Automorphisms of Certain Groups and Semigroups of Matrices [1]

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Abstract

We characterize automorphisms for semigroups of nonnegative matrices including doubly stochastic matrices, row (column) stochastic matrices, positive matrices, and nonnegative monomial matrices. Also, we present elementary proofs for the characterizations of automorphisms of the symmetric group and alternating group (realized as permutation matrices). Furthermore, for each of the above groups and semigroups of matrices, we relax the nonnegativity assumption to get a larger (semi)group of matrices, and characterize the automorphisms on the larger (semi)group and their subgroups (subsemigroups) as well. A number of general techniques are developed to study the problems.

1 Introduction

There has been considerable interest in studying linear and multiplicative maps that leave invariant some subsets, groups, and semigroups of matrices; see [13, Chapter 4] and also [2, 3, 6, 8, 9, 10, 11, 12, 14, 15]. Of course, if G is a (semi)group of matrices, and $\phi : G \to G$ is a multiplicative map, then ϕ is just a (semi)group homomorphism. While there are nice structural theorems for (semi)group homomorphisms for classical (semi)groups such as the group of invertible matrices, and for semigroups of matrices with ranks not exceeding a given positive integer (see [1, 5, 7]), there does not seem to be many known results for semigroups of nonnegative matrices. In this paper, we characterize automorphisms for semigroups of nonnegative matrices including doubly stochastic matrices, row (column) stochastic matrices, positive matrices, and nonnegative monomial matrices. Also, we present elementary proofs for the characterizations of the automorphisms of the symmetric group and alternating group (realized as permutation matrices). Furthermore, for each of the above groups and semigroups, we relax the nonnegativity assumption to get a larger (semi)group of matrices, and characterize the automorphisms on the larger (semi)group and their subgroups (subsemigroups) as well.

While some of our results on group automorphisms are known, we give elementary matrix proofs of them so that the paper is more self-contained. Moreover, some of the techniques

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are extended to obtain new results on semigroup automorphisms. In fact, our proofs utilize a lot of linear algebraic and geometric arguments. It would be interesting to find algebraic proofs for our results on semigroup automorphisms.

Our paper is organized as follows. In Section 2, we present elementary proofs for the characterizations of the automorphisms of the groups of permutation matrices, and even permutation matrices. These automorphisms always have the form

$$A \mapsto PAP^{-1}$$

for some permutation matrix P except for the groups of 6×6 permutation matrices and 6×6 even permutation matrices. Explicit constructions for all the automorphisms in these exceptional cases are also presented.

In Section 3, we consider generalized (even) permutation matrices, i.e., matrices obtained from (even) permutation matrices by changing the signs of some of the nonzero entries. We refine the techniques in Section 2 to characterize the group automorphisms of these groups and several other subgroups of the group of generalized permutation matrices.

In Section 4, we show that the automorphisms of nonnegative monomial matrices have the form

$$A = (a_{ij}) \mapsto \rho(|\det A|)T(\sigma(a_{ij}))T^{-1}$$

for some nonnegative monomial matrix T, multiplicative map ρ on $(0, \infty)$ and bijective multiplicative map σ on $[0, \infty)$ such that the map $\alpha \mapsto \rho(|\alpha|)^n \sigma(\alpha)$ is also bijective on $(0, \infty)$. We also treat the larger group of monomial matrices without the nonnegativity assumption.

Sections 5 and 6 concern the characterizations of the semigroup automorphisms of $n \times n$ doubly stochastic, row stochastic, and column stochastic matrices, and their generalizations obtained by removing the nonnegativity assumption on the entries of the matrices. We first characterize the semigroup automorphisms for the generalized (row, column, and doubly) stochastic matrices in Section 5. The results are then used to treat the problems for (row, column, and doubly) stochastic matrices in Section 6. In particular, we show that except for the case of 2×2 doubly stochastic matrices, the automorphisms must have the form

$$A \mapsto PAP^{-1}$$

for some permutation matrix P. The exceptional cases are also treated.

In Section 7, we determine the automorphisms for $n \times n$ positive matrices. The automorphisms have the form

$$A \mapsto TAT^{-1}$$

for some nonnegative monomial matrix T.

We use and develop a variety of techniques in our proofs. Very often, it is relatively easy to show that a mapping with a special form is a group automorphism. The non-trivial part is to show that a group automorphism has a certain special form. In our proofs, we frequently use the following facts concerning a group automorphism ϕ of a group G.

1. The order of $g \in G$ is the same as the order of $\phi(g)$.

2. If g is in the center of G, i.e., $gx = xg$ for all $x \in G$, then so is $\phi(g)$.

3. For $g \in G$, let $\mathcal{C}(g) = \{x \in G : gx = xg\}$. Then $\phi(\mathcal{C}(g)) = \mathcal{C}(\phi(g))$.

4. For $g \in G$, let $\mathcal{S}_g = \{s^{-1}gs : s \in G\}$. Then $\phi(\mathcal{S}_g) = \mathcal{S}_{\phi(g)}$.

5. If $H \subseteq G$ is a generating set of G, and there is $s \in G$ such that $\phi(x) = s^{-1}xs$ for all $x \in H$, then $\phi(x) = s^{-1}xs$ for all $x \in G$.

Additional proof techniques include:

6. Replace the automorphism ϕ by a mapping such as

$$x \mapsto s^{-1}\phi(x)s \quad \text{for some } s \in G,$$

   and show that the resulting map has some nice structures or properties, say, mapping a certain subset of G onto itself. For example, in our study of the automorphisms for the semigroup of $n \times n$ generalized column stochastic matrices, we use this technique to show that the modified map will send the semigroup of $n \times n$ generalized doubly stochastic matrices onto itself.

7. Reduce the problem to a simpler problem. For example, in our study of the automorphisms for the semigroup of $n \times n$ generalized doubly stochastic matrices, we reduce the problem to the study of automorphisms for the semigroup of $(n-1) \times (n-1)$ matrices.

8. Extend the semigroup automorphism $\phi : G \to G$ to a semigroup automorphism $\phi_1 : G_1 \to G_1$ for a certain overgroup G_1 of G. For example, in our study of the automorphisms for the semigroup of $n \times n$ doubly stochastic matrices, we extend the automorphism to an automorphism of the semigroup of $n \times n$ generalized stochastic matrices. The same technique is used to study the automorphisms for the semigroup of positive matrices.

2 Symmetric Group and Alternating Group

Denote by $\mathbf{P}_n$ the symmetric group of degree n. Hölder [6] characterized automorphisms of the symmetric group and showed that except for $n = 6$, an automorphism on $\mathbf{P}_n$ must be an inner automorphism, i.e., the automorphism must have the form $A \mapsto PAP^t$ for some permutation matrix P (see also [14]). When $n = 6$, there are other automorphisms on $\mathbf{P}_6$. However, the verification for the additional automorphisms on $\mathbf{P}_6$ was not explicit in [6]. In this section, we give a proof of the result with an explicit construction of the automorphisms on $\mathbf{P}_6$. Furthermore, we present the characterization of automorphisms of the alternating group $\mathbf{A}_n$.

In our discussion, we identify $\mathbf{P}_n$ with the group of $n \times n$ permutation matrices. If a permutation $\sigma \in \mathbf{P}_n$ is given in cycle product notation, we denote by P_σ the permutation matrix corresponding to σ. For example, if σ is the transposition $(1, 2)$, then $P_{(1,2)}$ is the permutation matrix obtained from the identity matrix I_n by interchanging the first and second rows. For notational convenience, we sometimes write the cycle $(i_1, \ldots, i_k)$ as $(i_1 \cdots i_k)$. We use $\mathrm{Aut}\,(\mathbf{P}_n)$ to denote the group of all automorphisms on $\mathbf{P}_n$.

Theorem 2.1 *A map $\phi : \mathbf{P}_n \to \mathbf{P}_n$ is a group automorphism if and only if one of the following holds.*

(a) *ϕ is an inner automorphism.*

(b) *$n = 6$, there is an automorphism $\Phi : \mathbf{P}_6 \to \mathbf{P}_6$ determined by*

$$\Phi(P_{(12)}) = P_{(12)(34)(56)}, \quad \Phi(P_{(13)}) = P_{(13)(25)(46)}, \quad \Phi(P_{(14)}) = P_{(14)(35)(26)},$$

$$\Phi(P_{(15)}) = P_{(15)(24)(36)}, \quad \Phi(P_{(16)}) = P_{(16)(45)(23)},$$

and ϕ is the composition of an inner automorphism with Φ.

Consequently, Aut$(\mathbf{P}_2)$ *is the trivial group; if $n \geq 3$ and $n \neq 6$ then* Aut$(\mathbf{P}_n)$ *is isomorphic to $\mathbf{P}_n$; if $n = 6$ then* $[\text{Aut}(\mathbf{P}_6) : \mathbf{P}_6] = 2$.

Proof. First, we show that there is an automorphism $\Phi : \mathbf{P}_6 \to \mathbf{P}_6$ having the images on $\mathcal{T} = \{P_{(1,2)}, \ldots, P_{(1,6)}\}$ as described in (b). To this end, for every $P \in \mathbf{P}_6$ we choose a representation of P as a product of elements in $\mathcal{T}$. Then for $P = T_1 \cdots T_k$ with $T_1, \ldots, T_k \in \mathcal{T}$, define $\Phi(P) = \Phi(T_1) \cdots \Phi(T_k)$. We then check that Φ is bijective and satisfies $\Phi(PQ) = \Phi(P)\Phi(Q)$ for the (6!)(6!) choices of (P, Q), say, with the help of computer. See the Matlab program in the appendix, which is also available at http://www.resnet.wm.edu/˜cklixx/S6isom.m.txt.

Note that in our Matlab program, we perform simple operations on permutation matrices, and numerical errors will not be an issue in the verification. Extra cautious readers may modify the Matlab program to use symbolic package such as Maple to do the verification.

Now, suppose $\phi \in$ Aut$(\mathbf{P}_n)$. We show that (using the arguments in [6] and [14]) either

(i) $\phi(T)$ is a transposition for each transposition $T \in \mathbf{P}_n$; or

(ii) $n = 6$ and $\phi(T)$ is a product of three disjoint transpositions for each transposition $T \in \mathbf{P}_6$.

To see this, let T be a transposition, and let $\mathcal{S}_T$ be the set of elements in $\mathbf{P}_n$ of the form PTP^{-1} with $P \in \mathbf{P}_n$. Then

$$\phi(\mathcal{S}_T) = \{\phi(P)\phi(T)\phi(P)^{-1} : P \in \mathbf{P}_n\} = \mathcal{S}_{\phi(T)}.$$

Note that for any $Q \in \mathbf{P}_n$, $Q \in \mathcal{S}_T$ if and only if Q is a transposition. So, $\mathcal{S}_T$ has $\binom{n}{2}$ elements. Since T has order 2 and so is $\phi(T)$, we see that $\phi(T)$ is a product of k disjoint transpositions for some positive integer k, and hence $\mathcal{S}_{\phi(T)}$ has $\frac{n!}{2^k k!(n-2k)!}$ elements. Because $\phi(\mathcal{S}_T) = \mathcal{S}_{\phi(T)}$, we have

$$\binom{n}{2} = \frac{n!}{2^k k!(n-2k)!}.$$

The above equation holds for all n when $k = 1$, or for $(n, k) = (6, 3)$.

Now, if (i) holds with $n = 2$ then ϕ is the identity map. For $n \geq 3$ consider $\mathcal{T} = \{P_{(1,2)}, \ldots, P_{(1,n)}\}$. Then the product of any two elements in $\mathcal{T}$ is a three cycle, and for $n \geq 4$ the product of any three elements in $\mathcal{T}$ is a four cycle. Thus, the same holds for

$$\phi(\mathcal{T}) = \{\phi(P_{(1,2)}), \ldots, \phi(P_{(1,n)})\}.$$

It follows that $\phi(P_{(1,k)}) = P_{(i_1,i_k)}$ for $k = 2, \ldots, n$, where $\{i_1, \ldots, i_n\} = \{1, \ldots, n\}$. As a result, if $\sigma(j) = i_j$ for $j = 1, \ldots, n$, then $\phi(P_{(1,k)}) = P_\sigma P_{(1,k)} P_\sigma^t$ for each k. Since $\{P_{(1,2)}, \ldots, P_{(1,n)}\}$ generates $\mathbf{P}_n$, it follows that $\phi(A) = P_\sigma A P_\sigma^t$ for all $A \in \mathbf{P}_n$.

Suppose (i) does not hold. Then $(n, k) = (6, 3)$ and there is a transposition T such that $\phi(T)$ is a product of three disjoint transpositions. Moreover, $\phi(\mathcal{S}_T) = \mathcal{S}_{\phi(T)}$. Note that $\mathcal{S}_T$ is the set of all transpositions and $\mathcal{S}_{\phi(T)}$ is the set of all products of three disjoint transpositions. Therefore, condition (ii) holds. Now, $\Phi^{-1} \circ \phi$ maps transpositions to transpositions, and we are back to (i). Hence, $\Phi^{-1} \circ \phi$ is an inner automorphism. $\square$

Using a similar proof, we can characterize automorphisms of the alternating group $\mathbf{A}_n$.

Theorem 2.2 *A map $\phi : \mathbf{A}_n \to \mathbf{A}_n$ is a group automorphism if and only if one of the following holds.*

(a) *There is $P \in \mathbf{P}_n$ such that*

$$\phi(A) = PAP^t \quad \text{for all} \quad A \in \mathbf{A}_n.$$

(b) *$n = 6$ and there is $P \in \mathbf{P}_n$ such that*

$$\phi(A) = \Phi(PAP^t) \quad \text{for all} \quad A \in \mathbf{A}_n,$$

where Φ is the mapping defined in Theorem 2.1 restricted to $\mathbf{A}_n$.

3 Generalized Permutation Matrices

Denote by Σ_n the group of signature matrices, that is, the diagonal matrices D with diagonal entries in $\{1, -1\}$. Let $\mathbf{GP}_n$ (respectively, $\mathbf{GA}_n$) be the group of all generalized (even) permutation matrices, that is, the set of all $n \times n$ matrices of the form DP with $D \in \Sigma_n$ and $P \in \mathbf{P}_n (P \in \mathbf{A}_n)$. In this section, we characterize the group automorphisms of $\mathbf{GP}_n$ and $\mathbf{GA}_n$. It turns out that we can use a unified proof to treat three other subgroups of $\mathbf{GP}_n$. To describe these subgroups, let $\delta(A)$ be the number of negative entries of a matrix A in $\mathbf{GP}_n$. Consider the subgroup

$$\mathbf{GA}_n^+ = \{A = DP \in \mathbf{GP}_n : D \in \Sigma_n, P \in \mathbf{A}_n \text{ and } \det(A) = 1\} \subseteq \mathbf{GP}_n.$$

Then $\mathbf{GP}_n$ is a disjoint union of the four cosets $\mathbf{GA}_n^+$, where the three cosets other than $\mathbf{GA}_n^+$ are

$$\begin{aligned}
\mathbf{GA}_n^- &= \{A = DP \in \mathbf{GP}_n : D \in \Sigma_n, P \in \mathbf{A}_n \text{ and } \det(A) = -1\}, \\
\overline{\mathbf{GA}}_n^+ &= \{A = DP \in \mathbf{GP}_n : D \in \Sigma_n, P \notin \mathbf{A}_n \text{ and } \det(A) = 1\}, \\
\overline{\mathbf{GA}}_n^- &= \{A = DP \in \mathbf{GP}_n : D \in \Sigma_n, P \notin \mathbf{A}_n \text{ and } \det(A) = -1\}.
\end{aligned}$$

Note that $\mathbf{GA}_n = \mathbf{GA}_n^+ \cup \mathbf{GA}_n^-$. Define

$$\mathbf{GP}_n^+ = \mathbf{GA}_n^+ \cup \overline{\mathbf{GA}}_n^+ = \{A \in \mathbf{GP}_n : \det(A) = 1\},$$

and

$$\mathbf{GP}_n^e = \mathbf{GA}_n^+ \cup \overline{\mathbf{GA}}_n^- = \{DP \in \mathbf{GP}_n : D \in \Sigma_n, P \in \mathbf{P}_n \text{ and } \det(D) = 1\}.$$

Then $\mathbf{GP}_n^+$ and $\mathbf{GP}_n^e$ are also subgroups in $\mathbf{GP}_n$. Evidently, $A \in \mathbf{GP}_n^e$ if and only if $\delta(A)$ is even, i.e., $(-1)^{\delta(A)} = 1$. We have the following characterizations for the automorphisms of the above groups.

Theorem 3.1 *Let $\mathcal{P} = \mathbf{GA}_n^+, \mathbf{GP}_n^+, \mathbf{GP}_n^e, \mathbf{GA}_n$, or $\mathbf{GP}_n$. Suppose*

$$M = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad N = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}.$$

A map $\phi : \mathcal{P} \to \mathcal{P}$ is an automorphism if and only if there exists $P \in \mathbf{GP}_n$ such that one of the following holds.

(a) *ϕ has the form*

$$A \mapsto PAP^t.$$

(b) *$\mathcal{P} = \mathbf{GP}_n$ and ϕ has the form*

$$A \mapsto (-1)^{\delta(A)}(\det A)PAP^t.$$

(c) *n is even, $\mathcal{P} \in \{\mathbf{GP}_n^e, \mathbf{GP}_n\}$ and ϕ has the form*

$$A \mapsto (\det A)PAP^t.$$

(d) *n is even, $\mathcal{P} \in \{\mathbf{GP}_n^+, \mathbf{GP}_n\}$ and ϕ has the form*

$$A \mapsto (-1)^{\delta(A)}PAP^t.$$

(e) *$\mathcal{P} \in \{\mathbf{GA}_2, \mathbf{GP}_2^e\}$ and ϕ can be any bijective map with $\phi(I_2) = I_2$.*

(f) $\mathcal{P} = \mathbf{GP}_2$ *and* ϕ *has the form*

$$A \mapsto PMAM^tP^t.$$

(g) $\mathcal{P} \in \{\mathbf{GA}_4^+, \mathbf{GP}_4^e\}$ *and* ϕ *has the form*

$$A \mapsto PNAN^tP^t.$$

(h) $\mathcal{P} = \mathbf{GP}_4^e$ *and* ϕ *has the form*

$$A \mapsto (\det A)PNAN^tP^t.$$

One can easily organize the results in terms of the five groups as follows.

(1) The automorphisms of $\mathbf{GA}_n^+$ have the form (a), (g).

(2) The automorphisms of $\mathbf{GP}_n^+$ have the form (a), (d).

(3) The automorphisms of $\mathbf{GP}_n^e$ have the form (a), (c), (e), (g), (h).

(4) The automorphisms of $\mathbf{GA}_n$ have the form (a), (c), (e).

(5) The automorphisms of $\mathbf{GP}_n$ have the form (a), (b), (c), (d), (f).

In the following, let D_i be the matrix obtained from I_n by replacing the i-th diagonal entry with -1. Also for any $1 \leq k \leq n$, we define $D_{i_1\ldots i_k}$ as $D_{i_1}\cdots D_{i_k}$ for any distinct $1 \leq i_1, \ldots, i_k \leq n$. Furthermore, let $\mathcal{D}_k$ be the set containing all such $D_{i_1\ldots i_k}$.

When $n = 2$, the theorem follows readily from standard results in elementary group theory if one observes that $\mathbf{GA}_2^+, \mathbf{GP}_2^+, \mathbf{GP}_2^e, \mathbf{GA}_2, \mathbf{GP}_2$ are isomorphic to the additive groups $\mathbb{Z}_2, \mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_4 \times \mathbb{Z}_2$, respectively. We include short proofs below for the sake of completeness.

Proof of the sufficiency part. Firstly, it is clear that the map ϕ described in (a) is an automorphism on $\mathcal{P}$.

Now suppose ϕ has the form described in (b), (c) or (d). Clearly, ϕ is multiplicative. It remains to show that ϕ is bijective. We may assume that $P = I_n$; otherwise we replace ϕ by $A \mapsto P^t\phi(A)P$.

We have the following observations.

Suppose A belongs to	$\mathbf{GA}_n^+$	$\mathbf{GA}_n^-$	$\overline{\mathbf{GA}}_n^+$	$\overline{\mathbf{GA}}_n^-$.
Then $(\det A)A$ equals	A	$-A$	A	$-A$;
$(-1)^{\delta(A)}A$ equals	A	$-A$	$-A$	A ;
$(-1)^{\delta(A)}(\det A)A$ equals	A	A	$-A$	$-A$.

Note that $\det(-A) = (-1)^n \det(A)$.

Suppose A belongs to	$\mathbf{GA}_n^+$	$\mathbf{GA}_n^-$	$\overline{\mathbf{GA}}_n^+$	$\overline{\mathbf{GA}}_n^-$.
If n is even, then $-A$ belongs to	$\mathbf{GA}_n^+$	$\mathbf{GA}_n^-$	$\overline{\mathbf{GA}}_n^+$	$\overline{\mathbf{GA}}_n^-$;
if n is odd, then $-A$ belongs to	$\mathbf{GA}_n^-$	$\mathbf{GA}_n^+$	$\overline{\mathbf{GA}}_n^-$	$\overline{\mathbf{GA}}_n^+$.

Thus, if n is even and ϕ has form (b)-(d), then

$$\left(\phi(\mathbf{GA}_n^+), \phi(\mathbf{GA}_n^-), \phi(\overline{\mathbf{GA}}_n^+), \phi(\overline{\mathbf{GA}}_n^-)\right) = \left(\mathbf{GA}_n^+, \mathbf{GA}_n^-, \overline{\mathbf{GA}}_n^+, \overline{\mathbf{GA}}_n^-\right);$$

if n is odd and ϕ has the form (b), then

$$\left(\phi(\mathbf{GA}_n^+), \phi(\mathbf{GA}_n^-), \phi(\overline{\mathbf{GA}}_n^+), \phi(\overline{\mathbf{GA}}_n^-)\right) = \left(\mathbf{GA}_n^+, \mathbf{GA}_n^-, \overline{\mathbf{GA}}_n^-, \overline{\mathbf{GA}}_n^+\right).$$

Therefore, ϕ is surjective and hence ϕ is bijective on the corresponding $\mathcal{P}$.

For case (e), note that $\mathbf{GA}_2 = \{\pm I_2, \pm D_1\}$ and $\mathbf{GP}_2^e = \{\pm I_2, \pm P_{(12)}\}$. Then for $\mathcal{P} \in \{\mathbf{GA}_2, \mathbf{GP}_2^e\}$, $A^2 = I_2$ for all $A \in \mathcal{P}$. Furthermore, all elements in $\mathcal{P}$ commute. Thus, any bijective map ϕ on $\mathcal{P}$ with $\phi(I_2) = I_2$ is multiplicative.

For case (f), it is sufficient to show that the map $\psi : \mathbf{GP}_2 \to \mathbf{GP}_2$ defined by $\psi(A) = MAM^t$ is an automorphism. Clearly, ψ is multiplicative and injective. It remains to show that $\psi(\mathbf{GP}_2) = \mathbf{GP}_2$. Note that $\psi(P_{(12)}) = D_1$ and $\psi(D_1) = P_{(12)}$. Since $\{P_{(12)}, D_1\}$ generates $\mathbf{GP}_2$, $\psi(\mathbf{GP}_2) \subseteq \mathbf{GP}_2$ and hence $\psi(\mathbf{GP}_2) = \mathbf{GP}_2$ as ψ is injective.

For case (g) and (h), it is sufficient to show that the map $\Psi(A) = NAN^t$ maps $\mathbf{GP}_4^e$ and $\mathbf{GA}_4^+$ onto themselves. In fact,

$$\Psi(P_{12}) = D_{34}P_{(34)}, \quad \Psi(P_{(13)}) = D_{23}P_{(23)}, \quad \Psi(P_{(14)}) = D_{24}P_{(24)},$$

$$\Psi(D_{12}) = -P_{(12)(34)}, \quad \Psi(P_{(123)}) = D_{34}P_{(234)}, \quad \Psi(P_{(124)}) = D_{34}P_{(243)}.$$

Since $\mathcal{T}_1 = \{P_{(12)}, P_{(13)}, P_{(14)}, D_{12}\}$ generates $\mathbf{GP}_4^e$, $\Psi(\mathcal{T}_1) \subseteq \mathbf{GP}_4^e$, and Ψ is injective, we see that $\Psi(\mathbf{GP}_4^e) = \mathbf{GP}_4^e$. Similarly, since $\mathcal{T}_2 = \{P_{(123)}, P_{(124)}, D_{12}\}$ generates $\mathbf{GA}_4^+$ and $\Psi(\mathcal{T}_2) \subseteq \mathbf{GA}_4^+$, we see that $\Psi(\mathbf{GA}_4^+) = \mathbf{GA}_4^+$. □

Proof of necessity part for $n = 2$. Since ϕ is a group automorphism, $\phi(I_2) = I_2$. When $\mathcal{P} \in \{\mathbf{GA}_2, \mathbf{GP}_2^e\}$, we are done. Also it is clear for the case when $\mathbf{GA}_2^+ = \{I_2, -I_2\}$.

For $\mathcal{P} = \mathbf{GP}_2^+ = \{\pm I_2, \pm D_1P_{(12)}\}$, observe that $-I_2$ is the only order 2 element in the center of $\mathbf{GP}_2^+$. Thus $\phi(-I_2) = -I_2$, and $\phi(\{D_1P_{(12)}, -D_1P_{(12)}\}) = \{D_1P_{(12)}, -D_1P_{(12)}\}$. Hence, ϕ is either the identity map or the map $A \mapsto D_1AD_1^t$.

Now, suppose $\mathcal{P} = \mathbf{GP}_2 = \{\pm I_2, \pm D_1, \pm P_{(12)}, \pm D_1 P_{(12)}\}$. Note that $\phi(I_2) = I_2$ and $\phi(-I_2) = -I_2$ as $-I_2$ is the only order 2 element in $\mathbf{GP}_2$ which commute with all elements. Also as $D_1^2 = I_2$, we see that either

$$\phi(\{D_1, -D_1\}) = \{D_1, -D_1\} \quad \text{or} \quad \phi(\{D_1, -D_1\}) = \{P_{(12)}, -P_{(12)}\}.$$

Suppose the first case holds. Then $\phi(\{P_{(12)}, -P_{(12)}\}) = \{P_{(12)}, -P_{(12)}\}$. So, ϕ has the form described in (a). If the second case holds, then the map $\psi(A) = M^t \phi(A) M$ maps $\{D_1, -D_1\}$ onto itself. So, ψ must be of the form (a) and hence ϕ has the form described in (f). □

For $n \geq 3$, we prove it by a sequence of lemmas. In the following, we always assume that $n \geq 3$ and $\phi : \mathcal{P} \to \mathcal{P}$ is an automorphism. Note that if $-I_n \in \mathcal{P}$, we must have $\phi(-I_n) = -I_n$ as $-I_n$ is the order 2 element in the center of $\mathcal{P}$.

Lemma 3.2 *Suppose $n \geq 3$. Then there is $P \in \mathbf{GP}_n$ such that one of the following holds.*

(a) $\phi(E) = PEP^t$ *for all* $E \in \mathcal{D}_2$.

(b) $\mathcal{P} \in \{\mathbf{GA}_4^+, \mathbf{GP}_4^e\}$ *and* $\phi(E) = PNEN^tP^t$ *for all* $E \in \mathcal{D}_2$, *where N is the* 4×4 *matrix defined in Theorem 3.1.*

Proof. We first claim that either

(i) $\phi(\mathcal{D}_2) = \mathcal{D}_2$, or

(ii) $\mathcal{P} \in \{\mathbf{GA}_4^+, \mathbf{GP}_4^e\}$ and $\phi(\mathcal{D}_2)$ equals to one of the following sets:

$$\{N^t E N : E \in \mathcal{D}_2\} \quad \text{or} \quad \{D_1^t N^t E N D_1 : E \in \mathcal{D}_2\}.$$

To prove our claim, for a given $A \in \mathcal{P}$ let $\mathcal{S}_A$ be the set of elements in $\mathcal{P}$ of the form TAT^t with $T \in \mathcal{P}$. Clearly, $\mathcal{S}_{D_{12}}$ contains all D_{ij} only, i.e., $\mathcal{S}_{D_{12}} = \mathcal{D}_2$. So $\mathcal{S}_{D_{12}}$ has $\binom{n}{2}$ elements.

On the other hand, since $\phi(D_{12})^2 = \phi(I_n) = I_n$, $\phi(D_{12})$ is permutationally similar to

$$\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes I_p\right] \oplus \left[-\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes I_q\right] \oplus -I_r \oplus I_{n-2(p+q)-r}$$

for some $p, q, r \geq 0$. Now for any $A \in \mathcal{P}$, $A \in \mathcal{S}_{\phi(D_{12})}$ if and only if A is permutationally similar to the matrix of the form

$$\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes I_{p'}\right] \oplus \left[-\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes I_{q'}\right] \oplus -I_r \oplus I_{n-2(p'+q')-r}$$

for some p' and q' such that $p' + q' = p + q$. Furthermore, if $\mathcal{P} \in \{\mathbf{GA}_n^+, \mathbf{GP}_n^e\}$ for even n with $(p+q, r) = (n/2, 0)$, $p - p'$ must be an even number. Since ϕ is bijective, we have $\phi(\mathcal{S}_A) = \mathcal{S}_{\phi(A)}$ for any $A \in \mathcal{P}$. Hence, $\phi(\mathcal{D}_2) = \phi(\mathcal{S}_{D_{12}}) = \mathcal{S}_{\phi(D_{12})}$, and

$$\binom{n}{2} = \begin{cases} \frac{n!}{2((n/2)!)} & \text{if } \mathcal{P} \in \{\mathbf{GA}_n^+, \mathbf{GP}_n^e\} \text{ for even } n \text{ with } (p+q, r) = (n/2, 0), \\ \frac{n!}{(p+q)!(n-2(p+q)-r)!r!} & \text{otherwise.} \end{cases}$$

Examining the above equation in the two cases, we see that the equality holds

(1) for an arbitrary n with $(p+q,r) \in \{(0,2),(0,n-2)\}$, or

(2) for $\mathcal{P} \in \{\mathbf{GA}_4^+, \mathbf{GP}_4^e\}$ with $(p+q,r) = (2,0)$.

Suppose $p+q=0$. Then $\phi(D_{12})$ is permutationally similar to $-I_r \oplus I_{n-r} \in \mathcal{D}_r$. When $r=2$, $\phi(\mathcal{D}_2) = \mathcal{S}_{\phi(D_{12})} = \mathcal{D}_2$. Then condition (i) in the claim holds. If $r = n-2$, then

$$\phi(\mathcal{D}_2) = \mathcal{S}_{\phi(D_{12})} = \mathcal{D}_{n-2} = -\mathcal{D}_2.$$

Suppose $\phi(D_{12}) = -D_{ij}$ and $\phi(D_{13}) = -D_{kl}$ for some i,j,k,l with $\{i,j\} \neq \{k,l\}$. Since $D_{12}D_{13} = D_{23} \in \mathcal{D}_2$,

$$D_{ij}D_{kl} = \phi(D_{12})\phi(D_{13}) = \phi(D_{23}) \in \mathcal{D}_{n-2}.$$

If the index i,j,k,l are all distinct, then $D_{ijkl} = D_{ij}D_{kl} \in \mathcal{D}_{n-2}$ implies that $n-2=4$, i.e., $n=6$. However, when $n=6$, there are 8 distinct elements E in $\mathcal{D}_2$ such that $D_{12}E \in \mathcal{D}_2$ while there are only 6 distinct elements F in $\mathcal{D}_4$ such that $\phi(D_{12})F = -D_{ij}F \in \mathcal{D}_4$. This contradicts the fact that ϕ is an automorphism.

Next, if $\{i,j\} \cap \{k,l\} \neq \emptyset$, we must have $D_{ij}D_{kl} \in \mathcal{D}_2$. Thus, $\mathcal{D}_2 \cap \mathcal{D}_{n-2}$ is nonempty. And this holds if and only if $n=4$. In this case, $r = n-2 = 2$, and so (i) holds.

Finally, suppose $\mathcal{P} \in \{\mathbf{GA}_4^+, \mathbf{GP}_4^e\}$ with $(p+q,r) = (2,0)$. Then $\phi(D_{12})$ is permutationally similar to either $P_{(12)(34)}$ $((p,q)=(2,0))$, $D_{34}P_{(12)(34)}$ $((p,q)=(1,1))$ or $-P_{(12)(34)}$ $((p,q)=(0,2))$. It follows that

$$\mathcal{S}_{\phi(D_{12})} = \begin{cases} \mathcal{T}_1 = \{\pm P_{(12)(34)}, \pm P_{(13)(24)}, \pm P_{(14)(23)}\} & \text{if } (p,q) \in \{(2,0),(0,2)\}, \\ \mathcal{T}_2 = \{\pm D_{12}P_{(12)(34)}, \pm D_{13}P_{(13)(24)}, \pm D_{14}P_{(14)(23)}\} & \text{if } (p,q) = (1,1). \end{cases}$$

Note that

$$\mathcal{T}_1 = \{N^tEN : E \in \mathcal{D}_2\} \quad \text{and} \quad \mathcal{T}_2 = \{D_1^tN^tEND_1 : E \in \mathcal{D}_2\}.$$

Therefore, condition (ii) in the claim holds.

Now, suppose condition (i) holds. Then $\mathcal{T} = \{D_{12}, D_{13}, \ldots, D_{1n}\} \subseteq \mathcal{D}_2$ and any product of two elements in $\mathcal{T}$ lies in $\mathcal{D}_2$. The set $\phi(\mathcal{T})$ has the same properties, and hence $\phi(\mathcal{T}) = \{D_{i_1i_2}, \ldots, D_{i_1i_n}\}$ with $\{i_1,\ldots,i_n\} = \{1,\ldots,n\}$. Define $\sigma(k) = i_k$ for $k=1,\ldots,n$. Then $\phi(D_{1k}) = P_\sigma D_{1k} P_\sigma^t$ for all $k \geq 2$. Since $\mathcal{T}$ generates $\mathcal{D}_2$, we have $\phi(E) = P_\sigma E P_\sigma^t$ for all $E \in \mathcal{D}_2$. So, condition (a) holds.

Suppose condition (ii) holds. Then either the map

$$A \mapsto N\phi(A)N^t \quad \text{or} \quad A \mapsto ND_1\phi(A)D_1^tN^t$$

maps $\mathcal{D}_2$ onto itself. Using a similar argument as in the last paragraph, we see that there is $P \in \mathbf{P}_4$ such that either $\phi(E) = NPEP^tN^t$ for all $E \in \mathcal{D}_2$, or $\phi(E) = D_1NPEP^tN^tD_1^t$ for all $E \in \mathcal{D}_2$. In either case, there is $Q \in \mathbf{GP}_4$ such that $QN = NP$. So, ϕ must satisfy (b). □

Lemma 3.3 *Suppose $n \geq 3$ and $\phi(E) = E$ for all $E \in \mathcal{D}_2$. Then there is $S \in \Sigma_n$ such that*

$$\phi(A) = SAS^t \quad \textit{for all} \quad A \in \mathbf{GA}_n^+.$$

Proof. Take any $P_{(ijk)} \in \mathbf{A}_n$. We claim that

$$\phi(P_{(ijk)}) \in \{P_{(ijk)}, D_i P_{(ijk)} D_i, D_j P_{(ijk)} D_j, D_k P_{(ijk)} D_k\}.$$

To see this, observe that for any $X \in \mathbf{GP}_n$, $D_{st} X = X D_{st}$ for all $D_{st} \in \mathcal{D}_2$ if and only if $X \in \Sigma_n$. Consequently, $X = AP_{(ijk)}^t \in \mathbf{GP}_n$ commutes with all matrices in $\mathcal{D}_2$ if and only if $A = DP_{(ijk)}$ for some $D \in \Sigma_n$. Now, for any $D_{st} \in \mathcal{D}_2$, $P_{(ijk)}^t D_{st} P_{(ijk)} \in \mathcal{D}_2$, we have

$$\phi(P_{(ijk)})^t D_{st} \phi(P_{(ijk)}) = \phi(P_{(ijk)}^t)\phi(D_{st})\phi(P_{(ijk)}) = \phi(P_{(ijk)}^t D_{st} P_{(ijk)}) = P_{(ijk)}^t D_{st} P_{(ijk)}.$$

Thus, $\phi(P_{(ijk)}) = DP_{(ijk)}$ for some $D \in \Sigma_n$.

Let $D = \operatorname{diag}(d_1, \dots, d_n)$. Since $(DP_{(ijk)})^3 = \phi(P_{(ijk)})^3 = I_n$, $d_i d_j d_k = 1$ and $d_l = 1$ for all $l \neq i, j, k$, Therefore, $D \in \{I_n, D_{ij}, D_{jk}, D_{ik}\}$. Note that

$$D_{ij} P_{(ijk)} = D_i P_{(ijk)} D_i, \quad D_{jk} P_{(ijk)} = D_j P_{(ijk)} D_j, \quad D_{ik} P_{(ijk)} = D_k P_{(ijk)} D_k.$$

Thus our claim holds.

Now replacing ϕ by $A \mapsto E^t \phi(A) E$ with $E = D_1, D_2$ or D_3, if necessary, we can further assume that $\phi(P_{(123)}) = P_{(123)}$. For any $k \geq 4$, $P_{(123)} P_{(12k)} P_{(123)} = P_{(1k2)}$. Suppose $\phi(P_{(12k)}) = D_{12} P_{(12k)}$. Then

$$\phi(P_{(1k2)}) = \phi(P_{(123)} P_{(12k)} P_{(123)}) = P_{(123)} D_{12} P_{(12k)} P_{(123)} = D_{23} P_{(1k2)}.$$

But then $\phi(P_{(1k2)})^3 \neq I_n$, which is impossible. Similarly, we see that $\phi(P_{(12k)}) \neq D_{2k} P_{(12k)}$. Let $s_1 = s_2 = s_3 = 1$ and for $k \geq 4$,

$$s_k = \begin{cases} 1 & \text{if } \phi(P_{(12k)}) = P_{(12k)}, \\ -1 & \text{if } \phi(P_{(12k)}) = D_{1k} P_{(12k)}. \end{cases}$$

Write $S = \operatorname{diag}(s_1, \dots, s_n)$. We see that $\phi(P_{(12k)}) = S\phi(P_{(12k)})S^t$ for all $k \geq 3$. Also, we have $\phi(E) = E = SES^t$ for all $E \in \mathcal{D}_2$. Since $\{P_{(123)}, \dots, P_{(12n)}\} \cup \mathcal{D}_2$ generates $\mathbf{GA}_n^+$, $\phi(P) = SPS^t$ for all $P \in \mathbf{GA}_n^+$. □

Lemma 3.4 *Suppose $n \geq 3$ and $\phi(A) = A$ for all $A \in \mathbf{GA}_n^+$. If $\mathcal{D}_1 \subseteq \mathcal{P}$, then either*

(a) *$\phi(E) = E$ for all $E \in \mathcal{D}_1$, or*

(b) *n is even with $\phi(E) = -E$ for all $E \in \mathcal{D}_1$.*

Proof. For $n = 3$, observe that $\mathcal{D}_2 = -\mathcal{D}_1$ and $\phi(-I_3) = -I_3$. It follows that $\phi(E) = E$ for all $E \in \mathcal{D}_1$.

Now suppose $n \geq 4$. Note that for any $A \in \mathcal{P}$, A commutes with all matrices of the form $[1] \oplus B$ in $\mathbf{GA}_n^+$ if and only if $A \in \{I_n, -I_n, D_1, -D_1\}$. Since $\phi(A) = A$ for all $\mathbf{GA}_n^+$,

$$\phi(D_1)([1] \oplus B) = \phi(D_1)\phi([1] \oplus B) = \phi([1] \oplus B)\phi(D_1) = ([1] \oplus B)\phi(D_1)$$

for all $[1] \oplus B \in \mathbf{GA}_n^+$. As $\phi(\{I_n, -I_n\}) = \{I_n, -I_n\}$, $\phi(D_1) \in \{D_1, -D_1\}$.

Suppose $\phi(D_1) = D_1$. For each $D_i \in \mathcal{D}_1$, $D_i = PD_1P^t$ for some $P \in \mathbf{A}_n \subseteq \mathbf{GA}_n^+$. Then

$$\phi(D_i) = \phi(P)\phi(D_1)\phi(P^t) = PD_1P^t = D_i.$$

Thus, condition (a) holds.

Now suppose $\phi(D_1) = -D_1$. Using a similar argument as in the last paragraph, we have $\phi(D_i) = -D_i$ for all $D_i \in \mathcal{D}_1$. Also observe that

$$\phi(-I_n) = \phi(D_1) \cdots \phi(D_n) = (-D_1) \cdots (-D_n) = (-1)^n(-I_n).$$

Then n must be even. □

Lemma 3.5 *Suppose $n \geq 3$ and $\phi(A) = A$ for all $A \in \mathbf{GA}_n^+$. If $X \in \{P_{(12)}, D_1P_{(12)}\} \cap \mathcal{P}$, then $\phi(X) \in \{X, -X\}$.*

Proof. Note that for any $X \in \mathbf{GP}_n$, we have $D_{st}X = XD_{st}$ for all $D_{st} \in \mathcal{D}_2$ if and only if $X \in \Sigma_n$. Consequently, $X = AP_{(12)} \in \mathbf{GP}_n$ commutes with all matrices in $\mathcal{D}_2$ if and only if $A = RP_{(12)}$ for some $R \in \Sigma_n$. Since ϕ fixes every matrix in $\mathcal{D}_2$, if $X \in \{P_{(12)}, D_1P_{(12)}\} \cap \mathcal{P}$, then $\phi(X) = RP_{(12)}$ for some $R \in \Sigma_n$.

Suppose $X = P_{(12)}$. Then for any $t = 3, \ldots, n$, $P_{(1t)} = P_{(12)}P_{(1t2)} \in \mathcal{P}$ and

$$\phi(P_{(1t)}) = \phi(P_{(12)}P_{(1t2)}) = \phi(P_{(12)})\phi(P_{(1t2)}) = RP_{(12)}P_{(1t2)} = RP_{(1t)}.$$

Since $\phi(P_{(1t)})^2 = I_n$ for all $t > 1$, one can conclude that $R = \pm I_n$.

Similarly, when $X = D_1P_{(12)}$, we have

$$\phi(D_1P_{(1t)}) = \phi(D_1P_{(12)}P_{(1t2)}) = \phi(D_1P_{(12)})\phi(P_{(1t2)}) = RP_{(12)}P_{(1t2)} = RP_{(1t)}$$

for $t = 3, \ldots, n$. Since $\phi(D_1P_{(1t)})^2 = D_{1t}$ for all $t > 1$, we can conclude that $R = \pm D_1$. In both cases, we see that $\phi(X) \in \{X, -X\}$. □

Proof of the necessity part for $n \geq 3$. By Lemma 3.2 and replacing ϕ by $A \mapsto P^t\phi(A)P$ or $A \mapsto N^tP^t\phi(A)PN$ for some $P \in \mathbf{GP}_n$, we may assume that $\phi(E) = E$ for all $E \in \mathcal{D}_2$. Then by Lemma 3.3, there is $S \in \Sigma_n$ such that $\phi(A) = SAS^t$ for all $\mathbf{GA}_n^+$. Thus, the result follows for $\mathcal{P} = \mathbf{GA}_n^+$.

For the other group $\mathcal{P}$, we may further assume that $\phi(A) = A$ for all $\mathbf{GA}_n^+$ by replacing ϕ by $A \mapsto S^t\phi(A)S$.

For $\mathcal{P} = \mathbf{GP}_n^e$ or $\mathbf{GP}_n^+$, let $X = P_{(12)}$ or $D_1 P_{(12)}$ according to $\mathcal{P} = \mathbf{GP}_n^e$ or $\mathbf{GP}_n^+$. Then $X \in \mathcal{P}$. By Lemma 3.5, $\phi(X) = X$ or $-X$. Since $\det(-X) = (-1)^n \det X$, $\phi(X) = -X$ only when n is even. In such case, we replace ϕ by $A \mapsto (\det A)A$, or $A \mapsto (-1)^{\delta(A)}A$, according to $\mathcal{P} = \mathbf{GP}_n^e$ or $\mathbf{GP}_n^+$, We then have $\phi(X) = X$. Since $\{X\} \cup \mathbf{GA}_n^+$ generates $\mathcal{P}$, it follows that $\phi(A) = A$ for all $A \in \mathcal{P}$. Then the result follows.

Now for the case when $\mathcal{P} = \mathbf{GA}_n$. Clearly, $\mathcal{D}_1 \subseteq \mathbf{GA}_n$. By Lemma 3.4, either $\phi(E) = E$ for all $E \in \mathcal{D}_1$, or n is even and $\phi(E) = -E$ for all $E \in \mathcal{D}_1$. For the second case, we can replace ϕ by the map $A \mapsto (\det A)A$. We then have $\phi(E) = E$ for all $E \in \mathcal{D}_1$. Since $\mathcal{D}_1 \cup \mathbf{GA}_n^+$ generates $\mathbf{GA}_n$, we see that $\phi(A) = A$ for all $A \in \mathbf{GA}_n$.

Finally, for $\mathcal{P} = \mathbf{GP}_n$, by Lemma 3.3 and 3.5 and replacing ϕ by the maps $A \mapsto (-1)^{\delta(A)}A$ and/or $A \mapsto (\det A)A$, we may assume that $\phi(P_{(12)}) = P_{(12)}$ and $\phi(E) = E$ for all $E \in \mathcal{D}_1$. Since $\{P_{(12)}\} \cup \mathcal{D}_1 \cup \mathbf{GA}_n^+$ generates $\mathbf{GP}_n$, we have $\phi(A) = A$ for all $A \in \mathbf{GP}_n$. The proof is completed. $\square$

4 Monomial Matrices

A matrix $A \in M_n(\mathrm{I\!R})$ with the property that each row and column has exactly one nonzero entry is said to be a monomial matrix. In other words, A is a product of some generalized permutation matrix P with some diagonal matrix D with positive diagonal entries. It is a nonnegative monomial matrix if all entries of A are nonnegative. This is equivalent to saying that P is a permutation matrix. Denote by $\mathbf{MN}_n$ and $\mathbf{NMN}_n$ be the set of all $n \times n$ monomial matrices and nonnegative monomial matrices respectively. We characterize the automorphisms of the groups $\mathbf{MN}_n$ and $\mathbf{NMN}_n$ in the following theorem. We continue to use $\delta(A)$ to denote the number of negative entries of A.

Theorem 4.1 *Let $\mathcal{M} = \mathbf{MN}_n$ or $\mathbf{NMN}_n$ and $\mathcal{R} = \mathrm{I\!R}$ or $[0, \infty)$ according to $\mathcal{M} = \mathbf{MN}_n$ or $\mathbf{NMN}_n$. Then $\phi : \mathcal{M} \to \mathcal{M}$ is a group automorphism if and only if there exist $T \in \mathcal{M}$, a multiplicative map ρ on $(0, \infty)$ and a bijective multiplicative map σ on $\mathcal{R}$ such that the map $\alpha \mapsto \rho(|\alpha|)^n\sigma(\alpha)$ is bijective on $\mathcal{R}\backslash\{0\}$ and one of the following holds.*

(a) *ϕ has the form*

$$A = (a_{ij}) \mapsto \rho(|\det A|)\ T\,(\sigma(a_{ij}))\,T^{-1},$$

(b) *$\mathcal{M} = \mathbf{MN}_n$ and ϕ has the form*

$$A = (a_{ij}) \mapsto \frac{(-1)^{\delta(A)}(\det A)}{|\det A|}\ \rho(|\det A|)\ T\,(\sigma(a_{ij}))\,T^{-1},$$

(c) $\mathcal{M} = \mathbf{MN}_n$, *n is even, and* ϕ *has the form*

$$A = (a_{ij}) \mapsto (-1)^{\delta(A)}\ \rho(|\det A|)\ T\,(\sigma(a_{ij}))\,T^{-1}$$

or

$$A = (a_{ij}) \mapsto \frac{(\det A)}{|\det A|}\ \rho(|\det A|)\ T\,(\sigma(a_{ij}))\,T^{-1}.$$

Proof. For the sufficiency part, one readily checks that ϕ defined in (a), (b) and (c) are multiplicative. It remains to show that ϕ is bijective.

Suppose ϕ has the form (a). Suppose $\phi(A) = \phi(B)$ with $A = (a_{ij})$ and $B = (b_{ij})$ in $\mathcal{M}$. Then

$$\rho(|\det A|)(\sigma(a_{ij})) = \rho(|\det B|)(\sigma(b_{ij})).$$

Note that $|\det A| = \left|\prod_{a_{ij}\neq 0} a_{ij}\right|$ for all $A \in \mathcal{M}$. Then

$$\rho(|\det A|)^n\sigma(|\det A|) = \left|\prod_{a_{ij}\neq 0} \rho(|\det A|)\sigma(a_{ij})\right|$$

$$= \left|\prod_{b_{ij}\neq 0} \rho(|\det B|)\sigma(b_{ij})\right| = \rho(|\det B|)^n\sigma(|\det B|).$$

Since $\alpha \mapsto \rho(|\alpha|)^n\sigma(\alpha)$ is bijective, we have $|\det A| = |\det B|$. Thus, $(\sigma(a_{ij})) = (\sigma(b_{ij}))$. As σ is bijective, $a_{ij} = b_{ij}$ for all i, j, i.e., $A = B$. Hence, ϕ is injective.

For any $B \in \mathcal{M}$, let $C = (c_{ij}) = T^{-1}BT$. Then there exists $\alpha \in \mathcal{R}\backslash\{0\}$ such that $\rho(|\alpha|)^n\sigma(\alpha) = \prod_{c_{ij}\neq 0} c_{ij}$. Furthermore, for any $1 \le i, j \le n$, there exists $a_{ij} \in \mathcal{R}$ such that $\sigma(a_{ij}) = \rho(|\alpha|)^{-1}c_{ij}$, i.e., $\rho(|\alpha|)\sigma(a_{ij}) = c_{ij}$. Let $A = (a_{ij}) \in \mathcal{M}$. Then

$$\sigma\left(\prod_{a_{ij}\neq 0} a_{ij}\right) = \prod_{a_{ij}\neq 0} \sigma(a_{ij}) = \prod_{c_{ij}\neq 0} \frac{c_{ij}}{\rho(|\alpha|)} = \frac{1}{\rho(|\alpha|)^n}\prod_{c_{ij}\neq 0} c_{ij} = \sigma(\alpha).$$

Hence, $\prod_{a_{ij}\neq 0} a_{ij} = \alpha$ and so $|\det A| = |\alpha|$. Thus,

$$\phi(A) = \rho(|\det A|)T(\sigma(a_{ij}))T^{-1} = \rho(|\alpha|)T(\sigma(a_{ij}))T^{-1} = T(c_{ij})T^{-1} = TCT^{-1} = B.$$

So ϕ is surjective and ϕ is an automorphism.

Now we can write the map ϕ in (b) in the following form

$$A \mapsto \frac{(-1)^{\delta(A)}(\det A)}{|\det A|}\psi(A),$$

where ψ is an automorphism of the form (a). Notice that $\psi(-A) = -\psi(A)$ for all $A \in \mathcal{M}$ as $\sigma(-\alpha) = -\sigma(\alpha)$ for all α. It follows that $\phi(-A) = -\phi(A)$ for all $A \in \mathcal{M}$. For any

$A, B \in \mathcal{M}$, suppose $\phi(A) = \phi(B)$. Then either $\psi(A) = \psi(B)$ or $\psi(A) = -\psi(B) = \psi(-B)$. As ψ is injective, either $A = B$ or $A = -B$. But the latter case cannot hold, otherwise

$$\phi(A) = \phi(B) = -\phi(-B) = -\phi(A).$$

Hence, ϕ is injective. On the other hand, for any $B \in \mathcal{M}$, there is $A \in \mathcal{M}$ such that $\psi(A) = B$. Then $\phi(\{A, -A\}) = \{B, -B\}$. So ϕ is surjective. The proof for (c) is similar.

Next we turn to the necessary part. Denote by $\mathbf{ND}_n$ the set of all nonnegative invertible diagonal matrices. Let $\mathcal{P} = \mathbf{GP}_n$ or $\mathbf{P}_n$ according to $\mathcal{M} = \mathbf{MN}_n$ or $\mathbf{NMN}_n$ respectively. Then $\mathbf{ND}_n$ is a normal subgroup of $\mathcal{M}$ and the quotient group $\mathcal{M}/\mathbf{ND}_n$ is isomorphic to $\mathcal{P}$. Hence, every automorphism ϕ of $\mathcal{M}$ will induce an automorphism ϕ_1 of $\mathcal{P}$. In the following, we restate these observations in matrix terms and give a short proof of the fact that ϕ_1 is an automorphism of $\mathcal{P}$ in Assertion 1. We then use the results on $\mathcal{P}$ to help finish our proof by establishing four additional assertions.

Every matrix A in $\mathcal{M}$ is a product of a matrix in $\mathbf{ND}_n$ and a matrix in $\mathcal{P}$. For any $P \in \mathcal{P}$, and let

$$\mathcal{P}(P) = \{DP : D \in \mathbf{ND}_n\}.$$

Then $A \in \mathcal{P}(P)$ if and only if the sign pattern of A and P are the same. Furthermore, distinct $\mathcal{P}(P)$ are disjoint and the union of all $\mathcal{P}(P)$ equals to $\mathcal{M}$. For each $P \in \mathcal{P}$, $\phi(P) \in \mathcal{P}(Q)$ for some $Q \in \mathcal{P}$. Define $\phi_1 : \mathcal{P} \to \mathcal{P}$ by $\phi_1(P) = Q$.

Assertion 1 *The map $\phi_1 : \mathcal{P} \to \mathcal{P}$ is an automorphism.*

Proof. Clearly, ϕ_1 is a multiplicative map. To show that ϕ_1 is injective, suppose $\phi_1(P_1) = \phi_1(P_2)$ for some $P_1, P_2 \in \mathcal{P}$. Then $\phi(P_1) = S_1 Q$ and $\phi(P_2) = S_2 Q$ for some $S_1, S_2 \in \mathbf{ND}_n$ and $Q \in \mathcal{P}$. Since $P_1 P_2^{-1} \in \mathcal{P}$, there exists some positive integer k such that $(P_1 P_2^{-1})^k = I_n$. Then

$$I_n = \phi(I_n) = \phi((P_1 P_2^{-1})^k) = [(S_1 Q)(S_2 Q)^{-1}]^k = [S_1 S_2^{-1}]^k.$$

Since $S_1 S_2^{-1}$ is a nonnegative invertible diagonal matrix, it must be the identity matrix, i.e., $S_1 = S_2$. Then $\phi(P_1) = \phi(P_2)$, and hence $P_1 = P_2$ as ϕ is injective. So, ϕ_1 is injective. Since $\mathcal{P}$ is finite, ϕ_1 is bijective. □

Assertion 2 *Replacing ϕ by one of the maps*

$$\text{(i) } A \mapsto Q^t \phi(A) Q,\ Q \in \mathcal{P}, \quad \text{(ii) } A \mapsto (-1)^{-\delta(A)} \phi(A), \quad \text{(iii) } A \mapsto \frac{|\det A|}{(-1)^{\delta(A)} \det A} \phi(A),$$

or their compositions, we may assume that $\phi_1(P) = P$ for every $P \in \mathcal{P}$.

Proof. By Theorems 2.1 or 3.1, there exists $Q \in \mathcal{P}$ such that one of the following holds.

(I) ϕ_1 has the form $P \mapsto QPQ^t$.

(II) $\mathcal{P} = \mathbf{GP}_n$ and ϕ_1 has the form $P \mapsto (-1)^{\delta(P)} (\det P) QPQ^t$.

(III) $\mathcal{P} = \mathbf{GP}_n$ with even n and ϕ_1 has the form $P \mapsto (-1)^{\delta(P)}QPQ^t$ or $P \mapsto (\det P)QPQ^t$.

The assertion follows. $\square$

From now on, we will assume that $\phi_1(P) = P$ for every $P \in \mathcal{P}$.

Assertion 3 *Replacing ϕ by $A \mapsto D^{-1}\phi(A)D$ for some $D \in \mathbf{ND}_n$, we may further assume that $\phi(P) = P$ for every $P \in \mathcal{P}$.*

Proof. Let τ be the permutation $(1, 2, \ldots, n)$ and P_τ be the corresponding permutation matrix. Then

$$P_\tau = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix} \quad \text{and} \quad \phi(P_\tau) = LP_\tau = \begin{pmatrix} 0 & \cdots & 0 & l_1 \\ l_2 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & l_n & 0 \end{pmatrix}$$

for some $L = \operatorname{diag}(l_1, \ldots, l_n) \in \mathbf{ND}_n$. Note that $P_\tau^n = I_n$ and $\phi(P_\tau)^n = (l_1 \cdots l_n)I_n$. Thus, $l_1 \cdots l_n = 1$.

Set $d_k = l_1 \cdots l_k$ and $D = \operatorname{diag}(d_1, \ldots, d_n) \in \mathbf{ND}_n$. Then

$$\phi(P_\tau)D = \begin{pmatrix} 0 & \cdots & 0 & l_1 \\ l_1 l_2 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & l_1 \cdots l_n & 0 \end{pmatrix} = DP_\tau.$$

Replacing ϕ by $A \mapsto D^{-1}\phi(A)D$, we may assume $\phi(P_\tau) = P_\tau$.

Let $P_{(1,2)}$ be the permutation matrix corresponding to the transposition $(1, 2)$. Then $\phi(P_{(1,2)}) = KP_{(1,2)}$ for some $K = \operatorname{diag}(k_1, \ldots, k_n) \in \mathbf{ND}_n$. Observe that

$$\operatorname{diag}(k_1k_2, k_1k_2, k_3^2, \ldots, k_n^2) = (KP_{(1,2)})^2 = \phi(P_{(1,2)})^2 = \phi(I_n) = I_n.$$

Then $k_1k_2 = 1$ and $k_j = 1$ for $j > 2$. On the other hand, we know that the permutation $(1, 2)(1, 2, \ldots, n)$ has order $n - 1$ and fixes 1. Then the $(1, 1)$-th entry of $(KP_{(1,2)}P_\tau)^{n-1}$ is k_1^{n-1}. Since $(KP_{(1,2)}P_\tau)^{n-1} = \phi(P_{(1,2)}P_\tau)^{n-1} = I_n$, $k_1 = k_2 = 1$. Hence, $\phi(P_{(1,2)}) = P_{(1,2)}$.

When $\mathcal{M} = \mathbf{MN}_n$, since $D_1 = \operatorname{diag}(-1, 1 \ldots, 1) \in \mathcal{P}$, we see that $\phi(D_1) = TD_1$ for some $T \in \mathbf{ND}_n$. Moreover, we have $T = I_n$ as $D_1^2 = I_n$. So, $\phi(D_1) = D_1$. Since the sets $\{P_{(1,2)}, P_\tau\}$ and $\{P_{(1,2)}, P_\tau, D_1\}$ generate $\mathbf{P}_n$ and $\mathbf{GP}_n$ respectively, we have $\phi(P) = P$ for all $P \in \mathcal{P}$. $\square$

In the rest of the proof, we assume that $\phi(P) = P$ for every $P \in \mathcal{P}$.

Assertion 4 *There exist multiplicative maps ρ and σ on $(0, \infty)$ such that*

$$\phi(D) = \rho(d_1 \cdots d_n)\operatorname{diag}(\sigma(d_1), \ldots, \sigma(d_n)) \quad \textit{for all} \quad D = \operatorname{diag}(d_1, \ldots, d_n) \in \mathbf{ND}_n.$$

Proof. For any $\alpha \in (0, \infty)$, let $R_\alpha = \text{diag}\,(\alpha, 1, \ldots, 1)$. Note that R_α commutes with all matrices of the form $[1] \oplus P$ in $\mathcal{P}$, it follows that $\phi(R_\alpha) = \text{diag}\,(\beta_\alpha, \gamma_\alpha, \ldots, \gamma_\alpha)$ for some nonzero β_α and γ_α. As

$$\text{diag}\,(\beta_\alpha, \gamma_\alpha, \ldots, \gamma_\alpha) = \phi(R_\alpha) = \phi(R_{\sqrt{\alpha}})^2 = \text{diag}\,(\beta^2_{\sqrt{\alpha}}, \gamma^2_{\sqrt{\alpha}}, \ldots, \gamma^2_{\sqrt{\alpha}}),$$

β_α and γ_α must be positive.

Define $\rho, \sigma : (0, \infty) \to (0, \infty)$ by $\rho(\alpha) = \gamma_\alpha$ and $\sigma(\alpha) = \beta_\alpha \gamma_\alpha^{-1}$ respectively. Clearly, both ρ and σ are multiplicative and

$$\phi(\text{diag}\,(\alpha, 1 \ldots, 1)) = \rho(\alpha)\text{diag}\,(\sigma(\alpha), 1, \ldots, 1).$$

Since $\phi(P) = P$ for all $P \in \mathcal{P}$,

$$\phi(\text{diag}\,(1, \ldots, 1, \underbrace{\alpha}_{i-\text{th}}, 1 \ldots, 1)) = \rho(\alpha)\text{diag}\,(1, \ldots, 1, \underbrace{\sigma(\alpha)}_{i-\text{th}}, 1 \ldots, 1).$$

Then the result follows. □

Assertion 5 *The conclusion of Theorem 4.1 holds.*

Proof. When $\mathcal{M} = \mathbf{NMN}_n$, we extend σ in Assertion 4 to $\sigma_1 : [0, \infty) \to [0, \infty)$ by $\sigma_1(\alpha) = \sigma(\alpha)$ for $\alpha > 0$ and $\sigma_1(0) = 0$. When $\mathcal{M} = \mathbf{MN}_n$, we can extend σ to $\sigma_1 : \mathbb{R} \to \mathbb{R}$ by

$$\sigma_1(\alpha) = \begin{cases} \sigma(\alpha) & \text{if } \alpha > 0, \\ 0 & \text{if } \alpha = 0, \\ -\sigma(-\alpha) & \text{if } \alpha < 0. \end{cases}$$

One can readily verify that σ_1 is multiplicative and extends σ. Rename σ_1 by σ. For each $A \in \mathcal{M}$, $A = DP$ for some $D \in \mathbf{ND}_n$ and $P \in \mathcal{P}$. Also, $\det D = |\det D| = |\det A|$. By Assertions 3 and 4, we conclude that

$$\phi(A) = \rho(|\det A|)(\sigma(a_{ij})) \quad \text{for all} \quad A = (a_{ij}) \in \mathcal{M}.$$

It remains to show that σ and ρ satisfy the conditions of the theorem. Notice that $\sigma(1) = 1 = \rho(1)$ and $\sigma(\alpha^{-1}) = \sigma(\alpha)^{-1}$ for all nonzero α. We first show that σ is injective. Suppose not, then exist two distinct nonzero $\alpha, \beta \in \mathbb{R}$ such that $\sigma(\alpha) = \sigma(\beta)$. Let $A = \text{diag}\,(\alpha, \alpha^{-1}, 1, \ldots, 1)$ and $B = \text{diag}\,(\beta, \beta^{-1}, 1, \ldots, 1)$. Then

$$\phi(A) = \text{diag}\,(\sigma(\alpha), \sigma(\alpha)^{-1}, 1, \ldots, 1) = \phi(B),$$

which contradicts that ϕ is injective.

On the other hand, for any nonzero β, there exists a diagonal matrix $D = \text{diag}\,(d_1, \ldots, d_n)$ in $\mathcal{M}$ such that $\phi(D) = \text{diag}\,(\beta, 1, \ldots, 1)$. Then we have

$$\rho(|\det D|)\sigma(d_1) = \beta \quad \text{and} \quad \rho(|\det D|)\sigma(d_2) = 1.$$

Thus, $\sigma(d_1/d_2) = \beta$. Hence, σ is surjective.

Define $\pi(\alpha) = \rho(|\alpha|)^n\sigma(\alpha)$ on $\mathcal{R}\backslash\{0\}$. Note that $A \in \mathcal{M}$ is a nonzero scalar matrix if and only if A commutes with all matrices in $\mathcal{P}$. Since $\phi(P) = P$ for all $P \in \mathcal{P}$, $\phi(A)$ is a nonzero scalar matrix if and only if A is. As $\phi(\alpha I_n) = \rho(|\alpha|^n)\sigma(\alpha)I_n = \pi(\alpha)I_n$, π must be bijective as ϕ is. □

5 Generalized Stochastic Matrices

A matrix $A \in M_n(\mathbb{R})$ with the properties that all entries of A are nonnegative and the sum of each row (column) equals to 1 is said to be a row (column) stochastic matrix. Set $\mathbf{1} = (1, \ldots, 1)^t \in \mathbb{R}^n$. Then a matrix A with nonnegative entries is row (column) stochastic if and only if $A\mathbf{1} = \mathbf{1}$ ($A^t\mathbf{1} = \mathbf{1}$). If $A \in M_n(\mathbb{R})$ is both row and column stochastic, then A is a doubly stochastic matrix. Denote by $\mathbf{RS}_n$, $\mathbf{CS}_n$, and $\mathbf{DS}_n$ the set of $n \times n$ row stochastic, column stochastic, and doubly stochastic matrices. One readily checks that they are semigroups. If the nonnegativity assumption is removed from the matrices in these semigroups, we get the semigroups of generalized row stochastic matrices, generalized column stochastic matrices, and generalized doubly stochastic matrices, respectively. We use $\mathbf{GRS}_n$, $\mathbf{GCS}_n$ and $\mathbf{GDS}_n$ to denote these semigroups, and we will characterize their automorphisms in this section. The result will be used to characterize the semigroup automorphisms of $\mathbf{RS}_n$, $\mathbf{CS}_n$ and $\mathbf{DS}_n$ in the next section.

Let J_n be the $n \times n$ matrix with $\frac{1}{n}$ as each of its entries, and for any matrices A and F, set

$$A_F = FAF^t.$$

The main theorem of this section is the following.

Theorem 5.1 *Let $\mathcal{G} = \mathbf{GDS}_n$, $\mathbf{GRS}_n$ or $\mathbf{GCS}_n$. Then $\phi : \mathcal{G} \to \mathcal{G}$ is an automorphism if and only if one of the following holds.*

(a) *There exists an invertible $T \in \mathcal{G}$ such that*

$$\phi(A) = TAT^{-1} \quad \text{for all} \quad A \in \mathcal{G}.$$

When $\mathcal{G} = \mathbf{GDS}_n$, one can assume that $T \in \mathbf{DS}_n$.

(b) *$\mathcal{G} = \mathbf{GDS}_2$ and there exists a bijective multiplicative map σ on $\mathbb{R}$ such that*

$$\phi(A) = \sigma(\det A)(I_2 - J_2) + J_2 \quad \text{for all} \quad A \in \mathbf{GDS}_2.$$

The proof depends heavily on the following proposition.

Proposition 5.2 *Let F be an orthogonal matrix with $\mathbf{1}/\sqrt{n}$ as the first column. Then*

(a) *$A \in \mathbf{GDS}_n$ if and only if $A = \begin{pmatrix} 1 & 0 \\ 0 & A_1 \end{pmatrix}_F$ for some $A_1 \in M_{n-1}(\mathbb{R})$.*

(b) $B \in \mathbf{GRS}_n$ *if and only if* $B = \begin{pmatrix} 1 & b^t \\ 0 & B_1 \end{pmatrix}_F$ *for some* $b \in \mathbb{R}^{n-1}$ *and* $B_1 \in M_{n-1}(\mathbb{R})$.

(c) $C \in \mathbf{GCS}_n$ *if and only if* $C = \begin{pmatrix} 1 & 0 \\ c & C_1 \end{pmatrix}_F$ *for some* $c \in \mathbb{R}^{n-1}$ *and* $C_1 \in M_{n-1}(\mathbb{R})$.

Proof. Note that $\mathbf{1}$ is the right (left) eigenvector of all row (column) stochastic matrices corresponding to the eigenvalue 1. Then (b) and (c) follow directly. Since a doubly stochastic matrix is both row and column stochastic, (a) also holds. □

By the above proposition, the semigroups $\mathbf{GDS}_n$ and $M_{n-1}(\mathbb{R})$ are isomorphic. We will use the known characterization of the automorphism on $M_{n-1}(\mathbb{R})$ to help solve our problem. Note that the automorphisms of $M_1(\mathbb{R}) = \mathbb{R}$ is different from that for $M_k(\mathbb{R})$ for $k > 1$. This explains why the case $n = 2$ is special in Theorem 5.1.

Furthermore, the semigroup $\mathbf{GCS}_n$ and the semigroup of affine maps on $\mathbb{R}^{n-1}$ are isomorphic under the map

$$\begin{pmatrix} 1 & 0 \\ c & C_1 \end{pmatrix}_F \mapsto [c \,|\, C_1],$$

where $[c \,|\, C_1]$ represents the affine map $x \mapsto C_1 x + c$. By Theorem 5.1, we see that the automorphisms of the semigroup of affine maps on $\mathbb{R}^{n-1}$ are given by

$$[c \,|\, C_1] \mapsto [Pc \,|\, PC_1P^{-1}]$$

for some $P \in GL_{n-1}(\mathbb{R})$.

We organize the proof of the theorem in several subsections.

5.1 Auxiliary lemmas

In the following, we fixed F to be some $n \times n$ orthogonal matrix with $\mathbf{1}/\sqrt{n}$ as its first column. Also for any $\alpha \in \mathbb{R}$ let

$$J(\alpha) = \alpha(I_n - J_n) + J_n.$$

Note that for any $\alpha, \beta \in \mathbb{R}$,

(i) $J(\alpha)J(\beta) = J(\alpha\beta)$,

(ii) $J(\alpha) \in \mathbf{DS}_n$ if and only if $\alpha \in [-1/(n-1), 1]$,

(iii) $J(1) = I_n = \begin{pmatrix} 1 & 0 \\ 0 & I_{n-1} \end{pmatrix}_F$, $J(0) = J_n = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_F$ and $J(\alpha) = \begin{pmatrix} 1 & 0 \\ 0 & \alpha I_{n-1} \end{pmatrix}_F$.

Lemma 5.3 *Let* $\mathcal{G} = \mathbf{GDS}_n$, $\mathbf{GRS}_n$ *or* $\mathbf{GCS}_n$. *Suppose* $A \in \mathcal{G}$. *Then there exists* $\epsilon \in (0, 1]$ *such that for any* $\alpha \in (0, \epsilon)$,

(a) $AJ(\alpha) \in \mathbf{DS}_n$ *if* $\mathcal{G} = \mathbf{GDS}_n$,

(b) $AJ(\alpha) \in \mathbf{RS}_n$ *if* $\mathcal{G} = \mathbf{GRS}_n$,

(c) $J(\alpha)A \in \mathbf{CS}_n$ *if* $\mathcal{G} = \mathbf{GCS}_n$.

Proof. We just prove (c), the proofs for (a) and (b) being similar. For any $A \in \mathbf{GCS}_n$, by Proposition 5.2, we write $A = \begin{pmatrix} 1 & 0 \\ c & A_1 \end{pmatrix}_F$ for some $A_1 \in M_{n-1}(\mathbb{R})$. Then there exists $\epsilon \in (0,1]$ such that for any $\alpha \in (0,\epsilon)$ all entries of $\begin{pmatrix} 0 & 0 \\ \alpha c & \alpha A_1 \end{pmatrix}_F$ have absolute values less than $1/n$. Thus

$$J(\alpha)A = \begin{pmatrix} 1 & 0 \\ 0 & \alpha I_{n-1} \end{pmatrix}_F \begin{pmatrix} 1 & 0 \\ c & A_1 \end{pmatrix}_F = \begin{pmatrix} 1 & 0 \\ \alpha c & \alpha A_1 \end{pmatrix}_F = J_n + \begin{pmatrix} 0 & 0 \\ \alpha c & \alpha A_1 \end{pmatrix}_F$$

has nonnegative entries only. Hence, $J(\alpha)A \in \mathbf{CS}_n$. It is clear that $J(\alpha)$ is invertible and is in $\mathbf{DS}_n$ as $0 < \alpha < 1$. □

Lemma 5.4 *For any $X \in \mathbf{GRS}_n$, X is a rank one idempotent if and only if*

$$AX = X \quad \textit{for all} \quad A \in \mathbf{GRS}_n.$$

Similarly, for any $X \in \mathbf{GCS}_n$, X is a rank one idempotent if and only if

$$XA = X \quad \textit{for all} \quad A \in \mathbf{GCS}_n.$$

Proof. We just prove for the case for $\mathbf{GRS}_n$. The other case is similar. The necessary part is clear. For the sufficiency part, let $x_1, \ldots, x_n$ be the column vectors of X. If $AX = X$, then $x_1, \ldots, x_n$ are the right eigenvectors of A corresponding to the eigenvalue 1. As the multiples of $\mathbf{1}$ are the only vectors satisfying $Ax = x$ for all $A \in \mathbf{GRS}_n$, for each i, $x_i = a_i \mathbf{1}$ for some $a_i \in \mathbb{R}$. Then $X = \mathbf{1}\, a^t$ where $a^t = (a_1, \ldots, a_n)$. Finally as $X \in \mathbf{GRS}_n$, $\mathbf{1}\, a^t \mathbf{1} = \mathbf{1}$. Then $X^2 = (\mathbf{1}\, a^t)(\mathbf{1}\, a^t) = \mathbf{1}\, a^t = X$. □

5.2 Proof of Theorem 5.1 for $\mathcal{G} = \mathbf{GDS}_n$

The sufficiency part is clear. For necessity part, by Proposition 5.2, $A \in \mathbf{GDS}_n$ if and only if $A = \begin{pmatrix} 1 & 0 \\ 0 & A_1 \end{pmatrix}_F$ for some $A_1 \in M_{n-1}(\mathbb{R})$. Then for any $A_1 \in M_{n-1}(\mathbb{R})$,

$$\phi\left(\begin{pmatrix} 1 & 0 \\ 0 & A_1 \end{pmatrix}_F\right) = \begin{pmatrix} 1 & 0 \\ 0 & B_1 \end{pmatrix}_F$$

for some $B_1 \in M_{n-1}(\mathbb{R})$. Define $\psi : M_{n-1}(\mathbb{R}) \to M_{n-1}(\mathbb{R})$ by $\psi(A_1) = B_1$. Then ψ is a semigroup automorphism on $M_{n-1}(\mathbb{R})$.

When $n = 2$, ψ is just a bijective multiplicative map on $\mathbb{R}$. Observe that for any $A \in \mathbf{GDS}_2$,

$$A = (\det A)(I_2 - J_2) + J_2 = \begin{pmatrix} 1 & 0 \\ 0 & \det A \end{pmatrix}_F .$$

Then ϕ satisfies Theorem 5.1 (b). When $n > 2$, from [7](see also [1]), there exists an invertible $S \in M_{n-1}(\mathbb{R})$ such that

$$\psi(A_1) = SA_1S^{-1} \quad \text{for all} \quad A_1 \in M_{n-1}(\mathbb{R}).$$

Thus,

$$\phi(A) = TAT^{-1} \quad \text{for all} \quad A \in \mathbf{GDS}_n,$$

where $T = ([1] \oplus S)_F \in \mathbf{GDS}_n$.

When $\mathcal{G} = \mathbf{GDS}_n$, we can assume that the matrix $T \in \mathbf{DS}_n$ for the following reason. By Lemma 5.3, there exists $J(\alpha) \in \mathbf{DS}_n$ such that $T_1 = TJ(\alpha) \in \mathbf{DS}_n$. As $J(\alpha)$ commutes with all matrices in $\mathbf{GDS}_n$,

$$T_1AT_1^{-1} = [TJ(\alpha)]A[TJ(\alpha)]^{-1} = TAJ(\alpha)J(\alpha)^{-1}T^{-1} = TAT^{-1}.$$

Thus, we may replace T by $T_1 \in \mathbf{DS}_n$. □

5.3 Proof of Theorem 5.1 for $\mathcal{G} = \mathbf{GCS}_n$ and $\mathbf{GRS}_n$

In order to prove the theorem for $\mathcal{G} = \mathbf{GCS}_n$, we first establish a sequence of assertions.

Assertion 1 *Replacing ϕ by $A \mapsto T^{-1}\phi(A)T$ for some $T \in \mathbf{GCS}_n$, we may assume that $\phi(J_n) = J_n$ and $\phi(\mathbf{GDS}_n) = \mathbf{GDS}_n$.*

Proof. Let $X = \phi(J_n)$. For any $A \in \mathbf{GCS}_n$, there exists $B \in \mathbf{GCS}_n$ such that $\phi(B) = A$. Then

$$XA = \phi(J_n)\phi(B) = \phi(J_nB) = \phi(J_n) = X.$$

By Lemma 5.4, X is a rank one idempotent, and by Proposition 5.2, $X = \begin{pmatrix} 1 & 0 \\ x & 0_{n-1} \end{pmatrix}_F$ for some $x \in \mathbb{R}^{n-1}$. Let $T = \begin{pmatrix} 1 & 0 \\ x & I_{n-1} \end{pmatrix}_F$. Then

$$T^{-1}XT = \begin{pmatrix} 1 & 0 \\ -x & I_{n-1} \end{pmatrix}_F \begin{pmatrix} 1 & 0 \\ x & 0_{n-1} \end{pmatrix}_F \begin{pmatrix} 1 & 0 \\ x & I_{n-1} \end{pmatrix}_F = \begin{pmatrix} 1 & 0 \\ 0 & 0_{n-1} \end{pmatrix}_F = J_n.$$

Replacing ϕ by $A \mapsto T^{-1}\phi(A)T$, we may assume that $\phi(J_n) = J_n$. Note that for any $A \in \mathbf{GCS}_n$, $A \in \mathbf{GDS}_n$ if and only if $AJ_n = J_n$. As

$$\phi(A)J_n = \phi(A)\phi(J_n) = \phi(AJ_n),$$

$\phi(A) \in \mathbf{GDS}_n$ if and only if $A \in \mathbf{GDS}_n$. Thus, $\phi(\mathbf{GDS}_n) = \mathbf{GDS}_n$. □

By the result in Subsection 3.2, we know that

$$\phi\left(\begin{pmatrix}1 & 0\\ 0 & X\end{pmatrix}_F\right) = \begin{pmatrix}1 & 0\\ 0 & \psi(X)\end{pmatrix}_F \quad \text{for all} \quad \begin{pmatrix}1 & 0\\ 0 & X\end{pmatrix}_F \in \mathbf{GDS}_n, \tag{1}$$

where ψ is a semigroup automorphism on $M_{n-1}(\mathbb{R})$. In fact, $\psi = \sigma$ is a bijective multiplicative map on $\mathbb{R}$ if $n = 2$ and $\psi(X) = SXS^{-1}$ for some invertible $S \in M_{n-1}(\mathbb{R})$ if $n > 2$. Replacing ϕ by

$$A \mapsto ([1] \oplus S^{-1})_F \phi(A)([1] \oplus S)_F, \tag{2}$$

if necessary, we may further assume that $\psi(A) = A$ for all $A \in \mathbf{GDS}_n$ when $n > 2$.

Assertion 2 *There exist nonzero $\alpha, \beta \in \mathbb{R}$ such that*

$$\phi\left(\begin{pmatrix}1 & 0\\ \mathbf{1} & I_{n-1}\end{pmatrix}_F\right) = \begin{pmatrix}1 & 0\\ \alpha\mathbf{1} & \beta I_{n-1}\end{pmatrix}_F.$$

Replacing ϕ by the map $A \mapsto T^{-1}\phi(A)T$, where $T = \begin{pmatrix}1 & 0\\ 0 & \alpha I_{n-1}\end{pmatrix}$, we may assume that $\alpha = 1$.

Proof. Clearly, the result holds when $n = 2$. When $n > 2$, let

$$\phi\left(\begin{pmatrix}1 & 0\\ \mathbf{1} & I_{n-1}\end{pmatrix}_F\right) = \begin{pmatrix}1 & 0\\ b & B_1\end{pmatrix}_F.$$

Observe that for any invertible $R \in \mathbf{GRS}_{n-1}$,

$$\begin{pmatrix}1 & 0\\ \mathbf{1} & I_{n-1}\end{pmatrix}_F = \begin{pmatrix}1 & 0\\ R\mathbf{1} & I_{n-1}\end{pmatrix}_F = \begin{pmatrix}1 & 0\\ 0 & R\end{pmatrix}_F \begin{pmatrix}1 & 0\\ \mathbf{1} & I_{n-1}\end{pmatrix}_F \begin{pmatrix}1 & 0\\ 0 & R^{-1}\end{pmatrix}_F.$$

Hence,

$$\begin{aligned}\begin{pmatrix}1 & 0\\ b & B_1\end{pmatrix}_F &= \phi\left(\begin{pmatrix}1 & 0\\ \mathbf{1} & I_{n-1}\end{pmatrix}_F\right) = \phi\left(\begin{pmatrix}1 & 0\\ 0 & R\end{pmatrix}_F\right)\phi\left(\begin{pmatrix}1 & 0\\ \mathbf{1} & I_{n-1}\end{pmatrix}_F\right)\phi\left(\begin{pmatrix}1 & 0\\ 0 & R^{-1}\end{pmatrix}_F\right)\\ &= \begin{pmatrix}1 & 0\\ 0 & R\end{pmatrix}_F \begin{pmatrix}1 & 0\\ b & B_1\end{pmatrix}_F \begin{pmatrix}1 & 0\\ 0 & R^{-1}\end{pmatrix}_F = \begin{pmatrix}1 & 0\\ Rb & RB_1R^{-1}\end{pmatrix}_F.\end{aligned}$$

Thus, $Rb = b$ and $RB_1R^{-1} = B_1$ for all invertible $R \in \mathbf{GRS}_{n-1}$. Consequently, $b = \alpha\mathbf{1}$ and $B_1 = \beta I_{n-1}$ for some nonzero $\alpha, \beta \in \mathbb{R}$. □

Assertion 3 *For every $\begin{pmatrix}1 & 0\\ a & A_1\end{pmatrix}_F \in \mathbf{GCS}_n$, there exist an invertible $B \in M_{n-1}(\mathbb{R})$ such that $B\mathbf{1} = a$. Moreover, for such a matrix B,*

$$\phi\left(\begin{pmatrix}1 & 0\\ a & A_1\end{pmatrix}_F\right) = \begin{pmatrix}1 & 0\\ \psi(B)\mathbf{1} & \beta\psi(A_1)\end{pmatrix}_F,$$

where β is defined as in Assertion 2 and ψ is defined as in (1).

Proof. First, observe that for any $A_1 \in M_{n-1}(\mathbb{R})$,

$$\begin{aligned}
\phi\left(\begin{pmatrix} 1 & 0 \\ \mathbf{1} & A_1 \end{pmatrix}_F\right) &= \phi\left(\begin{pmatrix} 1 & 0 \\ \mathbf{1} & I_{n-1} \end{pmatrix}_F\right)\phi\left(\begin{pmatrix} 1 & 0 \\ 0 & A_1 \end{pmatrix}_F\right) \\
&= \begin{pmatrix} 1 & 0 \\ \mathbf{1} & \beta I_{n-1} \end{pmatrix}_F \begin{pmatrix} 1 & 0 \\ 0 & \psi(A_1) \end{pmatrix}_F = \begin{pmatrix} 1 & 0 \\ \mathbf{1} & \beta\psi(A_1) \end{pmatrix}_F .
\end{aligned}$$

Now for any nonzero $a \in \mathbb{R}^{n-1}$, there is an invertible $B \in M_{n-1}(\mathbb{R})$ such that $a = B\mathbf{1}$. Then

$$\begin{aligned}
\phi\left(\begin{pmatrix} 1 & 0 \\ a & A_1 \end{pmatrix}_F\right) &= \phi\left(\begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix}_F\right)\phi\left(\begin{pmatrix} 1 & 0 \\ \mathbf{1} & B^{-1}A_1 \end{pmatrix}_F\right) \\
&= \begin{pmatrix} 1 & 0 \\ 0 & \psi(B) \end{pmatrix}_F \begin{pmatrix} 1 & 0 \\ \mathbf{1} & \beta\psi(B^{-1}A_1) \end{pmatrix}_F \\
&= \begin{pmatrix} 1 & 0 \\ \psi(B)\mathbf{1} & \beta\psi(A_1) \end{pmatrix}_F .
\end{aligned}$$

□

Assertion 4 *The conclusion of Theorem 5.1 holds for $\mathcal{G} = \mathbf{GCS}_n$.*

Proof. We continue to use the notations in Assertion 3. Suppose $n > 2$. Then by the replacement in (2) we have $\psi(B) = B$ and $\psi(A_1) = A_1$. Then $\psi(B)\mathbf{1} = B\mathbf{1} = a$. Since ϕ is multiplicative, β must equal to 1. Hence, $\phi(A) = A$ for all $A \in \mathbf{GCS}_n$.

When $n = 2$, we have $\mathbf{1} = 1$, $B = a$, and $\psi = \sigma$ is a bijective multiplicative map on $\mathbb{R}$. So, $\sigma(1) = 1$, and $\psi(B)\mathbf{1} = \sigma(a)$. Since ϕ is multiplicative, β must equal to 1, and

$$\phi\left(\begin{pmatrix} 1 & 0 \\ a & A_1 \end{pmatrix}_F\right) = \begin{pmatrix} 1 & 0 \\ \sigma(a) & \sigma(A_1) \end{pmatrix}_F \quad \text{for all} \quad \begin{pmatrix} 1 & 0 \\ a & A_1 \end{pmatrix}_F \in \mathbf{GCS}_2.$$

Moreover, for any $a, b \in \mathbb{R}$,

$$\begin{aligned}
\begin{pmatrix} 1 & 0 \\ \sigma(a+b) & 1 \end{pmatrix}_F &= \phi\left(\begin{pmatrix} 1 & 0 \\ a+b & 1 \end{pmatrix}_F\right) = \phi\left(\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}_F\right)\phi\left(\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}_F\right) \\
&= \begin{pmatrix} 1 & 0 \\ \sigma(a) & 1 \end{pmatrix}_F \begin{pmatrix} 1 & 0 \\ \sigma(b) & 1 \end{pmatrix}_F = \begin{pmatrix} 1 & 0 \\ \sigma(a)+\sigma(b) & 1 \end{pmatrix}_F .
\end{aligned}$$

Hence, σ is also additive, i.e., σ is a field-isomorphism of $\mathbb{R}$. Therefore, σ is the identity map and the result follows. □

The proof for generalized row stochastic matrices is similar.

6 Stochastic Matrices

Theorem 6.1 *Let $\mathcal{S} = \mathbf{DS}_n$, $\mathbf{RS}_n$ or $\mathbf{CS}_n$. Then $\phi : \mathcal{S} \to \mathcal{S}$ is an automorphism if and only if one of the following holds.*

(a) *There exists* $P \in \mathbf{P}_n$ *such that*

$$\phi(A) = PAP^t \quad \textit{for all} \quad A \in \mathcal{S}.$$

(b) $\mathcal{S} = \mathbf{DS}_2$ *and there exists a positive* λ *such that*

$$\phi(A) = (\det A)|\det A|^{\lambda-1}(I_2 - J_2) + J_2 \quad \textit{for all} \quad A \in \mathbf{DS}_2.$$

We first present some general lemmas.

Lemma 6.2 *Let* $\mathcal{S} = \mathbf{DS}_n$, $\mathbf{RS}_n$ *or* $\mathbf{CS}_n$. *A matrix* $X \in \mathcal{S}$ *has its inverse in* $\mathcal{S}$ *if and only if* $X \in \mathbf{P}_n$.

Proof. The sufficiency part is clear. For the necessary part, consider the cases when $\mathcal{S} = \mathbf{DS}_n$ or $\mathbf{RS}_n$. Suppose there is $X = (x_{ij}) \in \mathcal{S}$ such that $X \notin \mathbf{P}_n$ and X has its inverse $Y = (y_{ij}) \in \mathcal{S}$. Then for some column of X, there are at least two nonzero entries. Without loss of generality, assume the first column has more than one nonzero entry, say $x_{a1}, x_{b1} > 0$, for $a \neq b$. Since $YX = I_n$, for $i \neq 1$,

$$\sum_{k=1}^{n} y_{ik}x_{k1} = 0.$$

Since $x_{a1}, x_{b1} > 0$, it follows that $y_{ia} = 0 = y_{ib}$ for $i = 2, \ldots, n$. But then, Y cannot be invertible because the a-th and b-th column of Y are linear dependent, which is a contradiction. The proof for $\mathcal{S} = \mathbf{CS}_n$ is similar. □

Note that $[-1, 1]$ is a semigroup under the usual multiplication for real number.

Lemma 6.3 *If* $g : [-1, 1] \to [-1, 1]$ *is an automorphism, then there exists* $\lambda > 0$ *such that*

$$g(x) = x|x|^{\lambda-1} \quad \textit{for all} \quad x \in [-1, 1].$$

Proof. Since $g(0)^2 = g(0)$ and $g(1)^2 = g(1)$ and g is injective, $\{g(1), g(0)\} = \{0, 1\}$.

It is impossible to have $g(1) = 0$. Otherwise $g(0) = g(0)g(1) = 0$, and g is not injective. Therefore, we have $g(1) = 1$ and hence $g(0) = 0$.

Note that for any $x \geq 0$,

$$g(x) = g(\sqrt{x})g(\sqrt{x}) = [g(\sqrt{x})]^2 \geq 0.$$

On the other hand, for any $y \in [-1, 0)$, $g(y)^2 = g(y^2) = g(-y)^2$. Hence, $g(y) = -g(-y)$ as g is injective.

We consider the restriction map of g on $(0, 1]$. Note that g maps $(0, 1]$ onto $(0, 1]$. Extend g to $\hat{g} : (0, \infty) \to (0, \infty)$ by

$$\hat{g}(x) = \begin{cases} g(x) & \text{if } x \in (0, 1], \\ [g(x^{-1})]^{-1} & \text{if } x \in (1, \infty). \end{cases}$$

Then $\hat{g}$ is well-defined and multiplicative. Next, define $h : \mathbb{R} \to \mathbb{R}$ by $h(y) = \ln \hat{g}(e^y)$ for $y \in \mathbb{R}$. We see that h is an additive map on $\mathbb{R}$, so $h : \mathbb{R} \to \mathbb{R}$ is linear map over $\mathbb{Q}$. Since h maps $[0, \infty)$ to $[0, \infty)$, h is continuous, and therefore h is real linear. Thus, h has the form λy for some nonnegative λ. Then for $x \in (0, 1]$,

$$g(x) = \hat{g}(x) = e^{h(\ln x)} = e^{\lambda \ln x} = x^{\lambda}.$$

Since g is injective, λ must be positive. Finally, since $g(y) = -g(-y)$ for all $y \in [-1, 0)$, we get the conclusion. □

Proof of Theorem 6.1. Since ϕ is bijective, there exists $A \in \mathcal{S}$ such that $\phi(A) = I_n$. First, we have

$$\phi(I_n) = \phi(I_n)\phi(A) = \phi(I_n A) = \phi(A) = I_n,$$

i.e., $\phi(I_n) = I_n$. Now for any $P \in \mathbf{P}_n$, $\phi(P)\phi(P^t) = \phi(I_n) = I_n$, then $\phi(P)$ is invertible, and its inverse, $\phi(P^t)$, is in $\mathcal{S}$. By Lemma 6.2, $\phi(P) \in \mathbf{P}_n$. Therefore, $\phi(P_n) \subseteq P_n$. Since ϕ is injective, we have $\phi(P_n) = P_n$. Recall that $J(\alpha) = \alpha I_n + (1 - \alpha) J_n$. Let

$$\Omega = [-1/(n-1), 1] \quad \text{and} \quad \Gamma = \{J(\alpha) : \alpha \in \Omega\}.$$

Then $\Gamma \subseteq \mathcal{S}$. Because $X \in \mathcal{S}$ lies in Γ if and only if X commutes with all matrices in $\mathbf{P}_n$, we see that $\phi(\Gamma) = \Gamma$. Note that I_n and J_n are the only idempotents in Γ. Since $\phi(I_n) = I_n$, we have $\phi(J_n) = J_n$. Furthermore, J_n is the only singular matrix in Γ. Therefore, $\phi(X)$ is invertible for all invertible $X \in \Gamma$.

Assume that $\mathcal{S} = \mathbf{DS}_n$ or $\mathbf{RS}_n$. The proof for $\mathcal{S} = \mathbf{CS}_n$ is similar.

Let $\mathcal{G} = \mathbf{GDS}_n$ or $\mathbf{GRS}_n$ according to $\mathcal{S} = \mathbf{DS}_n$ or $\mathbf{RS}_n$. By Lemma 5.3, we know that for any $X \in \mathcal{G}$, there exists $J(\alpha) \in \Gamma$ such that $XJ(\alpha) \in \mathcal{S}$. We extend ϕ to an automorphism ϕ_1 on $\mathcal{G}$ by

$$\phi_1(X) = \phi(XJ(\alpha))\phi(J(\alpha))^{-1}.$$

We first show that ϕ_1 is well-defined. For any $X \in \mathcal{G}$ and nonzero $\alpha, \beta \in \Omega$. Suppose both $XJ(\alpha)$ and $XJ(\beta)$ are in $\mathcal{S}$. Then $XJ(\alpha)J(\beta) = XJ(\alpha\beta) \in \mathcal{S}$ and

$$\phi(XJ(\beta))\phi(J(\alpha)) = \phi(XJ(\alpha\beta)) = \phi(XJ(\alpha))\phi(J(\beta)).$$

Also, we see that $\phi(J(\alpha))^{-1}$ and $\phi(J(\beta))^{-1}$ exist and commute with each other. Hence, ϕ_1 is well-defined. Evidently, $\phi_1|_{\mathcal{S}} = \phi$.

Next we show that ϕ_1 is multiplicative. It is trivial if $\mathcal{G} = \mathbf{GDS}_n$ as $J(\alpha)$ commutes with all matrices in $\mathbf{GDS}_n$ and ϕ is multiplicative. Suppose $\mathcal{G} = \mathbf{GRS}_n$. For any $X, Y \in \mathbf{GRS}_n$, by Lemma 5.3, there exist nonzero $\alpha, \beta, \gamma \in \Omega$ such that $XJ(\alpha), YJ(\beta), XYJ(\gamma) \in \mathbf{RS}_n$. In fact, we may assume that $\alpha = \beta = \gamma$ by replacing all α, β, γ by $\min\{\alpha, \beta, \gamma\}$. By Proposition 5.2, we may write $X = \begin{pmatrix} 1 & x^t \\ 0 & X_1 \end{pmatrix}_F$ and $Y = \begin{pmatrix} 1 & y^t \\ 0 & Y_1 \end{pmatrix}_F$ respectively. Let $W = \begin{pmatrix} 1 & \gamma y^t \\ 0 & Y_1 \end{pmatrix}_F \in \mathbf{GRS}_n$. By Lemma 5.3, there exists a nonzero $\omega \in \Omega$ such that

$$\begin{pmatrix} 1 & \omega\gamma y^t \\ 0 & \omega Y_1 \end{pmatrix}_F = \begin{pmatrix} 1 & \gamma y^t \\ 0 & Y_1 \end{pmatrix}_F \begin{pmatrix} 1 & 0 \\ 0 & \omega I_{n-1} \end{pmatrix}_F = WJ(\omega) \in \mathbf{RS}_n.$$

Now, we have

$$\phi(YJ(\gamma))\ \phi(J(\omega)) = \phi(J(\gamma))\ \phi(WJ(\omega)) \tag{3}$$

and

$$\phi(XJ(\gamma))\ \phi(WJ(\omega)) = \phi(XYJ(\gamma))\ \phi(J(\omega)), \tag{4}$$

as

$$\phi\left(\begin{pmatrix} 1 & \gamma y^t \\ 0 & \gamma Y_1 \end{pmatrix}_F\right) \phi\left(\begin{pmatrix} 1 & 0 \\ 0 & \omega I_{n-1} \end{pmatrix}_F\right) = \phi\left(\begin{pmatrix} 1 & 0 \\ 0 & \gamma I_{n-1} \end{pmatrix}_F\right) \phi\left(\begin{pmatrix} 1 & \omega\gamma y^t \\ 0 & \omega Y_1 \end{pmatrix}_F\right)$$

and

$$\phi\left(\begin{pmatrix} 1 & \gamma x^t \\ 0 & \gamma X_1 \end{pmatrix}_F\right) \phi\left(\begin{pmatrix} 1 & \omega\gamma y^t \\ 0 & \omega Y_1 \end{pmatrix}_F\right) = \phi\left(\begin{pmatrix} 1 & \gamma(y^t + x^t Y_1) \\ 0 & \gamma X_1 Y_1 \end{pmatrix}_F\right) \phi\left(\begin{pmatrix} 1 & 0 \\ 0 & \omega I_{n-1} \end{pmatrix}_F\right).$$

As a result,

$$\begin{aligned}
\phi_1(X)\ \phi_1(Y)\ \phi(J(\omega)) &= \phi(XJ(\gamma))\ \phi(J(\gamma))^{-1}\ \phi(YJ(\gamma))\ \phi(J(\gamma))^{-1}\ \phi(J(\omega)) \\
&= \phi(XJ(\gamma))\ \phi(J(\gamma))^{-1}\ \phi(YJ(\gamma))\ \phi(J(\omega))\ \phi(J(\gamma))^{-1} \\
&= \phi(XJ(\gamma))\ \phi(WJ(\omega))\ \phi(J(\gamma))^{-1} \qquad (\text{ by } (3)\ ) \\
&= \phi(XYJ(\gamma))\ \phi(J(\omega))\ \phi(J(\gamma))^{-1} \qquad (\text{ by } (4)\ ) \\
&= \phi(XYJ(\gamma))\ \phi(J(\gamma))^{-1}\ \phi(J(\omega)) \\
&= \phi_1(XY)\ \phi(J(\omega)).
\end{aligned}$$

Since $\phi(J(w))$ is invertible, we conclude that ϕ_1 is multiplicative.

Because ϕ is injective, we see that ϕ_1 is injective. For any $Y \in \mathcal{G}$, by Lemma 5.3, there exists $J(\beta) \in \Gamma$ such that $YJ(\beta) \in \mathcal{S}$. Since $\phi(\Gamma) = \Gamma$ and $\phi(\mathcal{S}) = \mathcal{S}$, there exist $J(\alpha) \in \Gamma$ and $A \in \mathcal{S}$ such that $\phi(J(\alpha)) = J(\beta)$ and $\phi(A) = YJ(\beta)$. Let $X = AJ(\alpha)^{-1}$. Then $X \in \mathcal{G}$, $XJ(\alpha) = A$ and

$$\phi_1(X) = \phi(XJ(\alpha))\phi(J(\alpha))^{-1} = YJ(\beta)J(\beta)^{-1} = Y.$$

Hence, ϕ_1 is surjective. Then ϕ_1 is an automorphism on $\mathcal{G}$. By Theorem 5.1, either there exists $T \in \mathcal{G}$ such that $\phi_1(A) = TAT^{-1}$ for all $A \in \mathcal{G}$, or when $\mathcal{G} = \mathbf{GDS}_2$, there exists a bijective multiplicative map σ on $\mathbb{R}$ such that $\phi_1(A) = \sigma(\det A)(I_2 - J_2) + J_2$ for all $A \in \mathbf{GDS}_2$.

When $\mathcal{S} = \mathbf{DS}_2$ and $A \in \mathbf{GDS}_2$, we have $A \in \mathbf{DS}_2$ if and only if $\det A \in [-1, 1]$. Since $\phi(\mathbf{DS}_2) = \mathbf{DS}_2$, $\sigma([-1, 1]) = [-1, 1]$. Then by Lemma 6.3, there exists a positive λ such that $\sigma(x) = x|x|^{\lambda-1}$ for all $x \in [-1, 1]$. Hence,

$$\phi(A) = \phi_1(A) = (\det A)|\det A|^{\lambda-1}[I_2 - J_2] + J_2 \quad \text{for all} \quad A \in \mathbf{DS}_2.$$

For the remaining cases, we have

$$\phi(A) = \phi_1(A) = TAT^{-1} \quad \text{for all} \quad A \in \mathcal{S}.$$

Since $\phi(\mathbf{P}_n) = \mathbf{P}_n$, by Theorem 2.1 there exists $P \in \mathbf{P}_n$ such that $\phi(A) = PAP^t$ for all $A \in \mathbf{P}_n$. Here note that the exceptional case when $n = 6$ cannot happen because A and $\phi(A)(= TAT^{-1})$ always have the same set of eigenvalues.

It remains to show that one may choose T from $\mathbf{P}_n$. We replace ϕ and T by $A \mapsto P^t\phi(A)P$ and P^tT respectively, so that $\phi(A) = TAT^{-1}$ for all $A \in \mathcal{S}$ and $\phi(Q) = Q$ for all $Q \in \mathbf{P}_n$. Then we have $TQ = QT$ for all $Q \in \mathbf{P}_n$. Thus, T has the form $J(\alpha) = \alpha(I_n - J_n) + J_n$ for some $\alpha \in \mathbb{R}$.

When $\mathcal{S} = \mathbf{DS}_n$, as T commutes with all matrices in $\mathbf{DS}_n$, $TAT^{-1} = ATT^{-1} = A$. Hence, $\phi(A) = A$ for all $A \in \mathbf{DS}_n$.

Suppose $\mathcal{S} = \mathbf{RS}_n$. Note that ϕ and ϕ^{-1} have the form

$$A \mapsto J(\alpha)AJ(\alpha)^{-1} \quad \text{and} \quad A \mapsto J(\alpha)^{-1}AJ(\alpha),$$

and both are semigroup isomorphisms of $\mathbf{RS}_n$. If $J(\alpha) \neq I_n$, then $J(\alpha)$ or $J(\alpha)^{-1}$ has negative entries. Thus, for $A = \mathbf{1}\,(1, 0, \ldots, 0) \in \mathbf{RS}_n$, the matrix

$$J(\alpha)AJ(\alpha)^{-1} = \mathbf{1}\,(1, 0, \ldots, 0)J(\alpha)^{-1} \quad \text{or} \quad J(\alpha)^{-1}AJ(\alpha) = \mathbf{1}\,(1, 0, \ldots, 0)J(\alpha)$$

has negative entries, which is a contradiction. □

7 Positive Matrices

A matrix A is a positive (nonnegative) matrix if all entries of A are positive (nonnegative). Let $\mathbf{PM}_n$ and $\mathbf{NM}_n$ be the set of all $n \times n$ positive matrices and nonnegative matrices respectively. Automorphisms of $\mathbf{NM}_n$ are already determined in [1]. If one removes the positivity (nonnegativity) assumption on the entries of $\mathbf{PM}_n$ ($\mathbf{NM}_n$), one gets the semigroup of $n \times n$ real matrices $M_n(\mathbb{R})$. Using the result in [1] or [7], one easily shows that an automorphism $\phi : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ has the form

$$(a_{ij}) \mapsto S^{-1}(\sigma(a_{ij}))S$$

for some invertible matrix $S \in M_n(\mathbb{R})$ and field automorphism σ on $\mathbb{R}$. In the following, we characterize all the automorphisms of $\mathbf{PM}_n$.

Theorem 7.1 *A map $\phi : \mathbf{PM}_n \to \mathbf{PM}_n$ is a semigroup automorphism if and only if there exists $T \in \mathbf{NMN}_n$ such that*

$$\phi(A) = TAT^{-1} \quad \textit{for all} \quad A \in \mathbf{PM}_n.$$

We need some auxiliary lemmas to prove the main theorem of this section. The first lemma is a well-known result about positive matrices (e.g., see [4]).

Lemma 7.2 *Any positive matrix can be written as R_1AR_2 for some doubly stochastic matrix A and nonnegative invertible diagonal matrices R_1, R_2.*

By a similar proof as in Lemma 6.2, we have the following lemma about $\mathbf{NM}_n$.

Lemma 7.3 *A matrix $X \in \mathbf{NM}_n$ has its inverse in $\mathbf{NM}_n$ if and only if $X \in \mathbf{NMN}_n$.*

Lemma 7.4 *For any positive matrix A, $A = B_1C_1 = B_2C_2$ for some positive matrices B_1, B_2, C_1, C_2 such that B_1 and C_2 are invertible.*

Proof. For any positive matrix A, we check that there exists a sufficient small $0 < \epsilon < 1$ such that $[I_n - \epsilon J_n]A$ is still positive. Let $B_1 = I_n + [\epsilon(1-\epsilon)^{-1}]J_n$ and $C_1 = [I_n - \epsilon J_n]A$. Then both B_1 and C_1 are positive and B_1 is invertible. Furthermore, $A = B_1C_1$. Similarly, we show that $A = B_2C_2$ for some positive matrices B_2 and C_2 with C_2 invertible. $\square$

Now we are ready to prove Theorem 7.1. Again we prove it by establishing a sequence of assertions. Let

$$\mathcal{I} = \{X \in \mathbf{PM}_n : \phi(X) \text{ is invertible in } M_n(\mathbb{R})\}.$$

We have the following.

Assertion 1 *The set $\mathcal{I}$ contains some invertible elements in $M_n(\mathbb{R})$.*

Proof. Take a matrix $X \in \mathcal{I}$. By Lemma 7.4, $X = BC$ for some $B, C \in \mathbf{PM}_n$ with B invertible. Since $\phi(B)\phi(C) = \phi(X)$ is invertible, both $\phi(B)$ and $\phi(C)$ are invertible. Hence $\mathcal{I}$ contains some invertible element B. $\square$

The next assertion concerns the set

$$\mathcal{H} = \{H \in M_n(\mathbb{R}) : HA, AH \in \mathbf{PM}_n \text{ for all } A \in \mathbf{PM}_n\}.$$

Assertion 2 *The set $\mathcal{H}$ is a semigroup satisfying $\mathbf{PM}_n \subseteq \mathcal{H} \subseteq \mathbf{NM}_n$. Furthermore, we can extend ϕ to an automorphism $\phi_1 : \mathcal{H} \to \mathcal{H}$ by*

$$\phi_1(H) = \phi(X)^{-1}\phi(XH) = \phi(HX)\phi(X)^{-1}$$

for some $X \in \mathcal{I}$, i.e., ϕ is well-defined and $\phi_1|_{\mathbf{PM}_n} = \phi$.

Proof. Clearly, $\mathcal{H}$ is a semigroup containing $\mathbf{PM}_n$. Suppose $H \in M_n(\mathbb{R})$ has some negative entries, say the (i,j)-th entries of H is negative. Take a matrix $A = \alpha I_n + J_n$ with a large positive α. Then the (i,j)-th entry of AH is still negative. Thus, $\mathcal{H}$ only contains nonnegative matrices. In fact, we have

$$\mathcal{H} = \{H \in \mathbf{NM}_n : H \text{ has no zero rows or zero columns}\}.$$

Now we show that ϕ_1 is well-defined. For any $X_1, X_2 \in \mathcal{I}$, since

$$\phi(X_1)\phi(HX_2) = \phi(X_1HX_2) = \phi(X_1H)\phi(X_2),$$

we have

$$\phi(HX_2)\phi(X_2)^{-1} = \phi(X_1)^{-1}\phi(X_1H).$$

Thus, $\phi(X)^{-1}\phi(XH) = \phi(HX)\phi(X)^{-1}$ and ϕ_1 is independent of the choice of X. Furthermore, for any $A \in \mathbf{PM}_n$, there is $B \in \mathbf{PM}_n$ such that $\phi(B) = A$. Clearly, $HB \in \mathbf{PM}_n$ and so $\phi(HB) \in \mathbf{PM}_n$. Then

$$\phi_1(H)A = \phi(X)^{-1}\phi(XH)\phi(B) = \phi(X)^{-1}\phi(X)\phi(HB) = \phi(HB) \in \mathbf{PM}_n.$$

By a similar argument, we show that $A\phi_1(H) \in \mathbf{PM}_n$. Thus, $\phi_1(H) \in \mathcal{H}$. Therefore, ϕ_1 is well-defined.

For any $H_1, H_2 \in \mathcal{H}$ and $X \in \mathcal{I}$,

$$\phi(XH_1H_2)\phi(X) = \phi(XH_1H_2X) = \phi(XH_1)\phi(H_2X).$$

So

$$\phi_1(H_1H_2) = \phi(X)^{-1}\phi(XH_1H_2) = \phi(X)^{-1}\phi(XH_1)\phi(H_2X)\phi(X)^{-1} = \phi_1(H_1)\phi_1(H_2).$$

Hence, ϕ_1 is multiplicative. It remains to show that ϕ_1 is bijective. Applying the similar argument on ϕ^{-1}, we extend ϕ^{-1} to a multiplicative map $\psi : \mathcal{H} \to \mathcal{H}$ by

$$\psi(K) = [\phi^{-1}(Y)]^{-1}\phi^{-1}(YK) = \phi(KY)[\phi^{-1}(Y)]^{-1}$$

for some $Y \in \mathcal{I}^{-1} = \{Y : \phi^{-1}(Y) \text{ is invertible}\}$. Take some invertible $X \in \mathcal{I}$. Then $Y = \phi(X) \in \mathcal{I}^{-1}$. For any $H \in \mathcal{H}$,

$$\begin{aligned}\psi \circ \phi_1(H) &= \psi(\, \phi(X)^{-1}\phi(XH)\,) = [\phi^{-1}(Y)]^{-1}\, \phi^{-1}(\, Y\phi(X)^{-1}\phi(XH)\,)\\ &= X^{-1}\phi^{-1}(\phi(XH)) = X^{-1}XH = H.\end{aligned}$$

Thus, $\psi = \phi_1^{-1}$ and hence ϕ is bijective. □

The next three assertions concern the properties of $\phi_1 : \mathcal{H} \to \mathcal{H}$ defined in Assertion 2.

Assertion 3 *There exist $T \in \mathbf{NMN}_n$, a multiplicative map ρ on $(0, \infty)$ and a bijective multiplicative map σ on $[0, \infty)$ such that the map $\alpha \mapsto \rho(|\alpha|)^n\sigma(\alpha)$ is bijective on $(0, \infty)$ and*

$$\phi_1(N) = \rho(|\det N|)T(\sigma(n_{ij}))T^{-1} \quad \textit{for all} \quad N = (n_{ij}) \in \mathbf{NMN}_n.$$

Furthermore, replacing ϕ by $A \mapsto T^{-1}\phi(A)T$, we may assume that $T = I_n$.

Proof. Clearly, $\phi_1(I_n) = I_n$ and $\mathbf{NMN}_n \subseteq \mathcal{H}$. By Lemma 7.3, we have $\phi_1(\mathbf{NMN}_n) = \mathbf{NMN}_n$. Then by Theorem 4.1, ϕ_1 has the asserted form. □

Assertion 4 *For any $S \in \mathbf{DS}_n$, $\phi_1(S) = S$.*

Proof. By Assertion 3, we have $\phi_1(P) = P$ for all $P \in \mathbf{P}_n$. As $\phi_1(\mathbf{PM}_n) = \mathbf{PM}_n$ and J_n is the only idempotent in $\mathbf{PM}_n$ which commutes with all elements in $\mathbf{P}_n$, we have $\phi_1(J_n) = J_n$. Note that for any $A \in \mathcal{H} \subseteq \mathbf{NM}_n$, $A \in \mathbf{DS}_n$ if and only if $AJ_n = J_n = J_nA$. Then $\phi_1(\mathbf{DS}_n) = \mathbf{DS}_n$.

When $n > 2$, by Theorem 6.1, there exists $Q \in \mathbf{P}_n$ such that

$$\phi_1(S) = QSQ^t \quad \text{for all} \quad S \in \mathbf{DS}_n.$$

As $\phi_1(P) = P$ for all $P \in \mathbf{P}_n$, $Q = I_n$. Hence, $\phi_1(S) = S$ for all $S \in \mathbf{DS}_n$.

When $n = 2$, we deduce that $\phi_1(\mathbf{RS}_2 \cap \mathcal{H}) = \mathbf{RS}_2 \cap \mathcal{H}$. Note that all 2×2 row stochastic matrices are in $\mathcal{H}$ except $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$. Using the same proof of Theorem 6.1, we show that there is $Q \in \mathbf{P}_2$ such that

$$\phi_1(S) = QSQ^t \quad \text{for all} \quad S \in \mathbf{RS}_2 \cap \mathcal{H}.$$

As $QSQ^t = S$ for $S \in \mathbf{DS}_2$, the result follows. □

Assertion 5 *For any $D \in \mathbf{ND}_n$,*

$$\frac{m(\phi_1(D))}{m(D)} = \frac{\mathrm{tr}\,(\phi_1(D))}{\mathrm{tr}\,D},$$

where $m(D)$ is the smallest diagonal entries of D.

Proof. By Assertion 3, $\phi_1(D)$ is a scalar matrix if D is. Therefore, the assertion holds if D is a scalar matrix.

Now for any $\alpha \geq 0$ and nonscalar diagonal matrix $D = \mathrm{diag}\,(d_1, \ldots, d_n) \in \mathbf{ND}_n$, we claim that $J(\alpha)DJ_n$ is positive if and only if

$$\alpha < (\mathrm{tr}\,D)/(\mathrm{tr}\,D - n\; m(D)).$$

Note that $\mathrm{tr}\,D - n\; m(D) > 0$ as D is not a scalar matrix. Since

$$J(\alpha)DJ_n \quad = \quad (\alpha D + (1-\alpha)(\mathrm{tr}\,D/n)I_n)J_n,$$

$J(\alpha)DJ \in \mathbf{PM}_n$ if and only if $\alpha D + (1-\alpha)(\mathrm{tr}\,D/n)I_n$ has positive diagonal entries. This is equivalent to saying $\alpha d_i + (1-\alpha)(\mathrm{tr}\,D/n) > 0$ for all i, or simply, $\alpha\; m(D) + (1-\alpha)(\mathrm{tr}\,D/n) > 0$, i.e.,

$$\alpha < (\mathrm{tr}\,D)/(\mathrm{tr}\,D - n\; m(D)).$$

Next, we show that if $\alpha \geq 0$ and $J(\alpha)DJ_n$ is positive, then $\phi_1(J(\alpha)DJ_n) = J(\alpha)\phi_1(D)J_n$.

If $\alpha \leq 1$, then $J(\alpha) \in \mathbf{DS}_n$ and

$$\phi_1(J(\alpha)DJ_n) = \phi_1(J(\alpha))\phi_1(D)\phi_1(J_n) = J(\alpha)\phi_1(D)J_n.$$

If $\alpha > 1$, then $J(\alpha)^{-1} \in \mathbf{DS}_n$ and $\phi_1(J(\alpha)^{-1})\phi_1(J(\alpha)DJ_n) = \phi_1(DJ_n)$. By Assertion 4, $\phi_1(J(\alpha)^{-1}) = J(\alpha)^{-1}$. Thus,

$$\phi_1(J(\alpha)DJ_n) = J(\alpha)\phi_1(DJ_n) = J(\alpha)\phi_1(D)J_n.$$

Combining the above arguments, we see that $\phi_1(J(\alpha)DJ_n) = J(\alpha)\phi_1(D)J_n \in \mathbf{PM}_n$ whenever $\alpha \in [0, (\operatorname{tr} D)/(\operatorname{tr} D - n\ m(D)))$. Thus,

$$\frac{\operatorname{tr} D}{\operatorname{tr} D - n\ m(D)} \leq \frac{\operatorname{tr}(\phi_1(D))}{\operatorname{tr}(\phi_1(D)) - n\ m(\phi_1(D))}.$$

Applying the argument to ϕ_1^{-1}, we conclude that

$$\frac{\operatorname{tr} D}{\operatorname{tr} D - n\ m(D)} = \frac{\operatorname{tr}(\phi_1(D))}{\operatorname{tr}(\phi_1(D)) - n\ m(\phi_1(D))}.$$

Thus, we have the assertion. □

Assertion 6 *Suppose σ and ρ are the mappings in Assertion 3. Then either*

(i) *$\sigma(\alpha) = \alpha$ and $\rho(\alpha) = 1$ for all $\alpha > 0$, or*

(ii) *$n = 2$ with $\sigma(\alpha) = \alpha^{-1}$ and $\rho(\alpha) = \alpha$ for all $\alpha > 0$.*

Proof. Because $\sigma(\alpha^{-1}) = \sigma(\alpha)^{-1}$ and $\rho(\alpha^{-1}) = \rho(\alpha)^{-1}$ for all $\alpha > 0$, it is sufficient to prove the assertion for $0 < \alpha < 1$. For any $\alpha \in (0, 1)$ and $R_1 = \operatorname{diag}(\alpha, 1, \ldots, 1)$, we have $\phi_1(R_1) = \rho(\alpha)\operatorname{diag}(\sigma(\alpha), 1, \ldots, 1)$. By Assertion 5,

$$\frac{\min\{\rho(\alpha)\sigma(\alpha), \rho(\alpha)\}}{\alpha} = \frac{\rho(\alpha)[\sigma(\alpha) + (n-1)]}{\alpha + (n-1)},$$

i.e.,

$$\frac{\min\{\sigma(\alpha), 1\}}{\alpha} = \frac{\sigma(\alpha) + (n-1)}{\alpha + (n-1)}.$$

If $n = 2$, then either $\sigma(\alpha) = \alpha$ or $\sigma(\alpha) = \alpha^{-1}$. Since σ is a multiplicative map, either $\sigma(\beta) = \beta$ for all β, or $\sigma(\beta) = \beta^{-1}$ for all β.

When $n > 2$, we consider another matrix $R_2 = \operatorname{diag}(1, \alpha, \ldots, \alpha)$. By Assertion 5, we have

$$\frac{\min\{\sigma(\alpha), 1\}}{\alpha} = \frac{1 + (n-1)\sigma(\alpha)}{1 + (n-1)\alpha}.$$

Combining the two equations above, we see that $\sigma(\alpha) = \alpha$.

Now for any $0 < \alpha < 1$, take $\gamma = 1 + \sqrt{1-\alpha}$ and $R_3 = \operatorname{diag}(\gamma, 2-\gamma, 1, \ldots, 1)$. We check that $J_n R_3 J_n = J_n$. If (i) $\sigma(\alpha) = \alpha$ for all α, then

$$\phi_1(R_3) = \rho(\gamma(2-\gamma))\operatorname{diag}(\gamma, 2-\gamma, 1\ldots, 1) = \rho(\alpha)R_3.$$

It follows that

$$J_n = \phi_1(J_n) = \phi_1(J_n)\phi_1(R_3)\phi_1(J_n) = J_n\rho(\alpha)R_3J_n = \rho(\alpha)J_n.$$

Thus, $\rho(\alpha) = 1$.

If (ii) $n = 2$ with $\sigma(\alpha) = \alpha^{-1}$ for all α, then $\phi_1(R_3) = \rho(\alpha)\mathrm{diag}\,(\gamma^{-1}, (2-\gamma)^{-1})$. Thus,

$$J_n = \phi_1(J_n)\phi_1(R_3)\phi_1(J_n) = J_n\rho(\alpha)\mathrm{diag}\,(\gamma^{-1}, (2-\gamma)^{-1})J_n = [\rho(\alpha)\alpha^{-1}]J_n.$$

So $\rho(\alpha) = \alpha$. □

Assertion 7 *The conclusion of Theorem 7.1 holds.*

Proof. If Assertion 6(i) holds, then we have $\phi_1(N) = N$ for all $N \in \mathbf{NMN}_n$. By Lemma 7.2, every A in $\mathbf{PM}_n$ can be written as $A = R_1SR_2$ for some $R_1, R_2 \in \mathbf{ND}_n \subseteq \mathbf{NMN}_n$ and $S \in \mathbf{DS}_n$. Then by Assertion 3, 4 and 6,

$$\phi(A) = \phi_1(A) = \phi_1(R_1)\phi_1(S)\phi_1(R_2) = R_1SR_2 = A \quad \text{for all} \quad A \in \mathbf{PM}_n.$$

If Assertion 6(ii) holds, then $n = 2$ and for any $D = \mathrm{diag}\,(d_1, d_2) \in \mathbf{ND}_2$,

$$\phi_1(D) = (d_1d_2)\mathrm{diag}\,(d_1^{-1}, d_2^{-1}) = \mathrm{diag}\,(d_2, d_1) = QDQ^t,$$

where $Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Since $\phi_1(R) = R = QRQ^t$ for all $R \in \mathbf{DS}_2$, we see that $\phi(A) = QAQ^t$ for all $A \in \mathbf{PM}_n$. The result follows. □

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Appendix

Here is the Matlab program we used to check the special automorphism $\Phi : \mathbf{P}_6 \to \mathbf{P}_6$ described in Section 2. It took about 4 hours for a Pentium 4 1.99 GHz machine with 256 MB of RAM to confirm the result.

```
function stop = S6isom()
% S6isom will generate a 6x4320 matrix storing all the 720 permutation
% matrices in P_6 side by side,
% and another 6x4320 matrix storing all the 720 matrices F(A)
% side by side.
% It will then check that F(A(:,i)'*A(:,j))=FA(:,i)'*FA(:,j).
% Finally, it will check that F is one-one.
e1 =  [1 0 0 0 0 0]; e2 =  [0 1 0 0 0 0]; e3 =  [0 0 1 0 0 0];
e4 =  [0 0 0 1 0 0]; e5 =  [0 0 0 0 1 0]; e6 =  [0 0 0 0 0 1];
% Generate all useful transpositions and their images
X12 = [e2; e1; e3; e4; e5; e6]; Y12 = [e2; e1; e4; e3; e6; e5];
X13 = [e3; e2; e1; e4; e5; e6]; Y13 = [e3; e5; e1; e6; e2; e4];
X14 = [e4; e2; e3; e1; e5; e6]; Y14 = [e4; e6; e5; e1; e3; e2];
X15 = [e5; e2; e3; e4; e1; e6]; Y15 = [e5; e4; e6; e2; e1; e3];
X16 = [e6; e2; e3; e4; e5; e1]; Y16 = [e6; e3; e2; e5; e4; e1];
X23 = X12*X13*X12; Y23 = Y12*Y13*Y12;
X24 = X12*X14*X12; Y24 = Y12*Y14*Y12;
X25 = X12*X15*X12; Y25 = Y12*Y15*Y12;
X26 = X12*X16*X12; Y26 = Y12*Y16*Y12;
X34 = X13*X14*X13; Y34 = Y13*Y14*Y13;
X35 = X13*X15*X13; Y35 = Y13*Y15*Y13;
X36 = X13*X16*X13; Y36 = Y13*Y16*Y13;
X45 = X14*X15*X14; Y45 = Y14*Y15*Y14;
X46 = X14*X16*X14; Y46 = Y14*Y16*Y14;
X56 = X15*X16*X15; Y56 = Y15*Y16*Y15;
% building P_2 and F(P_2)
A(:,1:6) = eye(6); FA(:,1:6) = eye(6);
A(:,7:12) = X12;  FA(:,7:12) = Y12;
% building P_3 and F(P_3)
A(:,13:24) = X13*A(:,1:12); FA(:,13:24) = Y13*FA(:,1:12);
A(:,25:36) = X23*A(:,1:12); FA(:,25:36) = Y23*FA(:,1:12);
% building P_4 and F(P_4)
A(:,37:72) = X14*A(:,1:36);   FA(:,37:72) = Y14*FA(:,1:36);
A(:,73:108) = X24*A(:,1:36);  FA(:,73:108) = Y24*FA(:,1:36);
A(:,109:144) = X34*A(:,1:36); FA(:,109:144) = Y34*FA(:,1:36);
% building P_5 and F(P_5)
A(:,145:288) = X15*A(:,1:144);  FA(:,145:288) = Y15*FA(:,1:144);
A(:,289:432) = X25*A(:,1:144);  FA(:,289:432) = Y25*FA(:,1:144);
A(:,433:576) = X35*A(:,1:144);  FA(:,433:576) = Y35*FA(:,1:144);
A(:,577:720) = X45*A(:,1:144);  FA(:,577:720) = Y45*FA(:,1:144);
```

```
% building P_6 and F(P_6)
A(:,721:1440)  = X16*A(:,1:720);  FA(:,721:1440)  = Y16*FA(:,1:720);
A(:,1441:2160) = X26*A(:,1:720);  FA(:,1441:2160) = Y26*FA(:,1:720);
A(:,2161:2880) = X36*A(:,1:720);  FA(:,2161:2880) = Y36*FA(:,1:720);
A(:,2881:3600) = X46*A(:,1:720);  FA(:,2881:3600) = Y46*FA(:,1:720);
A(:,3601:4320) = X56*A(:,1:720);  FA(:,3601:4320) = Y56*FA(:,1:720);

% Check to see if our F is multiplicative, i.e, if A(i)*A(j)==A(m),
% then FA(i)*FA(j)==FA(m)

stop = 000;
for i=1:6:4315
    for j=1:6:4315
        for m=1:6:4315
            if stop == 111
                break
            else
                C=A(:,i:i+5)*A(:,j:j+5);
                D=A(:,m:m+5);
                if C(:)'*D(:) == 6
                    FC=FA(:,i:i+5)*FA(:,j:j+5);
                    FD=FA(:,m:m+5);
                    if FC(:)'*FD(:) ~= 6
                        stop = 111
                    end
                end
            end
        end
    end
end

% Here we check that F is one-one

for i=7:6:4315
            if stop == 222
                break
            else
                  if FA(:,i:i+5)'*eye(6) == 6
                        stop = 222
                    end
                end
            end
```