Optimization of the Spectral Radius of Nonnegative Matrices

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Abstract

In a recent paper by Axtell, Han, Hershkowitz, and the present authors, one of the main questions that was considered was finding $n \times n$ doubly stochastic matrices P and Q which solve the **multiplicative extremal spectral radius problems** $\min_{S\in\Omega_n} \rho(SA)$ and $\max_{S\in\Omega_n} \rho(SA)$. Here $A \in \mathbb{R}^{n,n}$ is an arbitrary, but fixed, $n \times n$ nonnegative matrix, $\rho(\cdot)$ is the spectral radius of a matrix, and Ω_n is the set of all $n\times n$ doubly stochastic matrices. It was shown there that the solution to both problems is attained at some permutation matrix. In this paper we consider an additive version of these problems, namely, of solving the **additive extremal spectral radius problems** $\min_{S\in\Omega_n} \rho(S+A)$ and $\max_{S\in\Omega_n} \rho(S+A)$. As a by product of, actually, solutions to more general spectral radius optimization problems, we obtain here that the solution to both additive spectral radius optimization problems is, once again, attained at some permutation matrix. One of the more general spectral radius optimization problems that we consider here is that of replacing the constrains that the optimization be done on the doubly stochastic matrices by the weaker constraint of optimizing just on the $n \times n$ column or row stochastic matrices.

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1 Introduction

In a recent paper [1] by Axtell, Han, Hershkowitz, Neumann, and Sze, the following *multiplicative spectral radius optimization problems* were considered: Let $A \in \mathbb{R}^{n,n}$ be a nonnegative and irreducible matrix and let Ω_n be the set of all $n \times n$ doubly stochastic matrices. Then determine the extremal values and the matrices on which they occur of:

$$\min_{S\in\Omega_n} \rho(SA) \quad \text{and} \quad \max_{S\in\Omega_n} \rho(SA), \tag{1}$$

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where $\rho(\cdot)$ denotes the spectral radius of a matrix. It was shown in that paper that the solution to both problems is always obtained on the set $\mathcal{P}_n$ of the $n \times n$ permutation matrices.

The work in this paper was motivated by parallel questions concerning the *additive spectral radius optimization problems*, namely, under the above notations, determine:

$$\min_{S \in \Omega_n} \rho(S + A) \quad \text{and} \quad \max_{S \in \Omega_n} \rho(S + A). \tag{2}$$

We shall indeed show that the extremal value of both problems is **always** attained at a permutation matrix. To us, this result was a surprise.

In fact we shall consider here optimization problems that extend the questions in both (1) and (2). Let $\mathcal{C}_n$ be the set of all $n \times n$ column stochastic matrices. Let $\mathcal{X}_n$ be, alternately, any one of Ω_n and $\mathcal{C}_n$. Then given any two arbitrary, but fixed, not necessarily irreducible nonnegative matrices $A, B \in \mathbb{R}^{n,n}$, determine

$$\min_{S \in \mathcal{X}_n} \rho(SA + B) \quad \text{and} \quad \max_{S \in \mathcal{X}_n} \rho(SA + B). \tag{3}$$

In all the extremal problems mentioned above the pattern of solution which we will obtain is the same. When $\mathcal{X}_n = \Omega_n$, then as will be shown in Section 2, the solutions are always attained at permutation matrices. When $\mathcal{X}_n = \mathcal{C}_n$, the solutions are attained at extremal points of $\mathcal{C}_n$, namely, on $\mathcal{E}_n$, which is the set of all nonnegative matrices having in each column exactly one nonzero entry equal to 1. This we do in Section 3. Furthermore, similar result to those we obtained for extremal problems on the column stochastic matrices, can be obtained for extremal problem on the row stochastic matrices.

Finally in Section 4, we shall extend the generality of the optimization problem (3) to the consideration of following problems: Given m nonnegative matrices $A_1, \ldots, A_m \in \mathbb{R}^{n,n}$, then find:

$$\min_{S_1,\ldots,S_m \in \mathcal{X}_n} \rho\left(\sum_{i=1}^{m} S_i A_i\right) \quad \text{and} \quad \max_{S_1,\ldots,S_m \in \mathcal{X}_n} \rho\left(\sum_{i=1}^{m} S_i A_i\right). \tag{4}$$

It should be commented that as the sets of matrices Ω_n and $\mathcal{C}_n$ are closed and bounded and as the spectral radius is a continuous function on $R^{n,n}$, all optimization problems here are attained in the sets on which they are considered. However, the spectral radius is **not** a convex function over these sets of matrices, yet, as we have claimed above, in all the problems considered here, the extremal spectral values are obtained at the extreme points of the sets on which they are considered.

Finally, much background material on nonnegative matrices can be found in the book by Berman and Plemmons [2]. Viewing some of the problems we consider here as perturbation problems, by a matrix of constant row or column sums or, indeed, a doubly stochastic matrix, of the spectral radius or Perron root of a nonnegative matrix, we should mention that other types of perturbation problems for the spectral radius of nonnegative matrices have been considered in the literature. To mention here a few we cite: Cohen [3], Deutsch and Neumann

[4], Elsner [5], Friedland [6], Golub and Meyer [7], Han, Neumann, and Tsatsomeros [8], and Jonson, Loewy, Olesky, and van den Driessche [9].

2 Doubly Stochastic Matrices

A key result to the developments in this paper is the following lemma which is actually a special case of Lemma 2.2 in [1].

Lemma 2.1 *Suppose T_1 and T_2 are irreducible nonnegative matrices in $\mathbb{R}^{n,n}$ such that* $\mathrm{rank}\,(T_1 - T_2) = 1$. *Then the map f_{T_1,T_2} defined by*

$$f_{T_1,T_2}(\alpha) \;:=\; \rho(\alpha T_1 + (1-\alpha)T_2), \quad \alpha \in [0,1],$$

is either a strictly monotone function or a constant function on $[0,1]$. Furthermore, if x and y are right and left Perron vectors of T_2, then:

(a) *f_{T_1,T_2} is strictly increasing if $y^t(T_1 - T_2)x > 0$.*

(b) *f_{T_1,T_2} is strictly decreasing if $y^t(T_1 - T_2)x < 0$.*

(c) *f_{T_1,T_2} is a constant function if $y^t(T_1 - T_2)x = 0$.*

Proof. In [1, Lemma 2.2], substitute $A = I_n$, $S_1 = T_1$, and $S_2 = T_2$, respectively. □

In our first result of this paper we consider the optimization problems (3) for the case when $\mathcal{X}_n = \Omega_n$.

Theorem 2.2 *Let $A, B \in \mathbb{R}^{n,n}$ be nonnegative matrices. Then there are permutation matrices P^* and Q^* such that*

$$\rho(P^*A + B) \;=\; \min_{S\in\Omega_n} \rho(SA + B) \quad \textit{and} \quad \rho(Q^*A + B) \;=\; \max_{S\in\Omega_n} \rho(SA + B). \tag{5}$$

Proof. We shall prove here only the left equality in (5), that is that the minimum of $\rho(SA + B)$ over Ω_n is attained at a permutation matrix, as the right equality can been proved along similar lines.

We first consider the case when B is irreducible. Suppose that $S^* \in \Omega_n$ is a matrix such that

$$\rho(S^*A + B) \;=\; \min_{S\in\Omega_n} \rho(SA + B)$$

and S^* is chosen so that among all matrices S satisfying the above equality, S^* has the maximum number of entries equal one. We claim that S^* has exactly n entires equal one and so it is a permutation matrix.

Suppose to the contrary that $S^* = (s_{i,j})$ has exactly k entries equal one, with $k < n$, at the positions $(i_1, j_1), \ldots, (i_k, j_k)$. Set $\mathcal{I} = \{i_1, \ldots, i_k\}$ and $\mathcal{J} = \{j_1, \ldots, j_k\}$. Let x

and $y = (y_1, \dots, y_n)^t$ be right and left Perron vectors of $S^*A + B$, respectively, and set $w = (w_1, \dots, w_n)^t = Ax$. Take p and q in $\{1, \dots, n\}$ such that

$$y_p = \max\{y_i : i \notin \mathcal{I}\} \quad \text{and} \quad w_q = \min\{w_j : j \notin \mathcal{J}\}. \tag{6}$$

Without loss of generality, we may assume that $p = q = 1$ and $\mathcal{I} = \mathcal{J} = \{n-k+1, \dots, n\}$. Otherwise, we can replace S^*, A, B, x, and y by $PS^*Q^t, QAP^t, PBP^t, Px$, and Py, respectively. Hence, S^* has the form $S_1^* \oplus I_k$ for some $S_1^* \in \Omega_{n-k}$. Note that all entries of S_1^*, or equivalently, all $s_{i,j}$, with $1 \le i, j \le n-k$, must be smaller than one.

Now let $S^\dagger = S^* + (1 - s_{1,1})^{-1} uv^t$ with

$$u = (s_{1,1} - 1, s_{2,1}, \dots, s_{n-k,1}, 0, \dots, 0)^t \quad \text{and} \quad v = (s_{1,1} - 1, s_{1,2}, \dots, s_{1,n-k}, 0, \dots, 0)^t.$$

Then $S^\dagger$ has the form $S_1^\dagger \oplus I_k$ with

$$S_1^\dagger = \begin{bmatrix} s_{1,1} & s_{1,2} & \cdots & s_{1,n-k} \\ s_{2,1} & & & \\ \vdots & & s_{i,j} & \\ s_{n-k,1} & & & \end{bmatrix} + \begin{bmatrix} 1 - s_{1,1} & -s_{1,2} & \cdots & -s_{1,n-k} \\ -s_{2,1} & & & \\ \vdots & & \frac{s_{i,1}s_{1,j}}{1-s_{1,1}} & \\ -s_{n-k,1} & & & \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & s_{i,j} + \frac{s_{i,1}s_{1,j}}{1-s_{1,1}} & \\ 0 & & & \end{bmatrix},$$

so that $S^\dagger$ is nonnegative with at least $k+1$ entries equal 1. Furthermore, as all the row and column sums of uv^t equal zero, the row and column sums of $S^\dagger$ coincide, respectively, with those of S^*. Hence $S^\dagger$ is a doubly stochastic matrix. As $S^\dagger$ has $k+1$ entries which are equal to one, we must have that $\rho(S^\dagger A + B) > \rho(S^* A + B)$.

Let $T_1 = S^\dagger A + B$ and $T_2 = S^* A + B$. Clearly, $T_1 - T_2 = (1 - s_{1,1})^{-1} uv^t A$ is a rank one matrix. Furthermore, as y_1 and w_1 satisfy (6), we have that

$$y^t u = \sum_{i=1}^{n-k} s_{i,1} y_i - y_1 \le \sum_{i=1}^{n-k} s_{i,1} y_1 - y_1 = 0$$

and

$$v^t w = \sum_{j=1}^{n-k} s_{1,j} w_j - w_1 \ge \sum_{j=1}^{n-k} s_{1,j} w_1 - w_1 = 0.$$

Thus, $(y^t u)(v^t w) \le 0$ and hence $y^t (T_1 - T_2) x = (1 - s_{1,1})^{-1} y^t u v^t w \le 0$. By Lemma 2.1, the map f_{T_1,T_2} is either a strictly decreasing function or a constant function. But this contradicts the fact that

$$f_{T_1,T_2}(0) = \rho(S^* A + B) < \rho(S^\dagger A + B) = f_{T_1,T_2}(1).$$

Therefore, the result holds when B is irreducible.

Now suppose B is not irreducible. If there is an $S^* \in \Omega_n \setminus \mathcal{P}_n$ such that

$$\min_{S\in\Omega_n} \rho(SA+B) \;=\; \rho(S^*A+B) < \min_{P\in\mathcal{P}_n} \rho(PA+B),$$

then a positive matrix $\tilde{B}$ can easily be found for which

$$\rho(S^*A+B+\tilde{B}) \;<\; \min_{P\in\mathcal{P}_n} \rho(PA+B) \;\le\; \min_{P\in\mathcal{P}_n} \rho(PA+B+\tilde{B}).$$

But this contradicts the fact that for the irreducible matrix $B+\tilde{B}$, there is $P^* \in \mathcal{P}_n$ such that

$$\rho(P^*A+(B+\tilde{B})) \;=\; \min_{S\in\Omega_n} \rho(SA+(B+\tilde{B})).$$

□

By taking $A = I_n$ and $B = 0_n$, respectively, Theorem 2.2, yields two corollaries.

Corollary 2.3 ([1, Theorem 2.1]) *Let $A \in \mathbb{R}^{n,n}$ be a nonnegative matrix. Then there are permutation matrices P^* and Q^* such that*

$$\rho(P^*A) \;=\; \min_{S\in\Omega_n} \rho(SA) \quad \textit{and} \quad \rho(Q^*A) \;=\; \max_{S\in\Omega_n} \rho(SA).$$

Corollary 2.4 *Let $B \in \mathbb{R}^{n,n}$ be a nonnegative matrix. Then there are permutation matrices P^* and Q^* such that*

$$\rho(P^*+B) \;=\; \min_{S\in\Omega_n} \rho(S+B) \quad \textit{and} \quad \rho(Q^*+B) \;=\; \max_{S\in\Omega_n} \rho(S+B). \tag{7}$$

Remark that in Corollary 2.3, we removed the assumption of irreducibility of A, which was imposted in [1, Theorem 2.1].

Before continuing, we note that Corollary 2.4 establishes our claim immediately following (2), a result which we called surprising. Let us provide here an example.

Example 2.5 Consider the matrix

$$B \;=\; \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{8}$$

As a point of information we find that $\rho(B) \approx 3.1149$. On computing the minimum and maximum of $\rho(P+B)$, as P runs over all permutations in $\mathcal{P}_5$, we find that for

$$P^* \;=\; \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad Q^* \;=\; \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

we have that

$$\min_{S\in\mathcal{S}_5}\rho(S+B) \;=\; \min_{P\in\mathcal{P}_5}\rho(P+B) \;=\; \rho(P^*+B) \;\approx\; 4.0050$$

and that

$$\max_{S\in\Omega_5}\rho(S+B) \;=\; \max_{P\in\mathcal{P}_5}\rho(P+B) \;=\; \rho(Q^*+B) \;\approx\; 4.1284.$$

For curiosity's sake, on generating using several Matlab commands the random doubly stochastic matrix:

$$S \;=\; \begin{bmatrix} 0.3833 & 0.01978 & 0.03607 & 0.2559 & 0.3049 \\ 0.02269 & 0.1668 & 0.2534 & 0.3569 & 0.2002 \\ 0.1564 & 0.3096 & 0.4718 & 0.02184 & 0.04031 \\ 0.1003 & 0.4457 & 0.01826 & 0.02990 & 0.4058 \\ 0.3373 & 0.05808 & 0.2204 & 0.3355 & 0.04870 \end{bmatrix},$$

we find that

$$\rho(S+B) \;=\; 4.0618.$$

We finally note that in this example, the identity matrix turned out to yield neither the minimum value nor the maximum value in (7) over all matrices in $\mathcal{P}_n$.

3 Column stochastic matrices

In this section we relax our requirement of the last section that our spectral radius optimization problems are carried out over the set of the doubly stochastic matrices. We replace this requirement by considering the optimization problems over the set of the row stochastic or column stochastic matrices. Since the pattern of solution over both classes of matrices is similar, we shall consider here only the optimization problems over the column stochastic matrices.

Recall our notation from Section 1. Let $\mathcal{C}_n$ be the set of $n\times n$ column stochastic matrices and $\mathcal{E}_n$ be the set of the extreme points of $\mathcal{C}_n$. The set $\mathcal{E}_n$ thus contains all $n\times n$ nonnegative matrices such that each column consists of exactly one entry with value one and all other entires zero.

Our first major result for this section is:

Theorem 3.1 *Let $A, B \in \mathbb{R}^{n,n}$ be nonnegative matrices. Then there are matrices E^* and F^* in $\mathcal{E}_n$ such that*

$$\rho(E^*A+B) \;=\; \min_{C\in\mathcal{C}_n}\rho(CA+B) \quad \textit{and} \quad \rho(F^*A+B) \;=\; \max_{C\in\mathcal{C}_n}\rho(CA+B). \tag{9}$$

Proof. It suffices to prove the case when B is irreducible. Once this case is done, the remaining case can be shown by a similar argument as at the end of the proof of Theorem 2.2. We will again show that one of the equalities in (9), say the right equality holds, as the proof of the other equality is similar.

Let $C^* \in \mathcal{C}_n$ be a matrix such that $\rho(C^*A+B) = \max_{C\in\mathcal{C}_n} \rho(CA+B)$ and such that among all maximizers of $\rho(CA+B)$ in $\mathcal{C}_n$, C^* has maximum number of zero entries. We claim that C^* has exactly $n(n-1)$ zero entries and hence C^* is in $\mathcal{E}_n$.

Suppose to the contrary, namely, $C^* = (c_{i,j})$ has fewer than $n(n-1)$ zero entries, then permutation matrices P and Q can be found such that the first column of PC^*Q has at least two nonzero entries, with the $(1,1)$–th entry being nonzero. Without loss of generality, we may assume that $P = Q = I_n$ by replacing C^*, A, and B with PC^*Q, Q^tAP^t, and PBP^t, respectively.

Consider the vector $c = (1-c_{1,1})^{-1}(0, c_{2,1}, c_{3,1}, \ldots, c_{n,1})^t \in \mathbb{R}^n$. Note that c is well defined as $0 < c_{1,1} < 1$. Furthermore, each entries of c is nonnegative and the sum of its entries is one. Now let C_1 and C_2 be the matrices obtained from C^* by replacing its first column with $(1, 0, \ldots, 0)^t$ and c, respectively. Clearly, both C_1 and C_2 are column stochastic matrices and both C_1 and C_2 have at least one more zero entries than C^*. But then, due to our assumptions on C^*, we must have that

$$\rho(C_1A+B) \;<\; \rho(C^*A+B) \quad \text{and} \quad \rho(C_2A+B) \;<\; \rho(C^*A+B).$$

Let $T_1 = C_1A+B$ and $T_2 = C_2A+B$. Now note that as $C^* = c_{1,1}C_1 + (1-c_{1,1})C_2$, we can write that

$$C^*A+B \;=\; c_{1,1}T_1 + (1-c_{1,1})T_2.$$

By the construction of C_1 and C_2, $T_1 - T_2 = (C_1 - C_2)A$ is a rank one matrix and so by Lemma 2.1, the map f_{T_1,T_2} is either strictly monotone or a constant function. But this contradicts the fact that

$$\max\{f_{T_1,T_2}(1), f_{T_1,T_2}(0)\} \;=\; \max\{\rho(C_1A+B), \rho(C_2A+B)\} \;<\; \rho(C^*A+B) \;=\; f_{T_1,T_2}(c_{1,1}).$$

□

Using a similar argument to those employed in Section 2, we have the following two corollaries.

Corollary 3.2 *Let $A \in \mathbb{R}^{n,n}$ be a nonnegative matrix. Then there exists matrices E^* and F^* in $\mathcal{E}_n$ such that*

$$\rho(E^*A) \;=\; \min_{C\in\mathcal{C}_n} \rho(CA) \quad \textit{and} \quad \rho(F^*A) \;=\; \max_{C\in\mathcal{C}_n} \rho(CA).$$

Corollary 3.3 *Let $B \in \mathbb{R}^{n,n}$ be a nonnegative matrix. Then there exists matrices E^* and F^* in $\mathcal{E}_n$ such that*

$$\rho(E^*+B) \;=\; \min_{C\in\mathcal{C}_n} \rho(C+B) \quad \textit{and} \quad \rho(F^*+B) \;=\; \max_{C\in\mathcal{C}_n} \rho(C+B).$$

Example 3.4 Let us consider the matrix B given in Example 2.5. After computing the values of $\rho(E+B)$, for all $5^5 = 3125$ of the elements E in $\mathcal{E}_5$, we find that for

$$E^* = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad F^* = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

we have

$$\min_{C \in \mathcal{C}_5} \rho(C+B) = \min_{E \in \mathcal{E}_5} \rho(E+B) = \rho(E^*+B) \approx 3.8662$$

and

$$\max_{C \in \mathcal{C}_5} \rho(C+B) = \max_{E \in \mathcal{E}_5} \rho(E+B) = \rho(F^*+B) \approx 4.4709.$$

Notice that both E^* and F^* are not permutation matrices and that

$$\rho(E^*+B) < \rho(P^*+B) < \rho(Q^*+B) < \rho(F^*+B),$$

where P^* and Q^* are the permutation matrices found in Example 2.5.

Next, consider an arbitrary but fixed positive vector $d = (d_1, \ldots, d_n)^t \in \mathbb{R}^n$. Let $\mathcal{C}_n(d)$ be the set of $n \times n$ matrix with the column sum equal to d, and let $\mathcal{E}_n(d)$ be the set of extreme points of $\mathcal{C}_n(d)$. Notice that $\mathcal{C}_n(d) = \{CD : C \in \Omega_n\}$ and $\mathcal{E}_n(d) = \{ED : E \in \mathcal{E}_n\}$, with $D = \operatorname{diag}(d_1, \ldots, d_n)$. Thus, we have the following.

Corollary 3.5 *Let $A, B \in \mathbb{R}^{n,n}$ be nonnegative matrices, and let $d \in \mathbb{R}^n$ be a positive vector. Then there are matrices E^* and F^* in $\mathcal{E}_n(d)$ such that*

$$\rho(E^*A+B) = \min_{C \in \mathcal{C}_n(d)} \rho(CA+B) \quad \text{and} \quad \rho(F^*A+B) = \max_{C \in \mathcal{C}_n(d)} \rho(CA+B).$$

Notice that results similar to Corollaries 3.2 and 3.3 can be also obtained for the set $\mathcal{C}_n(d)$.

4 Further Extension

In this section, we establish the claims made in display (4).

Theorem 4.1 *Let $\mathcal{X}_n = \Omega_n$ or $\mathcal{C}_n$ and $\mathcal{Y}_n = \mathcal{P}_n$ or $\mathcal{E}_n$ according to $\mathcal{X}_n$. Given m nonnegative matrices $A_1, \ldots, A_m \in \mathbb{R}^{n,n}$, there exist matrices $P_1^*, \ldots, P_m^*$ and $Q_1^*, \ldots, Q_m^*$ in $\mathcal{Y}_n$ such that*

$$\rho\left(\sum_{i=1}^m P_i^* A_i\right) = \min_{S_1, \ldots, S_m \in \mathcal{X}_n} \rho\left(\sum_{i=1}^m S_i A_i\right) \tag{10}$$

and

$$\rho\left(\sum_{i=1}^{m} Q_i^* A_i\right) \;=\; \max_{S_1,\ldots,S_m\in\mathcal{X}_n} \rho\left(\sum_{i=1}^{m} S_i A_i\right). \tag{11}$$

Proof. We will only prove (10), the proof of (11) is similar. Suppose there are $S_1^*, \ldots, S_m^*$ in $\mathcal{X}_n$ such that

$$\rho\left(\sum_{i=1}^{m} S_i^* A_i\right) \;=\; \min_{S_1,\ldots,S_m\in\mathcal{X}_n} \rho\left(\sum_{i=1}^{m} S_i A_i\right). \tag{12}$$

On applying Theorems 2.2 or 3.1 with $A = A_1$ and $B = \sum_{i=2}^{m} S_i^* A_i$, we see that there is a $P_1^* \in \mathcal{Y}_n$ such that

$$\rho\left(P_1^* A_1 + \sum_{i=2}^{m} S_i^* A_i\right) \;\le\; \rho\left(\sum_{i=1}^{m} S_i^* A_i\right).$$

Now suppose that the existence of $P_1^*, \ldots, P_{k-1}^* \in \mathcal{Y}_n$ has already been established. We now apply Theorems 2.2 or 3.1 with $A = A_k$ and $B = \sum_{i=1}^{k-1} P_i^* A_i + \sum_{i=k+1}^{m} S_i^* A_i$, and so there is $P_k^* \in \mathcal{Y}_n$ such that

$$\rho\left(\sum_{i=1}^{k} P_i^* A_i + \sum_{i=k+1}^{m} S_i^* A_i\right) \;\le\; \rho\left(\sum_{i=1}^{k-1} P_i^* A_i + \sum_{i=k}^{m} S_i^* A_i\right).$$

Thus there exist $P_1^*, \ldots, P_m^* \in \mathcal{Y}_n$ such that

$$\rho\left(\sum_{i=1}^{m} P_i^* A_i\right) \;\le\; \rho\left(\sum_{i=1}^{m} S_i^* A_i\right).$$

But then by (12), the above inequality is indeed an equality. □

We comment that clearly the above result also holds when $\mathcal{X}_n$ equals either the set of row stochastic matrices or the set $\mathcal{C}_n(d)$.

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