\gamma\tau_\ell}. & T < \tau_\ell \end{cases} \tag{39}$$

Similarly, in the weak-driving regime, we have

$$\xi = e^{-\frac{\gamma T}{2}} \left(\cosh\frac{\omega T}{2} + \frac{\gamma}{\omega} \sinh\frac{\omega T}{2}\right), \tag{40}$$

where $\omega = \sqrt{\gamma^2 - \Omega^2}$ and $\varphi = \operatorname{arctanh} \frac{\gamma}{\omega}$. The general ℓ-photon wavepackets are

$$\xi^{\tau_1,\ldots,\tau_\ell} = \begin{cases} \frac{(-i\sqrt{2\gamma})^\ell e^{-\frac{\gamma T}{2}}}{\cosh^{\ell+1}\varphi} \sinh\frac{\omega\delta_1}{2} \cdots \sinh\frac{\omega\delta_\ell}{2} \\ \quad \cosh\left(\frac{\omega(T-\tau_\ell)}{2} - \varphi\right), & \tau_\ell \le T \\ \frac{(-i\sqrt{2\gamma})^\ell e^{-\frac{\gamma T}{2}}}{\cosh^{\ell+1}\varphi} \sinh\frac{\omega\delta_1}{2} \cdots \sinh\frac{\omega\delta_{\ell-1}}{2} \\ \quad \sinh\frac{\omega(T-\tau_{\ell-1})}{2} e^{-\gamma(\tau_\ell - T)}. & T < \tau_\ell \end{cases} \tag{41}$$

Based on the above calculation, we display in Fig. 3(a) the shape $|\xi^\tau|^2$ of the single-photon component generated by soft π-pulses (i.e., $\Omega T = \pi$) in the strong, balanced and weak driving regimes, respectively. All these wavepackets gradually rise from zero and the steepness of the rising slope is proportional to the driving strength Ω. After the driving pulse is turned off, they spontaneously decay to zero. We also calculate the probabilities of generating 0, 1, 2 and more photons according to Eq. (9) when the pulse power Ω increases from weak- to strong-driving regimes. As is shown in Fig. 3(b), soft π-pulses do not generate perfect single flying qubits due to the coexisting multi-photon emission processes, unless in the hard-pulse limit that Ω approaches infinity.

We also plot in Fig. 4(a)–(c) the dependence of probabilities of generating $0 \sim 2$ flying qubits on the pulse area ΩT. It can be seen that more flying qubits are released when ΩT grows, while the probabilities of observing fewer flying qubits gradually decay to zero. In the strong driving regime ($\Omega = 2\gamma$), the probability of observing ℓ flying qubits reaches its peak when the pulse area is around $\ell\pi$ (corresponding to the required energy for generating ℓ flying qubits), but the trend is less apparent in the weak-driving regime ($\Omega = 0.5\gamma$).

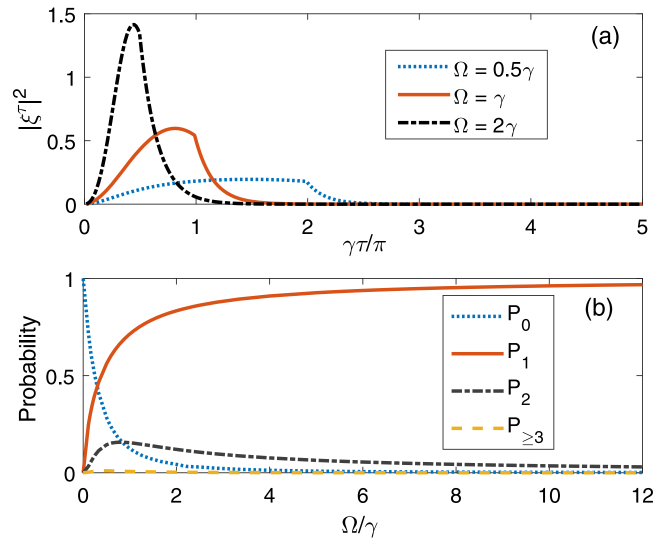

Fig. 3. (a) The shapes of single-photon wavepackets generated by π-pulses in the strong, balanced and weak driving regimes; (b) the probabilities of generating 0, 1, 2 and more flying qubits under π-pulses with driving strength varying from weak to strong coupling regimes.

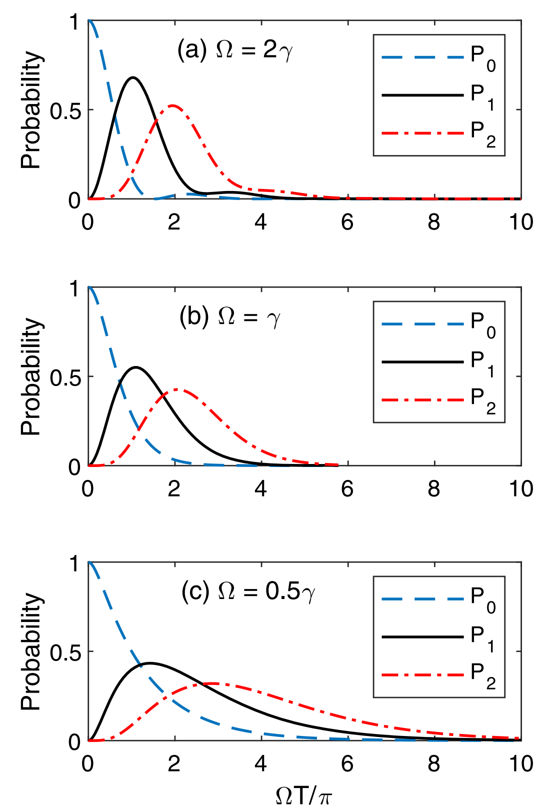

Fig. 4. The dependence of zero- to two-photon emission probabilities on the pulse area ΩT: (a) strong driving case $\Omega = 2\gamma$; (b) balanced driving case $\Omega = \gamma$; (c) weak driving case $\Omega = 0.5\gamma$.

4.2. The catching of flying qubits

The flying-qubit catching process is different from the generation process in that the waveguide is non-empty in the beginning, but is required to be empty by the end of the catching process. Here, we use the proposed approach to reproduce and generalize an existing result obtained by Nurdin et al. (2016).

Proposition 4.2. *Suppose that the standing qubit is initially prepared at the ground state* $|0\rangle$*, and has tunable Hamiltonian* $H(t) = \epsilon(t)\sigma_+\sigma_-$ *and coupling operator* $L(t) = \sqrt{2\gamma(t)}\sigma_-$*. Then, the standing qubit can perfectly catch an incoming flying qubit at state* $|1_\xi\rangle$*, where* $\xi(t) = \nu(t)e^{-i\phi(t)}$*, when*

$$2\gamma(t) = \frac{\nu^2(t)}{\int_0^t \nu^2(s)\mathrm{d}s}, \quad \epsilon(t) = \frac{d\phi(t)}{\mathrm{d}t}. \tag{42}$$

Proof. We introduce the auxiliary system A shown in Fig. 2 to generate the incoming flying qubit, whose Hamiltonian $H_0(t) = \epsilon_0(t)\sigma_+\sigma_-$ and coupling operator $L_0(t) = \sqrt{2\gamma_0(t)}\sigma_-$ are chosen according to Proposition 4.1. Then the joint AB-system's initial state is $|\psi_{AB}(0)\rangle = |10\rangle$, which, under the ordered basis $\{|00\rangle, |11\rangle, |10\rangle, |01\rangle\}$, has the vector form $[0\ 0\ 1\ 0]^\top$.

According to Eq. (27), the equivalent Hamiltonian

$$\begin{aligned} H_{\text{eff}}(t) = -i[&\gamma_A(t)\sigma_+\sigma_- \otimes \mathbb{I}_2 + \mathbb{I}_2 \otimes \gamma_B(t)\sigma_+\sigma_- \\ &+ 2\sqrt{\gamma_0(t)\gamma(t)}\sigma_- \otimes \sigma_+], \end{aligned} \tag{43}$$

and the coupling operator

$$L(t) = \begin{bmatrix} 0 & 0 & \sqrt{2\gamma_0(t)} & \sqrt{2\gamma(t)} \\ 0 & 0 & 0 & 0 \\ 0 & \sqrt{2\gamma(t)} & 0 & 0 \\ 0 & \sqrt{2\gamma_0(t)} & 0 & 0 \end{bmatrix}, \tag{44}$$

under the ordered basis $\{|00\rangle, |11\rangle, |10\rangle, |01\rangle\}$, where $\gamma_A(t) = \gamma_0(t) + i\epsilon_0(t)$ and $\gamma_B(t) = \gamma(t) + i\epsilon(t)$. Applying Theorem 3.1 to calculate the correlated system states $|\psi^{\tau_1,\ldots,\tau_\ell}(t)\rangle$, we have

$$|\psi(t)\rangle = e^{-\Gamma_A(t)}|10\rangle - \Xi(t)e^{-\Gamma_B(t)}|01\rangle, \tag{45}$$

$$|\psi^\tau(t)\rangle = \left[\sqrt{2\gamma_0(\tau)}e^{-\Gamma_A(\tau)} - \sqrt{2\gamma(\tau)}e^{-\Gamma_B(\tau)}\Xi(\tau)\right]|00\rangle, \tag{46}$$

where

$$\begin{aligned} \Gamma_{A,B}(t) &= \int_0^t \gamma_{A,B}(s)\mathrm{d}s, \\ \Xi(t) &= 2\int_0^t \sqrt{\gamma_0(s)\gamma(s)}e^{\Gamma_B(s)-\Gamma_A(s)}\mathrm{d}s. \end{aligned} \tag{47}$$

The rest terms $|\psi^{\tau_1,\ldots,\tau_\ell}(t)\rangle \equiv 0$ vanish for all $\ell \geq 2$.

The above equations show that at most one flying qubit can leak into the waveguide from the system B. Since we want the flying qubit to be fully absorbed, the term $|\psi^\tau(t)\rangle$ corresponding to the leakage must be kept zero, i.e.,

$$\sqrt{2\gamma_0(\tau)}e^{-\Gamma_A(\tau)} - \sqrt{2\gamma(\tau)}e^{-\Gamma_B(\tau)}\Xi(\tau) \equiv 0. \tag{48}$$

This condition leaves only the term $|\psi(t)\rangle \otimes |\Omega\rangle_F$ in the joint state $|\Psi(t)\rangle$, which implies that $|\psi(t)\rangle$ must be a normalized vector:

$$1 \equiv \langle\psi(t)|\psi(t)\rangle = e^{-2\Gamma_0(t)} + |\Xi(t)|^2 e^{-2\Gamma(t)}, \tag{49}$$

where $\Gamma_0(t) = \int_0^t \gamma_0(s)\mathrm{d}s$ and $\Gamma(t) = \int_0^t \gamma(s)\mathrm{d}s$.

Combining (48) and (49), we can cancel the term $\Xi(t)$ and use (33) to obtain the following condition

$$2\gamma(t) = \frac{2\gamma_0(t)e^{-2\Gamma_0(t)}}{1 - e^{-2\Gamma_0(t)}} = \frac{\nu^2(t)}{\int_0^t \nu^2(s)\mathrm{d}s}, \tag{50}$$

for the coupling strength in order to completely catch the flying qubit. Moreover, it is easy to examine that (48) and (49) hold when $\epsilon(t) = \epsilon_0(t)$, from which the phase condition can be obtained from Proposition 4.1. □

4.3. The transformation of flying qubits

Now let us see how a standing qubit responses to an incoming flying qubit at state $|1_{\xi_0}\rangle$, where ξ_0^τ is its wavepacket. Suppose that the standing qubit is prepared at the ground state $|0\rangle$ and is controlled by the tunable coupling $\gamma(t)$ and the coherent driving $u(t)$. We introduce the auxiliary two-level system A shown in Fig. 2 to generate the incoming flying qubit, which is initially excited and its Hamiltonian $H_0(t) = \epsilon_0(t)\sigma_+\sigma_-$ and coupling operator $L_0(t) = \sqrt{2\gamma_0(t)}\sigma_-$ are chosen according to Proposition 4.1. From Eq. (27), the effective Hamiltonian of the AB system is

$$\begin{aligned} H_{\text{eff}}(t) = \mathbb{I}_2 \otimes &\left[\frac{u(t)}{2}\sigma_+ + \frac{u^*(t)}{2}\sigma_-\right] - i[\gamma_A(t)\sigma_+\sigma_- \otimes \mathbb{I}_2 \\ &+ \mathbb{I}_2 \otimes \gamma_B(t)\sigma_+\sigma_- + 2\sqrt{\gamma_0(t)\gamma(t)}\sigma_- \otimes \sigma_+], \end{aligned} \tag{51}$$

and the coupling operator $L(t) = \sqrt{2\gamma_0(t)}\sigma_- \otimes \mathbb{I}_2 + \sqrt{2\gamma(t)}\mathbb{I}_2 \otimes \sigma_-$. The joint AB system is initially prepared at state $|\psi_{AB}(0)\rangle = |10\rangle$.

Consider the simple case that the coherent driving is a hard π pulse (i.e., $u(t) = \pi\delta(t)$) under which the system B is instantaneously flipped from $|0\rangle$ to $|1\rangle$, and hence $|\psi_{AB}(t)\rangle$ is flipped from $|10\rangle$ to $|11\rangle$ at $t = 0$. Hence, we only need to calculate the joint system's dynamics starting from the flipped state $|\psi_{AB}(0)\rangle = |11\rangle$. It can be verified from Corollary 3.1 that the wavepacket functions $\xi^{\tau_1,\ldots,\tau_\ell}$ in the steady state all vanish except when $\ell = 2$, and the two-photon wavepacket can be expressed as follows:

$$\begin{aligned} \xi^{\tau_1,\tau_2} =& 2\sqrt{\gamma_0(\tau_2)\gamma(\tau_1)}e^{-\Gamma_A(\tau_2)-\Gamma_B(\tau_1)} \\ &+ 2\sqrt{\gamma_0(\tau_1)\gamma(\tau_2)}e^{-\Gamma_A(\tau_1)-\Gamma_B(\tau_2)} \\ &+ 2\sqrt{\gamma(\tau_1)\gamma(\tau_2)}e^{-\Gamma_B(\tau_1)-\Gamma_B(\tau_2)}\left[\Xi(\tau_1) - \Xi(\tau_2)\right], \end{aligned} \tag{52}$$

where $\Gamma_{A,B}(\tau)$ and $\Xi(\tau)$ are defined by Eq. (47).

Recall that the system A is chosen to generate the desired single-photon wavepacket, i.e., $\xi_0^\tau = \sqrt{2\gamma_0(\tau)}e^{-\Gamma_A(\tau)}$. Let $\xi_1^\tau = \sqrt{2\gamma(\tau)}e^{-\Gamma_B(\tau)}$ be the wavepacket of a single photon spontaneously emitted from the system B driven by vacuum input. We can rewrite Eq. (52) in a more compact form:

$$\xi^{\tau_1,\tau_2} = \xi_0^{\tau_1}\xi_1^{\tau_2} + \xi_0^{\tau_2}\xi_1^{\tau_1} + \xi_1^{\tau_1}\xi_1^{\tau_2}\int_{\tau_1}^{\tau_2}\xi_0^s\xi_1^s\left[\int_0^s |\xi_1^\alpha|^2\mathrm{d}\alpha\right]^{-1}. \tag{53}$$

This form indicates that the incoming single photon knocks out an additional photon from the excited qubit, as is also called stimulated emission (Rephaeli & Fan, 2012), and the two single photons are entangled in a complex manner.

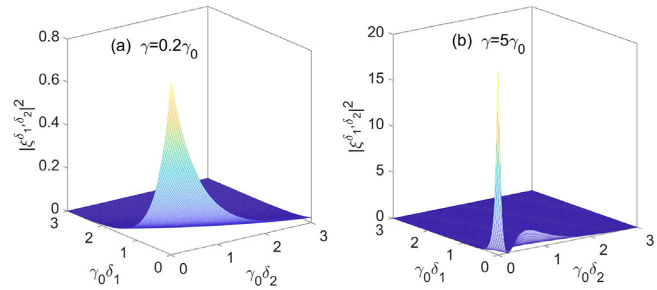

Fig. 5. The stimulated two-photon emission from an excited standing qubit by an incoming flying qubit: (a) the qubit decays slower than the incoming flying qubit pulse; (b) the standing qubit decays faster than the incoming flying qubit pulse.

We display the two-photon wavepackets in Fig. 5, which are reparameterized by independent variables $\delta_1 = \tau_1$ and $\delta_2 = \tau_2 - \tau_1$. In the case (a) that $\gamma < \gamma_0$ (i.e., the system's decay rate γ is slower than that of the incoming single-photon wavepacket ξ_0^τ), the two-photon emission probability monotonically decreases with both δ_1 and δ_2. In the case (b) that $\gamma \geq \gamma_0$ (i.e., the system's decay rate γ is faster than that of the incoming single-photon wavepacket ξ_0^τ), and the two flying qubits are not allowed to be separated at certain values of δ_2 (e.g., $\xi^{\tau_1,\tau_2} = 0$ when $\delta_2 = \gamma_0^{-1}$ and $\gamma_0 = \gamma$).

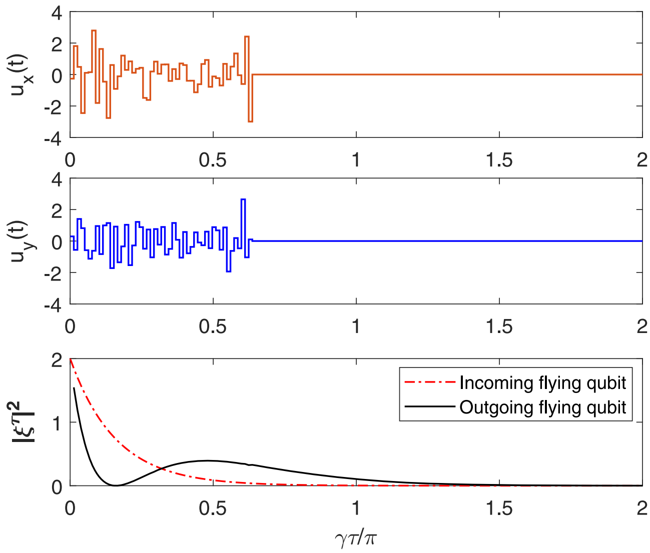

Fig. 6. The transformation of a flying qubit: (a)–(b) optimized control pulses; (c) the shapes of the incoming and outgoing flying qubits.

Now consider the case when soft driving control pulses are applied. We assume that the incoming single photon has exponentially decaying waveform $\xi_0^\tau = \sqrt{2\gamma_0}e^{-\gamma_0\tau}$ (corresponding to $\epsilon(t) = 0$ and $\gamma_0(t) \equiv \gamma_0$ for the auxiliary system A). The coherent driving control $u(t) = u_x(t) + iu_y(t)$ on the system B is turned on over a fixed time interval to reshape the flying qubit, and the joint system is initially prepared at state $|\Psi(0)\rangle = |10\rangle$.

We apply the genetic algorithm to optimize the waveform of $u(t)$ that maximizes the single-photon emission probability P_1. Numerical simulations show that the coherent control can improve the single-photon generation probability to 95.22% after 200 generations, yielding the optimized control pulse displayed in Fig. 6(a). As is shown in Fig. 6(b), the shape of the outgoing single-photon component is suppressed in the beginning and then revives, showing the ability of the driving control on reshaping the incoming flying qubit.

5. Conclusion

To conclude, we have developed a QSDE-based modeling method for the control of flying qubits in networked quantum information processing systems. Under the time-ordered photon-number basis, the proposed model can describe most general cases when the standing quantum system is steered by coherent or incoherent classical controls, and the quantum inputs can be either vacuum or non-vacuum. As illustrated by the examples, the states of the outgoing flying qubits can be efficiently computed by the derived low-dimensional non-unitary Schrödinger equation and the associated quantum jumps. This QSDE-based approach can be naturally applied to the analysis of flying-qubit control dynamics in quantum networks by combining the (S, L, H) formulation.

The established framework also lays the foundation for practical design of flying-qubit control systems, such as the precise shaping of flying qubits, the suppression of undesired multi-photon emissions, or robust control of flying qubits. As is exemplified in the numerical example, these design problems can be formulated as properly defined optimal control problems, and efficient algorithms need to be developed. These problems will be considered in our future studies.

The proposed model can be naturally extended to cases when the standing quantum system has multiple energy levels or is coupled to multiple waveguides. In principle, the analysis of flying qubit catching and transformation problems can still be handled by introducing multiple auxiliary systems coupled to these waveguides. However, the resulting calculation will become extremely hard as the equivalent system defined on the tensor of multiple Hilbert spaces becomes much larger. Hence, a direct method that is free of auxiliary systems is highly demanded. This has been shown to be possible from quantum scattering theory (Hurst & Kok, 2018; Trivedi et al., 2018), but the QSDE-based approach, which is potentially advantageous in the applications to complex networks, is still an open question. This will be also explored in our future works.

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