Every invertible matrix is diagonally equivalent to a matrix with distinct eigenvalues

Man-Duen Choi[*], Zejun Huang[†], Chi-Kwong Li[‡], and Nung-Sing Sze[§]

Abstract

We show that for every invertible $n \times n$ complex matrix A there is an $n \times n$ diagonal invertible D such that AD has distinct eigenvalues. Using this result, we affirm a conjecture of Feng, Li, and Huang that an $n \times n$ matrix is not diagonally equivalent to a matrix with distinct eigenvalues if and only if it is singular and all its principal minors of size $n-1$ are zero.

AMS Subject Classification. 15A18.

Keywords. Invertible matrices, diagonal matrices, distinct eigenvalues.

1 Introduction

Denote by M_n the set of $n \times n$ complex matrices. In [1], the authors pointed out that matrices with distinct eigenvalues have many nice properties. They then raised the question whether every invertible matrix in M_n is diagonally equivalent to a matrix with distinct eigenvalues, and conjectured that a matrix in M_n is not diagonally equivalent to a matrix with distinct eigenvalues if and only if it is singular and every principal minor of size $n-1$ is zero. They provided a proof for matrices in M_n with $n \leq 3$, and demonstrated the complexity of the problem for matrices in M_4 using their approach. In this note, we affirm their conjecture by proving the following theorem.

Theorem 1.1 *Suppose $A \in M_n$ is invertible. There is an invertible diagonal $D \in M_n$ such that AD has distinct eigenvalues.*

Once this result is proved, we have the following corollary.

[*]Department of Mathematics, University of Toronto, Toronto, Ontario, Canada M5S 2E4. (choi@math.toronto.edu)

[†]Department of Applied Mathematics, The Hong Kong Polytechnic University, Hung Hom, Kowloon, Hong Kong.(huangzejun@yahoo.cn)

[‡]Department of Mathematics, College of William & Mary, Williamsburg, Virginia 23187-8795, USA. This research was done while he was visiting The Hong Kong University of Science & Technology in 2011 under the support of a Fulbright Fellowship. Li is an honorary professor of The University of Hong Kong, Taiyuan University of Technology, and Shanghai University. (ckli@math.wm.edu)

[§]Department of Applied Mathematics, The Hong Kong Polytechnic University, Hung Hom, Kowloon, Hong Kong. (raymond.sze@inet.polyu.edu.hk)

Corollary 1.2 *Let $A \in M_n$. The following are equivalent.*

(a) *A is not diagonally equivalent to a matrix with distinct eigenvalues.*

(b) *There is no diagonal matrix D such that AD has distinct eigenvalues.*

(c) *The matrix A is singular and all principal minors of size $n-1$ are zero.*

Proof. The implication (a) $\Rightarrow$ (b) is clear. Suppose condition (c) does not hold. Then either A is invertible or A has an invertible principal submatrix of size $n-1$. Assume the former case holds. There is an invertible diagonal matrix D such that AD has distinct eigenvalues by Theorem 1.1. If the latter case holds, we may assume without loss of generality that the leading principal submatrix $A_1 \in M_{n-1}$ is invertible. By Theorem 1.1, there is an invertible diagonal matrix $D_1 \in M_{n-1}$ such that $A_1 D_1$ has distinct (nonzero) eigenvalues. Let $D = D_1 \oplus [0]$. Then AD has distinct eigenvalues including zero as an eigenvalue. Thus, (b) cannot hold. So, we have proved (a) $\Rightarrow$ (b) $\Rightarrow$ (c).

Recall that the characteristic polynomial of a matrix $B \in M_n$ has the form $\det(xI_n - B) = x^n + b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \cdots + b_1 x + b_0$, where $(-1)^j b_{n-j}$ is the sum of $j \times j$ principal minors of B. Suppose condition (c) holds. Since the principal minors of $D_1 A D_2$ are scalar multiples of the corresponding principal minors of A, then $D_1 A D_2$ has characteristic polynomial of the form $\det(xI_n - D_1 A D_2) = x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2$ so that 0 is a root with multiplicity at least two. Thus, $D_1 A D_2$ cannot have n distinct eigenvalues. So, the implication (c) $\Rightarrow$ (a) is proved. ∎

Note that the set of diagonal matrices is an n-dimensional subspace in M_n. We can extend Theorem 1.1 to the following.

Corollary 1.3 *Suppose $\mathcal{V}$ is a subspace of matrices in M_n.*

(a) *If there are invertible matrices R and S such that $R\mathcal{V}S = \{RXS : X \in \mathcal{V}\}$ contains the subspace of diagonal matrices, then for any invertible $A \in M_n$ there is $X \in \mathcal{V}$ such that AX has distinct eigenvalues.*

(b) *If there are invertible matrices R and S such that RXS has zero first row and zero last column for every $X \in \mathcal{V}$, then $A = SR$ is invertible and AX is similar to RXS which cannot have distinct eigenvalues for any $X \in \mathcal{V}$.*

Proof. (a) Suppose A is invertible. Then there is a diagonal matrix D such that $S^{-1}AR^{-1}D$ has distinct eigenvalues by Theorem 1.1. Set $X = R^{-1}DS^{-1} \in \mathcal{V}$. Notice that AX has distinct eigenvalues as $S^{-1}(AX)S = S^{-1}(AR^{-1}DS^{-1})S = (S^{-1}AR^{-1})D$.

Assertion (b) can be verified readily. ∎

2 Proof of Theorem 1.1

We will prove Theorem 1.1 by induction on n. The result is clear if $A \in M_1$. Assume that the result holds for all $k \times k$ invertible matrices with $1 \leq k < n$. Suppose $A \in M_n$ is invertible. We consider two cases.

Case 1. If all $k \times k$ principal minors of A are singular for $k = 1, \ldots, n-1$, then the characteristic polynomial of A has the form $x^n - a_0$ and has n distinct roots. So, the result holds with $D = I_n$.

Case 2. Suppose A has an invertible $k \times k$ principal minor. Without loss of generality, we may assume that $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ such that $A_{11} \in M_k$ is invertible for some $1 \le k < n$. Then the Schur complement of A_{22} equals $B = A_{22} - A_{21}A_{11}^{-1}A_{12}$ which is invertible; see [2, pp. 21-22]. By induction assumption, there are diagonal invertible $D_1 \in M_k$ and $D_2 \in M_{n-k}$ such that each of $A_{11}D_1$ and BD_2 has distinct nonzero eigenvalues, say, $\lambda_1, \ldots, \lambda_k$ and $\lambda_{k+1}, \ldots, \lambda_n$, respectively. Thus, $A_{11}D_1$ and BD_2 are diagonalizable and there are invertible $S_1 \in M_k$ and $S_2 \in M_{n-k}$ such that $S_1 A_{11} D_1 S_1^{-1} = \Lambda_1 = \operatorname{diag}(\lambda_1, \ldots, \lambda_k)$ and $S_2 B D_2 S_2^{-1} = \Lambda_2 = \operatorname{diag}(\lambda_{k+1}, \ldots, \lambda_n)$. Let $D_{r,s} = rD_1 \oplus sD_2$. The proof is complete if one can find some subitable r and s so that $AD_{r,s}$ has distinct eigenvalues. Notice that $AD_{r,s}$ has the same eigenvalues as

$$\begin{aligned} \tilde{A} &= \begin{bmatrix} S_1 & 0 \\ 0 & sS_2 \end{bmatrix} \begin{bmatrix} I_k & 0 \\ -A_{21}A_{11}^{-1} & I_{n-k} \end{bmatrix} AD_{r,s} \begin{bmatrix} I_k & 0 \\ A_{21}A_{11}^{-1} & I_{n-k} \end{bmatrix} \begin{bmatrix} S_1^{-1} & 0 \\ 0 & s^{-1}S_2^{-1} \end{bmatrix} \\ &= \begin{bmatrix} r\Lambda_1 + sS_1A_{12}D_2A_{21}A_{11}^{-1}S_1^{-1} & S_1A_{12}D_2S_2^{-1} \\ s^2 S_2 B D_2 A_{21} A_{11}^{-1} S_1^{-1} & s\Lambda_2 \end{bmatrix}. \end{aligned}$$

Denote by $D(a, d)$ the closed disk in $\mathbb{C}$ centered at a with radius $d \ge 0$. Suppose the $k \times k$ matrix $S_1A_{12}D_2A_{21}A_{11}^{-1}S_1^{-1}$ has diagonal entries $\mu_1, \ldots, \mu_k$ and let

$$d_1 = k\|S_1A_{12}D_2A_{21}A_{11}^{-1}S_1^{-1}\|, \quad d_2 = (n-k)\|S_1A_{12}D_2S_2^{-1}\|, \quad \text{and} \quad d_3 = k\|S_2BD_2A_{21}A_{11}^{-1}S_1^{-1}\|,$$

where $\|\cdot\|$ is the operator norm. By Geršgorin disk result (see [2, pp.344-347]), the eigenvalues of $\tilde{A}$ must lie in the union of the n Geršgorin disks, which is a subset of the union of n disks

$$D(r\lambda_1 + s\mu_1, sd_1 + d_2), \ldots, D(r\lambda_k + s\mu_k, sd_1 + d_2),\ D(s\lambda_{k+1}, s^2 d_3), \ldots, D(s\lambda_n, s^2 d_3).$$

We can choose sufficiently large $r > 0$ and sufficiently small $s > 0$ so that these disks are disjoint, and hence $\tilde{A}$ has n disjoint Geršgorin disks. Then $\tilde{A}$ has distinct eigenvalues. ■

We thank Editor Zhan for sending us the two related references [3, 4]. In these papers, the author proved following. Suppose A is an $n \times n$ matrix and $a_1, \ldots, a_n$ are complex numbers. Then there is a diagonal matrix E such that $A + E$ has eigenvalues $a_1, \ldots, a_n$. Moreover, if all principal minors of A are nonzero, then there is a diagonal matrix D such that AD has eigenvalues $a_1, \ldots, a_n$.

Note that the assumption on the principal minors of A is important in the second assertion. Obviously, if $\det(A) = 0$, then one cannot find diagonal D such that AD has n nonzero eigenvalues. Even if we remove this obvious obstacle and assume that A is invertible, one may not be able to find diagonal D so that AD has prescribed eigenvalues. For example, if $\{E_{1,1}, E_{1,2}, \ldots, E_{n,n}\}$ is the standard basis for M_n and $A = E_{1,2} + \cdots + E_{n-1,n}$, then the eigenvalues of AD always have the form $z, zw, \ldots, zw^{n-1}$ for some $z \in \mathbb{C}$, where w is the primitive nth root of unity.

It is interesting to determine the condition on A so that for any complex numbers $a_1, \ldots, a_n$, one can find a diagonal D such that AD has $a_1, \ldots, a_n$ as eigenvalues.

Acknowledgment

Research of Choi was supported by a NSERC grant. Research of Huang was supported by a HK RGC grant. Research of Li was supported by a USA NSF grant, and a HK RGC grant. Research of Sze was supported by HK RGC grants.

References

[1] X.-L. Feng, Z. Li and T.-Z. Huang, Is every nonsingular matrix diagonally equivalent to a matrix with all distinct eigenvalues? Linear Algebra Appl. 436 (2012), 120-125.

[2] R.A. Horn and C.R. Johnson, Matrix Analysis, Cambridge University Press, Cambridge, 1985.

[3] S. Friedland, On inverse multiplicative eigenvalue problems for matrices, Linear Algebra Appl. 12 (1975), 127-137.

[4] S. Friedland, Inverse eigenvalue problems, Linear Algebra Appl. 17 (1977), 15-51.