Linear maps transforming H-Unitary Matrices

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Dedicated to Professor Yung-Chow Wong on the occasion of his 90-th birthday

Abstract

Let H_1 be an $n \times n$ invertible Hermitian matrix, and let $\mathbf{U}(H_1)$ be the group of $n \times n$ H_1-unitary matrices, i.e., matrices A satisfying $A^*H_1A = H_1$. Suppose H_2 is an $m \times m$ invertible Hermitian matrix. We show that a linear transformation $\phi : M_n \to M_m$ satisfies $\phi(\mathbf{U}(H_1)) \subseteq \mathbf{U}(H_2)$ if and only if there exist invertible matrices $S \in M_m$, $U, V \in \mathbf{U}(H_2)$ such that

$$S^*H_2S = [(I_a \oplus -I_b) \otimes H_1] \oplus [(I_c \oplus -I_d) \otimes (H_1^{-1})^t],$$

and ϕ has the form

$$A \mapsto US[(I_{a+b} \otimes A) \oplus (I_{c+d} \otimes A^t)]S^{-1}V,$$

where a, b, c and d are nonnegative integers satisfying $(a+b+c+d)n = m$. Assume H_1 has inertia (p, q) and H_2 has inertia (r, s). Then there is a linear transformation mapping $\mathbf{U}(H_1)$ into $\mathbf{U}(H_2)$ if and only if there are nonnegative integers u and v such that $(r, s) = u(p, q) + v(q, p)$. These results generalize those of Marcus, Cheung and Li.

AMS Subject Classifications 15A04, 15A57, 15A63.
Keywords Linear transformations, H-unitary matrices, indefinite inner product.

1 Introduction

Let M_n be the algebra of $n \times n$ matrices. Suppose H is an invertible Hermitian matrix. A matrix $A \in M_n$ is *H-unitary* if $A^*HA = H$. Let $\mathbf{U}(H)$ be the set of H-unitary matrices. One readily checks that $\mathbf{U}(H)$ is a group, and $\mathbf{U}(I_n)$ is the usual unitary group. The study of H-unitary matrices arises from the study of indefinite inner product spaces. To see the connection, let $\langle \cdot, \cdot \rangle$ be the usual inner product, i.e. $\langle x, y \rangle = y^*x$ for $x, y \in \mathbb{C}^n$. An indefinite inner product in $\mathbb{C}^n$ is defined by

$$[x, y] = \langle Hx, y \rangle \quad \text{for any} \quad x, y \in \mathbb{C}^n.$$

Then $A \in M_n$ is *H-Hermitian* if $[Ax, y] = [x, Ay]$ for all $x, y \in \mathbb{C}^n$, equivalently, $HA = A^*H$; U is H-unitary if $[x, y] = [Ux, Uy]$ for all $x, y \in \mathbb{C}^n$, equivalently, $H^{-1}U^*HU = I_n$. We refer the readers to [2, 5] for general background of indefinite inner product spaces.

The purpose of this paper is to characterize linear transformations sending H_1-unitary matrices in M_n to H_2-unitary matrices in M_m for two given invertible Hermitian matrices $H_1 \in M_n$ and $H_2 \in M_m$. Denote by $X \otimes Y$ the matrix $(x_{ij}Y)$ for two matrices $X = (x_{ij})$ and Y. We have the following.

[1]Research partially supported by NSF.
[2]Thanks are due to Dr. Jor-Ting Chan for his guidance and encouragement.

Theorem 1 *Let $H_1 \in M_n$ and $H_2 \in M_m$ be invertible Hermitian matrices. A linear transformation $\phi : M_n \to M_m$ satisfies $\phi(\mathbf{U}(H_1)) \subseteq \mathbf{U}(H_2)$ if and only if there exist invertible matrices $S \in M_m$, $U, V \in \mathbf{U}(H_2)$ such that*

$$S^* H_2 S = [(I_a \oplus -I_b) \otimes H_1] \oplus [(I_c \oplus -I_d) \otimes (H_1^{-1})^t], \tag{1}$$

and ϕ has the form

$$A \mapsto US[(I_{a+b} \otimes A) \oplus (I_{c+d} \otimes A^t)]S^{-1}V, \tag{2}$$

where a, b, c and d are nonnegative integers satisfying $(a + b + c + d)n = m$.

Given two invertible Hermitian matrices $H_1 \in M_n$ and $H_2 \in M_m$, there does not always exist an invertible $S \in M_m$ satisfying (1). In such case, there will not be a linear transformation $\phi : M_n \to M_m$ such that $\phi(\mathbf{U}(H_1)) \subseteq \mathbf{U}(H_2)$. The next result show that the existence of an invertible $S \in M_m$ satisfying (1) is equivalent to the existence of a linear transformation $\phi : M_n \to M_m$ such that $\phi(\mathbf{U}(H_1)) \subseteq \mathbf{U}(H_2)$. Moreover, these conditions can be easily determined by the inertias of the matrices H_1 and H_2. (We say that the inertia of H_i is (p, q) if H_i has p positive eigenvalues and q negative eigenvalues.)

Theorem 2 *Let $H_1 \in M_n$ and $H_2 \in M_m$ be invertible Hermitian matrices such that H_1 has inertia (p, q) and H_2 has inertia (r, s). The following conditions are equivalent.*

(a) *There exists a linear transformation $\phi : M_n \to M_m$ such that $\phi(\mathbf{U}(H_1)) \subseteq \mathbf{U}(H_2)$.*

(b) *There exists an invertible matrix $S \in M_m$ satisfying* (1).

(c) *There are nonnegative integers u and v such that $(r, s) = u(p, q) + v(q, p)$.*

(d) *Either* (i) *$p - q = r - s = 0$ and $(u + v)p = r$, or*
(ii) *$p \neq q$ and $(u, v) = (pr - qs, ps - qr)/(p^2 - q^2)$ is a pair of nonnegative integers.*

Example 3 Suppose $(m, n) = (6, 3)$. If $(r, s) = (5, 1)$, and $(p, q) = (2, 1)$, then there does not exists (u, v) such that $(r, s) = u(p, q) + v(q, p)$. If we change (r, s) to $(4, 2)$, then $(u, v) = (2, 0)$ is the unique solution for the equation $(r, s) = u(p, q) + v(q, p)$.

When $(H_1, H_2) = (I_n, I_m)$, our results reduce to the following theorem in [4].

Corollary 4 *There is a linear transformation $\phi : M_n \to M_m$ such that $\phi(\mathbf{U}(I_n)) \subseteq \mathbf{U}(I_m)$ if and only if m is a multiple of n, and there exist $U, V \in \mathbf{U}(I_m)$ such that ϕ has the form $A \mapsto U[(I_u \otimes A) \oplus (I_v \otimes A^t)]V$, where u and v are nonnegative integers satisfying $(u+v)n = m$.*

When $H_1 = H_2 = I_n$, our results reduce to that of Marcus [7], see also [3, 6].

Corollary 5 *A linear transformation $\phi : M_n \to M_n$ satisfies $\phi(\mathbf{U}(I_n)) \subseteq \mathbf{U}(I_n)$ if and only if there exist $U, V \in \mathbf{U}(I_n)$ such that ϕ has the form $A \mapsto UAV$ or $A \mapsto UA^tV$.*

2 Auxiliary Results and Proofs

Proof of Theorem 1.

Let $J_{p,q} = I_p \oplus -I_q$ for any nonnegative integers p and q, and let $\{E_{ij} : 1 \le i, j \le n\}$ be the standard basis of M_n.

Consider the ($\Leftarrow$) part. Note that $A \in \mathbf{U}(H_1)$ if and only if $A^t \in \mathbf{U}((H_1^{-1})^t)$. Let $u = a + b$ and $v = c + d$. If $A \in \mathbf{U}(H_1)$, then

$$\begin{aligned}
& S^*(U^{-1}\phi(A)V^{-1})^* H_2 (U^{-1}\phi(A)V^{-1}) S \\
= \;& [(I_u \otimes A^*) \oplus (I_v \otimes (A^t)^*)](S^* H_2 S)[(I_u \otimes A) \oplus (I_v \otimes A^t)] \\
= \;& [(I_u \otimes A^*) \oplus (I_v \otimes (A^t)^*)][(J_{a,b} \otimes H_1) \oplus (J_{c,d} \otimes (H_1^{-1})^t)][(I_u \otimes A) \oplus (I_v \otimes A^t)] \\
= \;& [(J_{a,b} \otimes A^* H_1 A) \oplus (J_{c,d} \otimes (A^t)^*(H_1^{-1})^t A^t)] \\
= \;& [(J_{a,b} \otimes H_1) \oplus (J_{c,d} \otimes (H_1^{-1})^t)] \\
= \;& S^* H_2 S.
\end{aligned}$$

Thus, $U^{-1}\phi(A)V^{-1} \in \mathbf{U}(H_2)$ and hence $\phi(A) \in \mathbf{U}(H_2)$ as well.

Next, consider the ($\Rightarrow$) part. Assume that $\phi : M_n \to M_m$ is a linear map satisfying $\phi(\mathbf{U}(H_1)) \subseteq \mathbf{U}(H_2)$. We will establish a sequence of assertions, which allow us to impose extra conditions on the transformation ϕ, after we replace ϕ by a mapping of the form

$$A \mapsto V\phi(UAU^{-1})V^{-1} \tag{3}$$

for suitable $U \in \mathbf{U}(H_1)$ and $V \in \mathbf{U}(H_2)$. We always assume the extra condition once the triggering assertion is proved.

Assertion 1 *Replacing ϕ by the mapping $A \mapsto \phi(I_n)^{-1}\phi(A)$, we may assume $\phi(I_n) = I_m$.*

Assertion 2 *We may assume that $H_1 = J_{p,q}$ and $H_2 = J_{r,s}$ with $p \ge q$ and $r \ge s$.*

Proof. Let $S_1 \in M_n$ be invertible such that $S_1^* H_1 S_1 = J_{p,q}$ for some nonnegative integers p and q satisfying $p + q = n$. We may assume that $p \ge q$ because $\mathbf{U}(H_1) = \mathbf{U}(-H_1)$. Then $X \in M_n$ is H_1-unitary if and only if $S_1^{-1} X S_1$ is $J_{p,q}$-unitary. Similarly, there is an invertible S_2 such that $Y \in M_m$ is H_2-unitary if and only if $S_2^{-1} Y S_2$ is $J_{r,s}$-unitary. Note that a linear map ϕ satisfies $\phi(\mathbf{U}(H_1)) \subseteq \mathbf{U}(H_2)$ if and only if the mapping ψ defined by $A \mapsto S_2^{-1}\phi(S_1 A S_1^{-1}) S_2$ satisfies $\psi(\mathbf{U}(J_{p,q})) \subseteq \mathbf{U}(J_{r,s})$. Furthermore, ϕ has the asserted form if and only if ψ has the same form. ■

Assertion 3 *The linear map ϕ sends $J_{p,q}$-Hermitian matrices to $J_{r,s}$-Hermitian matrices.*

Proof. Suppose A is $J_{p,q}$-Hermitian. Then for any $t \in \mathbb{R}$,

$$J_{p,q}(e^{itA})^* J_{p,q} e^{itA} = e^{-itJ_{p,q}A^*J_{p,q}} e^{itA} = e^{-itA} e^{itA} = I_n.$$

So, $e^{itA} \in \mathbf{U}(J_{p,q})$. Now,

$$\phi(e^{itA}) = I_m + it\phi(A) - t^2\phi(A^2)/2! + \cdots$$

is $J_{r,s}$-unitary, i.e.,

$$I_m = J_{r,s}\phi(e^{itA})^* J_{r,s}\phi(e^{itA}) = I_m + it(\phi(A) - J_{r,s}\phi(A)^* J_{r,s}) + \cdots.$$

Thus, $\phi(A) - J_{r,s}\phi(A)^* J_{r,s} = 0$, i.e., $\phi(A)$ is $J_{r,s}$-Hermitian. ∎

Assertion 4 *Suppose*

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \in M_m$$

is $J_{r,s}$-unitary and $J_{r,s}$-Hermitian, where $B_{11} \in M_r$ and $B_{22} \in M_s$. Then there exists a unitary matrix $X = X_1 \oplus X_2 \in M_m$ with $X_1 \in M_r$ and $X_2 \in M_s$ such that

$$X^*BX = \begin{pmatrix} Z\sqrt{I_k + D^2} & 0 & D & 0 \\ 0 & Z_1 & 0 & 0 \\ -D & 0 & -Z\sqrt{I_k + D^2} & 0 \\ 0 & 0 & 0 & Z_2 \end{pmatrix}, \qquad (4)$$

where $Z, D \in M_k$, $Z_1 \in M_{r-k}$ and $Z_2 \in M_{s-k}$ such that D is a diagonal matrix with positive diagonal entries and Z, Z_1 and Z_2 are diagonal matrices with diagonal entries in $\{1, -1\}$. Consequently, there is $S \in \mathbf{U}(J_{r,s})$ such that $S^{-1}BS$ is a diagonal matrix with diagonal entries in $\{1, -1\}$.

Proof. Since B is $J_{r,s}$-Hermitian, we have $B_{11} = B_{11}^*$, $B_{22} = B_{22}^*$ and $-B_{21} = B_{12}^*$. Let $U_1 \in M_r$ and $U_2 \in M_s$ be unitary such that

$$R = U_1^* B_{12} U_2 = \begin{pmatrix} D & 0_{k,s-k} \\ 0_{r-k,k} & 0_{r-k,s-k} \end{pmatrix},$$

where $D \in M_k$ is a diagonal matrix with positive diagonal entries arranged in descending order. Set $U = U_1 \oplus U_2$ and

$$\tilde{B} = J_{r,s} U^* B U = \begin{pmatrix} P & R \\ R^* & Q \end{pmatrix}.$$

Then $P = P^*$, $Q = Q^*$, $\tilde{B} \in \mathbf{U}(J_{r,s})$. So, $P^2 = I_r + RR^*$, $Q^2 = I_s + R^*R$ and $PR = RQ$. Hence, $P = P_1\sqrt{I_k + D^2} \oplus P_2$ and $Q = Q_1\sqrt{I_k + D^2} \oplus Q_2$, where $P_1, Q_1 \in M_k$ are unitary such that $P_1^2 = Q_1^2 = I_k$, $P_2 \in M_{r-k}$ and $Q_2 \in M_{s-k}$ satisfy $P_2^2 = I_{r-k}$, $Q_2^2 = I_{s-k}$, and

$\{P_1, Q_1, D, \sqrt{I_k + D^2}\}$ is a commuting family. Since $PR = RQ$, we see that $P_1 = Q_1$. Thus, there exist unitary matrices $W_0 \in M_k$, $W_1 \in M_{r-k}$ and $W_2 \in M_{s-k}$ such that $W_0 D = DW_0$ and all the matrices $W_1^* P_2 W_1$, $W_2^* Q_2 W_2$, and $W_0^* P_1 W_0 = W_0^* Q_1 W$ are in diagonal forms. Let $W = W_0 \oplus W_1 \oplus W_0 \oplus W_2$ and $X = UW$. Then $X^* BX$ has the asserted form.

Now, $X^* BX$ is permutationally similar to a direct sum of $Z_1 = W_1^* P_2 W_1$, $Z_2 = W_2^* Q_2 W_2$ and 2×2 matrices of the form

$$\text{(i)} \quad C = \begin{pmatrix} \sqrt{1+d^2} & d \\ -d & -\sqrt{1+d^2} \end{pmatrix} = \tilde{S}^{-1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tilde{S}$$

or

$$\text{(ii)} \quad C = \begin{pmatrix} -\sqrt{1+d^2} & d \\ -d & \sqrt{1+d^2} \end{pmatrix} = \tilde{S}^{-1} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tilde{S},$$

where

$$\tilde{S} = \begin{pmatrix} \sqrt{1+s^2} & s \\ s & \sqrt{1+s^2} \end{pmatrix} \quad \text{with} \quad s = \begin{cases} \{[\sqrt{1+d^2} - 1]/2\}^{1/2} & \text{if (i) holds,} \\ -\{[\sqrt{1+d^2} - 1]/2\}^{1/2} & \text{if (ii) holds.} \end{cases}$$

∎

Assertion 5 *Replacing ϕ by a mapping $A \mapsto S\phi(A)S^{-1}$ for some $S \in \mathbf{U}(J_{r,s})$, we may assume that for any diagonal matrix $D = \operatorname{diag}(d_1, \ldots, d_n)$,*

$$\phi(D) = d_1 I_{k_1} \oplus \cdots \oplus d_n I_{k_n} \oplus d_1 I_{k_{n+1}} \oplus \cdots \oplus d_n I_{k_{2n}},$$

where $k_1, \ldots, k_{2n}$ are nonnegative integers with $k_1 + \cdots + k_n = r$ and $k_{n+1} + \cdots + k_{2n} = s$.

Proof. To get the desired conclusion, it suffices to show that there is $S \in \mathbf{U}(J_{r,s})$ such that the mapping defined by $A \mapsto S\phi(A)S^{-1}$ satisfies

$$I_t \oplus 0_{n-t} \mapsto I_{k_1+\cdots+k_t} \oplus 0_{k_{t+1}+\cdots+k_n} \oplus I_{k_{n+1}+\cdots+k_{n+t}} \oplus 0_{k_{n+t+1}+\cdots+k_{2n}} \tag{5}$$

for any $1 \leq t \leq n$. We prove this claim by induction on t. For $t = 1$, let $A = J_{1,n-1}$. Since A is $J_{p,q}$-Hermitian and $J_{p,q}$-unitary, by Assertions 3 and 4, $\phi(A) = S_1^{-1} D_1 S_1$, where $S_1 \in \mathbf{U}(J_{r,s})$ and $D_1 = I_{k_1} \oplus -I_{r-k_1} \oplus I_{k_{n+1}} \oplus -I_{s-k_{n+1}}$ for some nonnegative integers k_1 and k_{n+1}. Replacing ϕ by the mapping $X \mapsto S_1 \phi(X) S^{-1}$, we have $\phi(J_{1,n-1}) = D_1$. Since $\phi(I_n) = I_m$, we see that $\phi([1] \oplus 0_{n-1})$ has the asserted form.

Now, we assume that (5) holds for $t - 1$. Let $A = J_{t,n-t}$ and $K = I_{t-1,n-t+1}$. By induction assumption and the fact that $\phi(I_n) = I_m$, we may assume that $L = \phi(K) = I_a \oplus -I_{r-a} \oplus I_b \oplus -I_{s-b}$, where $a = k_1 + \cdots + k_{t-1}$ and $b = k_{n+1} + \cdots + k_{n+t}$. Let $B = \phi(A)$. Since $(A \pm iK)/\sqrt{2} \in \mathbf{U}(J_{p,q})$, it follows that $(B \pm iL)/\sqrt{2} \in \mathbf{U}(J_{r,s})$. So,

$$2I_m = J_{r,s}(B \pm iL)^* J_{r,s}(B \pm iL) = (B \mp iL)(B \pm iL) = B^2 + L^2 \pm i(BL - LB).$$

Thus $BL = LB$, i.e., $LBL = B$. Since $L = I_a \oplus -I_{r-a} \oplus I_b \oplus -I_{s-b}$, B has the form

$$\begin{pmatrix} B_{11} & 0 & B_{13} & 0 \\ 0 & B_{22} & 0 & B_{24} \\ B_{31} & 0 & B_{33} & 0 \\ 0 & B_{42} & 0 & B_{44} \end{pmatrix},$$

according to the block structure of L. On the other hand, both A and $A - 2[I_{t-1} \oplus 0_{n-t+1}]$ are $J_{p,q}$-unitary. Thus, B and $\tilde{B} = B - 2(I_a \oplus 0_{r-a} \oplus I_b \oplus 0_{s-b})$ are $J_{r,s}$-unitary, i.e., $B^* J_{r,s} B J_{r,s} = I_m = \tilde{B}^* J_{r,s} \tilde{B} J_{r,s}$. It follows that $B_{11} = I_a$, $B_{33} = I_b$, B_{13} and B_{31} are zero, and $\begin{pmatrix} B_{22} & B_{24} \\ B_{42} & B_{44} \end{pmatrix} \in \mathbf{U}(J_{r-a,s-b})$. By Assertion 4, there exists

$$\begin{pmatrix} S_{22} & S_{24} \\ S_{42} & S_{44} \end{pmatrix} \in \mathbf{U}(J_{r-a,s-b}) \quad \text{such that} \quad S_t = \begin{pmatrix} I_a & 0 & 0 & 0 \\ 0 & S_{22} & 0 & S_{24} \\ 0 & 0 & I_b & 0 \\ 0 & S_{42} & 0 & S_{44} \end{pmatrix} \in \mathbf{U}(J_{r,s})$$

and

$$S_t^{-1} \phi(A) S_t = S_t^{-1} B S_t = I_a \oplus I_{k_t} \oplus -I_{r-a-k_t} \oplus I_b \oplus I_{k_{n+t}} \oplus -I_{s-b-k_{n+t}}.$$

Now, we can replace ϕ by $X \mapsto S_t^{-1} \phi(X) S_t$ and assume that $\phi(I_t \oplus 0_{n-t})$ has the desired form. ■

We need some more notations and definitions in the rest of our proof. We have to consider different cases according to the following three types of ordered pair (u, v) of integers with $1 \le u < v \le n$:

$$\text{I}: 1 \le u \le p < v \le n; \qquad \text{II.a}: 1 \le u < v \le p; \qquad \text{II.b}: p < u < v \le n.$$

For any $B \in M_n$ and $C \in M_2$, let

$$A = B(C; [u, v])$$

be the matrix in M_n obtained from B by replacing $\begin{pmatrix} b_{uu} & b_{uv} \\ b_{vu} & b_{vv} \end{pmatrix}$ by C. Similarly, for any $C \in M_k$, and $B = (B_{ij}) \in M_m$, where $B_{ij} \in M_{k_i \times k_j}$ and $k = k_u + k_v + k_{n+u} + k_{n+v}$, let

$$A = B(C; \{u, v\})$$

be the matrix in M_m obtained from B by replacing its submatrix

$$\begin{pmatrix} B_{uu} & B_{uv} & B_{u(n+u)} & B_{u(n+v)} \\ B_{vu} & B_{vv} & B_{v(n+u)} & B_{v(n+v)} \\ B_{(n+u)u} & B_{(n+u)v} & B_{(n+u)(n+u)} & B_{(n+u)(n+v)} \\ B_{(u+v)u} & B_{(n+v)v} & B_{(n+v)(n+u)} & B_{(n+v)(n+v)} \end{pmatrix}$$

by C. For any $1 \le u < v < w \le n$, we define $B(C; [u, v, w])$ and $B(C; \{u, v, w\})$ in a similar way. Furthermore, we need the matrices in the following table in our proofs.

Table

Type	I	II.a	II.b
$M_B =$	$\begin{pmatrix} \sqrt{2} & \pm i \\ \pm i & -\sqrt{2} \end{pmatrix}$	$\begin{pmatrix} \frac{1}{\sqrt{2}} & \pm\frac{1}{\sqrt{2}} \\ \pm\frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix}$	same as Type II.a
$M_C =$	$\begin{pmatrix} \sqrt{2} & \pm 1 \\ \mp 1 & -\sqrt{2} \end{pmatrix}$	$\begin{pmatrix} \frac{1}{\sqrt{2}} & \pm\frac{i}{\sqrt{2}} \\ \mp\frac{i}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix}$	same as Type II.a
$M_+ =$	$\begin{pmatrix} \sqrt{2} & \pm\frac{i-1}{\sqrt{2}} \\ \pm\frac{i+1}{\sqrt{2}} & -\sqrt{2} \end{pmatrix}$	$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1+i}{2} \\ \frac{1-i}{2} & \frac{-1}{\sqrt{2}} \end{pmatrix}$	same as Type II.a
$J =$	$J_{2\alpha,\alpha} \oplus J_{2\beta,\beta}$	$I_{3\alpha} \oplus I_{3\beta}$	$J_{\alpha,2\alpha} \oplus J_{\beta,2\beta}$
$X =$	$\begin{pmatrix} \frac{\sqrt{3}+1}{2} & \frac{\sqrt{3}-1}{2} & 1 \\ \frac{\sqrt{3}-1}{2} & \frac{\sqrt{3}+1}{2} & 1 \\ 1 & 1 & \sqrt{3} \end{pmatrix}$	$\begin{pmatrix} \frac{\sqrt{3}-1}{2\sqrt{3}} & \frac{-\sqrt{3}-1}{2\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{-\sqrt{3}-1}{2\sqrt{3}} & \frac{\sqrt{3}-1}{2\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}$	$\begin{pmatrix} \sqrt{3} & 1 & 1 \\ 1 & \frac{\sqrt{3}+1}{2} & \frac{\sqrt{3}-1}{2} \\ 1 & \frac{\sqrt{3}-1}{2} & \frac{\sqrt{3}+1}{2} \end{pmatrix}$

Assertion 6 *Suppose $k_1, \ldots, k_{2n}$ have the meaning in Assertion 5. For any $1 \le u < v \le n$, let $B = \phi(E_{uv} + E_{vu})$ and $C = \phi(E_{uv} - E_{vu})$. The following conclusions hold.*

1. *If (u, v) is of Type I, then $k_u = k_{n+v}$ and $k_{n+u} = k_v$. Moreover, for $K = iB$, C or $(iB - C)/\sqrt{2}$, $K = 0_m(R; \{u, v\})$, where*

$$R = \begin{pmatrix} 0 & R_1 & 0 & R_2 \\ R_1^* & 0 & R_3 & 0 \\ 0 & -R_3^* & 0 & R_4 \\ -R_2^* & 0 & R_4^* & 0 \end{pmatrix} \tag{6}$$

and $R^2 = -I_k$.

2. *If (u, v) is of Type II.a/II.b, then $k_u = k_v$ and $k_{n+u} = k_{n+v}$. Moreover, for $K = B$, iC or $(B + iC)/\sqrt{2}$, $K = 0_m(R; \{u, v\})$, where R has the form (6) and $R^2 = I_k$.*

Consequently,

$$k_1 = \cdots = k_p = k_{n+p+1} = \cdots = k_{2n} = \alpha \quad \text{and} \quad k_{p+1} = \cdots = k_n = k_{n+1} = \cdots = k_{n+p} = \beta.$$

Proof. Suppose (u, v) is of Type I. Let $A = I_n(M; [u, v])$ with $M = M_B$, M_C or M_+, where M_B, M_C and M_+ are the matrices of Type I in the Table. Then A is both $J_{p,q}$-unitary and $J_{p,q}$-Hermitian. Write $A = (a_{ij})$ and $A_d = \text{diag}(a_{11}, \ldots, a_{nn})$. By Assertion 5, $\phi(A_d) = D = I_m(\tilde{D}; \{u, v\})$ with

$$\tilde{D} = a_{uu} I_{k_u} \oplus a_{vv} I_{k_v} \oplus a_{uu} I_{k_{n+u}} \oplus a_{vv} I_{k_{n+v}}. \tag{7}$$

Moreover, $\phi(A) = D \pm |a_{uv}| K = D \pm K$ are $J_{r,s}$-Hermitian as well as $J_{r,s}$-unitary with $K = iB$, C or $(iB - C)/\sqrt{2}$ depending on $M = M_B$, M_C or M_+. Hence,

$$I_m = J_{r,s}(D \pm K)^* J_{r,s}(D \pm K) = (D \pm K)^2 = D^2 + K^2 \pm (DK + KD).$$

Thus $DK + KD = 0$; by the block structure of $D = I_m(\tilde{D}; \{u, v\})$, only the following eight blocks

$$K_{uv},\ K_{u(n+v)},\ K_{vu},\ K_{v(n+u)},\ K_{(n+u)v},\ K_{(n+u)(n+v)},\ K_{(n+v)u},\ K_{(n+v)(n+u)},$$

can be nonzero. As K is $J_{r,s}$-Hermitian,

$$K_{vu} = K_{uv}^*, \quad K_{(n+v)u} = -K_{u(n+v)}^*, \quad K_{(n+u)v} = -K_{v(n+u)}^*, \quad K_{(n+v)(n+u)} = K_{(n+u)(n+v)}^*.$$

Hence, $K = 0_m(R; \{u, v\})$, where R has the form (6). Since $I_m = D^2 + K^2$, it follows that $I_k = \tilde{D}^2 + R^2$, where $\tilde{D}$ is the matrix in (7) and $k = k_u + k_v + k_{n+u} + k_{n+v}$. Since $\tilde{D}^2 = 2I_k$, we have $R^2 = -I_k$.

Suppose (u, v) is of Type II.a/II.b. Let $A = I_n(M; [u, v]) \in \mathbf{U}(J_{p,q})$ with $M = M_B$, M_C or M_+ of Type II.a/II.b in the Table. Then $\phi(A) = D \pm (K/\sqrt{2})$ with $D = \phi(\text{diag}(a_{11}, \ldots, a_{nn}))$ and $K = iB$, C or $(B + iC)/\sqrt{2}$. One can carry out a similar analysis as in the Type I case and conclude that $K = 0_m(R; \{u, v\})$, where R has the form (6) with $R^2 = I_k$. ∎

In the following, we always assume that $\alpha, \beta, k_1, \ldots, k_{2n}$ have the meaning in Assertions 5 and 6. We show that additional assumptions can be imposed on the matrices $\phi(E_{uv} + E_{vu})$ and $\phi(E_{uv} - E_{vu})$ for $1 \leq u < v \leq n$. However, it is inconvenient to use the block forms arising from Assertions 5 and 6. Instead, we introduce the following block permutation matrices:

$$P = \begin{pmatrix} I_{\alpha p} & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{\alpha q} \\ 0 & 0 & I_{\beta p} & 0 \\ 0 & I_{\beta q} & 0 & 0 \end{pmatrix}, \quad Q_1 = \begin{pmatrix} I_\alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & I_\alpha \\ 0 & 0 & I_\beta & 0 \\ 0 & I_\beta & 0 & 0 \end{pmatrix} \quad \text{and} \quad Q_2 = \begin{pmatrix} 0 & 0 & I_\alpha & 0 \\ 0 & 0 & 0 & I_\alpha \\ I_\beta & 0 & 0 & 0 \\ 0 & I_\beta & 0 & 0 \end{pmatrix},$$

where $P \in M_m$ and $Q_1, Q_2 \in M_{2(\alpha+\beta)}$. Then, for $D = \text{diag}(d_1, \ldots, d_n) \in M_n$ in Assertion 5, we have $P^*\phi(D)P = (D \otimes I_\alpha) \oplus (D \otimes I_\beta)$. Moreover, with the matrix R in Assertion 6, we have

$$P^*[0_m(R; \{u, v\})]P = \begin{cases} 0_m(Q_1^* R Q_1; \{u, v\}) & \text{if } (u, v) \text{ is of Type I,} \\ 0_m(R; \{u, v\}) & \text{if } (u, v) \text{ is of Type II.a,} \\ 0_m(Q_2^* R Q_2; \{u, v\}) & \text{if } (u, v) \text{ is of Type II.b,} \end{cases}$$

where $Q_2^* R Q_2$ also has the form (6) after relabeling the blocks $R_1, \ldots, R_4$, and $Q_1^* R Q_1$ has the form

$$T = \begin{pmatrix} 0 & T_1 & 0 & T_2 \\ -T_1^* & 0 & T_3 & 0 \\ 0 & T_3^* & 0 & T_4 \\ T_2^* & 0 & -T_4^* & 0 \end{pmatrix}. \tag{8}$$

Furthermore, it is clear that R has the form (6) if iR has the form (8). Thus, Assertion 6 can be rewritten as follows.

Assertion 7 *For any $1 \le u < v \le n$, let $B = \phi(E_{uv} + E_{vu})$ and $C = \phi(E_{uv} - E_{vu})$. Then*

$$P^*BP = 0_m(R; \{u, v\}), \;\; P^*CP = 0_m(T; \{u, v\}), \;\; \textit{and } P^*[(B + iC)/\sqrt{2}]P = 0_m(\tilde{R}; \{u, v\}),$$

where R and $\tilde{R}$ have the form (6) and T has the form (8) such that $R^2 = \tilde{R}^2 = I_k = -T^2$.

Assertion 8 *Replacing ϕ by the mapping $A \mapsto S\phi(A)S^{-1}$ for some $S \in \mathbf{U}(J_{r,s})$, we may assume that for any $A = E_{uv} + E_{vu}$ with $1 \le u < v \le n$,*

$$P^*\phi(A)P = (A \otimes I_\alpha) \oplus (A \otimes I_\beta).$$

Proof. Let

$$F = \begin{pmatrix} 0 & I_\alpha \\ I_\alpha & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & I_\beta \\ I_\beta & 0 \end{pmatrix} \in M_{2(\alpha+\beta)}. \tag{9}$$

By Assertion 7, $P^*\phi(E_{uv} + E_{vu})P = 0_m(R; \{u, v\})$, where R has the form (6). Since $R^2 = I_k$, it follows that

$$\begin{pmatrix} R_1 & R_2 \\ -R_3^* & R_4 \end{pmatrix} \in \mathbf{U}(J_{\alpha,\beta}) \quad \text{and} \quad U = \begin{pmatrix} I_\alpha & 0 & 0 & 0 \\ 0 & R_1 & 0 & R_2 \\ 0 & 0 & I_\beta & 0 \\ 0 & -R_3^* & 0 & R_4 \end{pmatrix} \in \mathbf{U}(J_{2\alpha,2\beta}).$$

Moreover, $R = U^{-1}FU$. Set $S_{uv} = P[I_m(U; \{u, v\})]P^*$. Then S_{uv} is $J_{r,s}$-unitary and

$$\begin{aligned} S_{uv}\phi(E_{uv} + E_{vu})S_{uv}^{-1} &= P[I_m(U; \{u, v\})][0_m(R; \{u, v\})][I_m(U^{-1}; \{u, v\})]P^* \\ &= P[0_m(URU^{-1}; \{u, v\})]P^* \\ &= P[0_m(F; \{u, v\})]P^*. \end{aligned}$$

By the block structure of S_{uv}, one can see that for any $A \in M_m$, all blocks of $S_{uv}AS_{uv}^{-1}$ are the same as those of A except for the blocks indexed by v and $n + v$. Hence,

$$S_{1v}P[0_m(R; \{1, v'\})]P^*S_{1v}^{-1} = P[0_m(R; \{1, v'\})]P^* \quad \text{for all} \quad v' \neq v.$$

Replacing ϕ by the mapping $A \mapsto S\phi(A)S^{-1}$ with $S = S_{12}\cdots S_{1n}$, we may assume that the assertion holds for $(u,v) = (1,w)$ with $2 \le w \le n$. Note also that the conclusion of Assertion 5 is not affected.

It remains to show that the conclusion holds for other (u,v) if $n > 2$. Let $A = I_n(X;[1,u,v])$, where $X = (x_{ij})$ is defined in the Table, depending on the type of (u,v). Assertion 7 ensures that $P^*\phi(E_{uv}+E_{vu})P = I_m(R;\{u,v\})$, where R has the form in (6). Together with Assertions 5 and 7, $P^*\phi(A)P = I_m(Y;\{1,u,v\})$, where

$$Y = \begin{pmatrix} x_{11}I_\alpha & x_{12}I_\alpha & x_{13}I_\alpha & 0 & 0 & 0 \\ x_{21}I_\alpha & x_{22}I_\alpha & x_{23}R_1 & 0 & 0 & x_{23}R_2 \\ x_{31}I_\alpha & x_{32}R_1^* & x_{33}I_\alpha & 0 & x_{32}R_3 & 0 \\ 0 & 0 & 0 & x_{11}I_\beta & x_{12}I_\beta & x_{13}I_\beta \\ 0 & 0 & -x_{23}R_3^* & x_{21}I_\beta & x_{22}I_\beta & x_{23}R_4 \\ 0 & -x_{32}R_2^* & 0 & x_{31}I_\beta & x_{32}R_4^* & x_{33}I_\beta \end{pmatrix}.$$

Since $\phi(A) \in \mathbf{U}(J_{r,s})$, it follows that Y is J-unitary, where J is defined in the Table depending on the type of (u,v). Comparing the $(1,3)$-th, $(1,5)$-th, $(1,6)$-th and $(4,6)$-th blocks on both sides of the equation $Y^*JY = J$, we see that $R_1 = I_\alpha$, $R_3 = 0$, $R_2 = 0$ and $R_4 = I_\beta$, respectively. Hence $R = F$. ∎

The next two assertions deal with $\phi(E_{uv} - E_{vu})$.

Assertion 9 *Replacing ϕ by the mapping $A \mapsto S\phi(A)S^{-1}$ for some $S \in \mathbf{U}(J_{r,s})$, we may further assume that*

$$P^*\phi(E_{12}-E_{21})P = [(E_{12}-E_{21}) \otimes J_{p_1,p_2}] \oplus [(E_{12}-E_{21}) \otimes J_{q_1,q_2}],$$

where p_1, p_2, q_1 and q_2 are nonnegative integers satisfying $p_1+p_2 = \alpha$ and $q_1+q_2 = \beta$.

Proof. Let

$$G = \begin{pmatrix} 0 & J_{p_1,p_2} \\ -J_{p_1,p_2} & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & J_{q_1,q_2} \\ -J_{q_1,q_2} & 0 \end{pmatrix} \in M_{2(\alpha+\beta)}. \tag{10}$$

Suppose $B = \phi(E_{12}+E_{21})$ and $C = \phi(E_{12}-E_{21})$. By assertion 7, we may write

$$P^*CP = 0_m(T;\{1,2\}) \quad \text{and} \quad P^*(B+iC)P = \sqrt{2}[0_m(\tilde{R};\{1,2\})],$$

where T and $\tilde{R}$ have the form (8) and (6) respectively, such that $\tilde{R}^2 = I_k = -T^2$. Since $P^*BP = 0_m(F;\{1,2\})$, we have $F + iT = \sqrt{2}\tilde{R}$. With $F^2 = I_k$, we see that $FT + TF = 0$, i.e.,

$$\begin{pmatrix} 0 & -T_1^* & 0 & T_3 \\ T_1 & 0 & T_2 & 0 \\ 0 & T_2^* & 0 & -T_4^* \\ T_3^* & 0 & T_4 & 0 \end{pmatrix} = FTF = -T = -\begin{pmatrix} 0 & T_1 & 0 & T_2 \\ -T_1^* & 0 & T_3 & 0 \\ 0 & T_3^* & 0 & T_4 \\ T_2^* & 0 & -T_4^* & 0 \end{pmatrix}.$$

So, $T_2 = -T_3$ and both T_1 and T_4 are Hermitian.

Now there exist unitary matrices $U_1 \in M_\alpha$ and $U_2 \in M_\beta$ such that $\tilde{T}_2 = U_1 T_2 U_2^* = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}$, where $D \in M_l$ is a diagonal matrix with positive entries in descending order, i.e., $D = d_1 I_{h_1} \oplus \cdots \oplus d_t I_{h_t}$ with $d_1 > \cdots > d_t$ and $h_1 + \cdots + h_t = l$. Let $U = U_1 \oplus U_1 \oplus U_2 \oplus U_2$. Then UTU^* has the same block form as T. Let

$$\tilde{T} = UTU^* = \begin{pmatrix} 0 & \tilde{T}_1 & 0 & \tilde{T}_2 \\ -\tilde{T}_1 & 0 & -\tilde{T}_2 & 0 \\ 0 & -\tilde{T}_2^* & 0 & \tilde{T}_4 \\ \tilde{T}_2^* & 0 & -\tilde{T}_4 & 0 \end{pmatrix}.$$

Then $\tilde{T}_1^2 = I_\alpha + \tilde{T}_2 \tilde{T}_2^*$; so, $\tilde{T}_1$ has the form

$$\tilde{T}_1 = \sqrt{1+d_1^2} X_1 \oplus \cdots \oplus \sqrt{1+d_t^2} X_t \oplus X_{t+1},$$

where $X_i \in M_{h_i}$ and $X_{t+1} \in M_{\alpha-l}$ are unitary and Hermitian. Similarly,

$$\tilde{T}_4 = \sqrt{1+d_1^2} Y_1 \oplus \cdots \oplus \sqrt{1+d_t^2} Y_t \oplus Y_{t+1},$$

where $Y_i \in M_{h_i}$ and $Y_{t+1} \in M_{\beta-l}$ are unitary and Hermitian. Set $X = X_1 \oplus \cdots \oplus X_t$ and $Y = Y_1 \oplus \cdots \oplus Y_t$. Since $\tilde{T}_1 \tilde{T}_2 + \tilde{T}_2 \tilde{T}_4 = 0$, it follows that $X = -Y$. Then

$$\tilde{T}_1 = \sqrt{I_l + D^2} X \oplus X_{t+1}, \quad \tilde{T}_4 = \sqrt{I_l + D^2}(-X) \oplus Y_{t+1},$$

and $\{X, D, \sqrt{I_l + D^2}\}$ is a commuting family. Therefore, there exist unitary $V_0 \in M_l$, $V_1 \in M_{\alpha-l}$ and $V_2 \in M_{\beta-l}$ such that $Z = V_0 X V_0^*$, $Z_1 = V_1 X_{t+1} V_1^*$ and $Z_2 = V_2 Y_{t+1} V_2^*$ are diagonal matrices with diagonal entries in $\{1, -1\}$. Let $W_1 = (V_0 \oplus V_1)U_1$, $W_2 = (V_0 \oplus V_2)U_2$ and $W = W_1 \oplus W_1 \oplus W_2 \oplus W_2$. Then

$$WTW^* = \begin{pmatrix} 0 & D_1 & 0 & D_2 \\ -D_1 & 0 & -D_2 & 0 \\ 0 & -D_2 & 0 & -D_4 \\ D_2 & 0 & D_4 & 0 \end{pmatrix}$$

with $D_1 = (Z\sqrt{I_l + D^2}) \oplus Z_1$, $D_4 = (Z\sqrt{I_l + D^2}) \oplus Z_2$ and $D_2 = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}$. Let

$$L = \begin{pmatrix} Z[(\sqrt{I_l + D^2} - I_l)/2]^{1/2} & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$K = \begin{pmatrix} \sqrt{I_l + LL^*} & 0 & L & 0 \\ 0 & \sqrt{I_l + LL^*} & 0 & L \\ L^* & 0 & \sqrt{I_l + L^*L} & 0 \\ 0 & L^* & 0 & \sqrt{I_l + L^*L} \end{pmatrix}.$$

Then $K \in \mathbf{U}(J_{2\alpha,2\beta})$, $K^{-1} = J_{2\alpha,2\beta} K J_{2\alpha,2\beta}$, and

$$WTW^* = K^{-1} \begin{pmatrix} 0 & Z \oplus Z_1 & 0 & 0 \\ -(Z \oplus Z_1) & 0 & 0 & 0 \\ 0 & 0 & 0 & -(Z \oplus Z_2) \\ 0 & 0 & Z \oplus Z_2 & 0 \end{pmatrix} K.$$

Now, there exist permutation matrices Q_1 and Q_2 such that $Q_1(Z \oplus Z_1)Q_1^* = J_{p_1,p_2}$ and $Q_2(Z \oplus Z_2)Q_2^* = J_{q_1,q_2}$ for some nonnegative integers p_1, p_2, q_1 and q_2 satisfying $p_1 + p_2 = \alpha$ and $q_1 + q_2 = \beta$. Let $Q = Q_1 \oplus Q_1 \oplus Q_2 \oplus Q_2$ and $S = QKW$. Then $STS^{-1} = G$.

Define $\tilde{W} = (I_n \otimes W_1) \oplus (I_n \otimes W_2)$, $\tilde{Q} = (I_n \otimes Q_1) \oplus (I_n \otimes Q_2)$,

$$\tilde{K} = \begin{pmatrix} I_n \otimes \sqrt{I_l + LL^*} & I_n \otimes L \\ I_n \otimes L^* & I_n \otimes \sqrt{I_l + L^*L} \end{pmatrix}$$

and $\tilde{S} = \tilde{Q}\tilde{K}\tilde{W}$. Then

$$\begin{aligned} (P\tilde{S}P^*)C(P\tilde{S}P^*)^{-1} &= P\tilde{S}[0_m(T;\{1,2\})]\tilde{S}^{-1}P^* \\ &= P[0_m(STS^{-1};\{1,2\})]P^* \\ &= P[0_m(G;\{u,v\})]P^*. \end{aligned}$$

On the other hand, we have $SFS^{-1} = F$. Hence, for any $1 \le u < v \le n$,

$$(P\tilde{S}P^*)\phi(E_{uv} + E_{vu})(P\tilde{S}P^*)^{-1} = P[0_m(SFS^{-1};\{u,v\})]P^* = P[0_m(F;\{u,v\})]P^*.$$

Since $P\tilde{Q}P^*$, $P\tilde{K}P^*$ and $P\tilde{W}P^*$ are all $J_{r,s}$-unitary, $P\tilde{S}P^* \in \mathbf{U}(J_{r,s})$. Replacing ϕ by $A \mapsto (P\tilde{S}P^*)\phi(A)(P\tilde{S}P^*)^{-1}$, we get the desired result. ∎

Assertion 10 *For any $A = E_{uv} - E_{vu}$ with $1 \le u < v \le n$, we have*

$$P^*\phi(A)P = (A \otimes J_{p_1,p_2}) \oplus (A \oplus J_{q_1,q_2}),$$

where p_1, p_2, q_1, q_2 are defined as in Assertion 9.

Proof. Define G as in (10). We first consider $(u,v) = (1,w)$. When $w = 2$, the result is valid by Assertion 9. Consider $w > 2$. By Assertion 7, $P^*\phi(E_{1w} - E_{w1})P = 0_m(T;\{1,w\})$, where T has the form (8). Let $A = I_n((XD_1);[1,2,w])$, where $D_1 = \mathrm{diag}(-1,1,1)$ and $X = (x_{ij})$ is defined in the Table, depending on the type of $(2,w)$. By Assertions 8 and 9, $P^*\phi(A)P = I_m(Y;\{1,2,w\})$, where

$$Y = \begin{pmatrix} -x_{11}I_\alpha & x_{12}J_{p_1,p_2} & x_{13}T_1 & 0 & 0 & x_{13}T_2 \\ -x_{21}J_{p_1,p_2} & x_{22}I_\alpha & x_{23}I_\alpha & 0 & 0 & 0 \\ -x_{31}T_1^* & x_{32}I_\alpha & x_{13}I_\alpha & x_{13}T_3 & 0 & 0 \\ 0 & 0 & x_{13}T_3^* & -x_{11}I_\beta & x_{12}J_{q_1,q_2} & x_{13}T_4 \\ 0 & 0 & 0 & -x_{21}J_{q_1,q_2} & x_{22}I_\beta & x_{23}I_\beta \\ x_{13}T_2^* & 0 & 0 & -x_{31}T_4^* & x_{32}I_\beta & x_{33}I_\beta \end{pmatrix}.$$

Since $\phi(A) \in \mathbf{U}(J_{r,s})$, it follows that Y is J-unitary, where J is defined in the Table depending on the type of (u, v). Comparing the $(2,3)$-th, $(2,4)$-th, $(2,6)$-th and $(5,6)$-th blocks on both sides of $Y^*JY = J$, we see that $T = G$.

For those (u, v) with $1 < u < v \leq n$, we can apply the preceding analysis to the matrix $A = I_n((D_2X); [1, u, v])$ with $D_2 = \text{diag}(1, 1, -1)$ to get the conclusion. ∎

By Assertions 5, 8–10, we see that for any $1 \leq i, j \leq n$,

$$\begin{aligned}
P^*\phi(E_{ii})P &= [E_{ii} \otimes I_\alpha] \oplus [E_{ii} \otimes I_\beta], \\
P^*\phi(E_{ij} + E_{ji})P &= [(E_{ij} + E_{ji}) \otimes I_\alpha] \oplus [(E_{ij} + E_{ji}) \otimes I_\beta], \\
P^*\phi(E_{ij} - E_{ji})P &= [(E_{ij} - E_{ji}) \otimes J_{p_1,p_2}] \oplus [(E_{ij} - E_{ji}) \otimes J_{q_1,q_2}].
\end{aligned}$$

Thus, there exists a permutation matrix $Q \in M_m$ such that for $E = E_{ii}$, $(E_{ij} + E_{ji})$ and $(E_{ij} - E_{ji})$,

$$Q^*\phi(E)Q = (I_{p_1+q_1} \otimes E) \oplus (I_{p_2+q_2} \otimes E^t).$$

Set $u = p_1 + q_1$ and $v = p_2 + q_2$. Then for any $A \in M_n$, we have

$$\phi(A) = Q[(I_u \otimes A) \oplus (I_v \otimes A^t)]Q^*.$$

It remains to prove that Q can be chosen to satisfy

$$Q^*J_{r,s}Q = (J_{a,b} \otimes J_{p,q}) \oplus (J_{c,d} \otimes J_{p,q})$$

for some nonnegative integers a, b, c and d satisfying $a + b = u$ and $c + d = v$. To this end, let

$$Q^*J_{r,s}Q = J_1 \oplus \cdots \oplus J_t,$$

where $t = m/n = u + v$ and $J_i \in M_n$ are diagonal matrices with diagonal entries in $\{1, -1\}$. Note that for any $A \in \mathbf{U}(J_{p,q})$, the matrix $Q^*\phi(A)Q$ is $Q^*J_{r,s}Q$-unitary. It follows that A is J_i-unitary for all i. Thus, $\mathbf{U}(J_{p,q}) \subseteq \mathbf{U}(J_i)$, and hence $J_i = \pm J_{p,q}$. Now, we may further permute the blocks J_i and assume that Q satisfies $Q^*J_\gamma Q = (J_{a,b} \otimes J_{p,q}) \oplus (J_{c,d} \otimes J_{p,q})$ with $a + b = u$ and $c + d = v$. The proof of Theorem 1 is complete. ∎

Proof of Theorem 2.

The equivalence of (c) and (d) can be verified readily.

(a) ⇒ (c): Suppose there is a linear transformation $\phi : M_n \to M_m$ such that $\phi(\mathbf{U}(H_1)) \subseteq \mathbf{U}(H_2)$. By Theorem 1, we have $(a + b + c + d)n = m$, i.e., m is a multiple of n. Moreover, comparing the inertias of the matrices on both side of (1), we see that $(r, s) = u(p, q) + v(q, p)$, where $u = a + c$ and $v = b + d$.

(c) ⇒ (b): Suppose there are nonnegative integers u, v such that $(r, s) = u(p, q) + v(q, p)$. Then $r + s = (u + v)(p + q)$, hence m is a multiple of n. Also, H_2 and $(I_u \otimes -I_v) \otimes H_1$ will have the same inertia. Thus, there exists $S \in M_m$ satisfying (1) with $(a, b, c, d) = (u, v, 0, 0)$.

(b) ⇒ (a): Suppose there exists $S \in M_m$ satisfying (1). We can construct ϕ of the form (2) with $U = V = I_m$ so that $\phi(\mathbf{U}(H_1)) \subseteq \mathbf{U}(H_2)$. ∎

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