Multiplicative maps preserving the higher rank numerical ranges and radii

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Dedicated to Professor Leiba Rodman on the occasion of his 60th birthday.

Abstract

Let $\mathbf{M}_n$ be the semigroup of $n \times n$ complex matrices under the usual multiplication, and let $\mathcal{S}$ be different subgroups or semigroups in $\mathbf{M}_n$ including the (special) unitary group, (special) general linear group, the semigroups of matrices with bounded ranks. Suppose $\Lambda_k(A)$ is the rank-k numerical range and $r_k(A)$ is the rank-k numerical radius of $A \in \mathbf{M}_n$. Multiplicative maps $\phi : \mathcal{S} \to \mathbf{M}_n$ satisfying $r_k(\phi(A)) = r_k(A)$ are characterized. From these results, one can deduce the structure of multiplicative preservers of $\Lambda_k(A)$.

Keywords Multiplicative preservers, higher rank numerical ranges.
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1 Introduction

Let $\mathbf{M}_n$ be the algebra of $n \times n$ complex matrices regarded as linear operators acting on the n-dimensional Hilbert space $\mathbb{C}^n$. In the context of quantum information theory, if the quantum states are represented as matrices in $\mathbf{M}_n$, then a *quantum channel* is a trace preserving completely positive linear map $L : \mathbf{M}_n \to \mathbf{M}_n$, that is, we have the following operator sum representation

$$L(A) = \sum_{j=1}^{r} E_j A E_j^*,$$

where $E_1, \ldots, E_r \in \mathbf{M}_n$ satisfy $\sum_{j=1}^r E_j^* E_j = I_n$; see [4, 5, 10, 11, 21]. The matrices $E_1, \ldots, E_r$ are known as *error operators* of the quantum channel L. A subspace V of $\mathbb{C}^n$ is a *quantum error correction code* for the channel L if there is another quantum channel $R : M_n \to M_n$ such that the composite map $R \circ L$ maps A to a multiple of A for any $A \in M_n$ satisfying $PAP = A$ where $P \in \mathbf{M}_n$ is the orthogonal projection with range space V. By the result in [10] (see also [21]), the channel R exists if and only if $PE_i^* E_j P = \gamma_{ij} P$ for all $i, j \in \{1, \ldots, r\}$. In this connection, for $1 \le k < n$ researchers define the *rank-k numerical range* of $A \in \mathbf{M}_n$ by

$$\Lambda_k(A) = \{\lambda \in \mathbb{C} : PAP = \lambda P \text{ for some rank } k\text{-orthogonal projection } P\},$$

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and the *joint rank-k numerical range* of $A_1, \ldots, A_m \in \mathbf{M}_n$ by $\Lambda_k(A_1, \ldots, A_m)$ to be the collection of complex vectors $(a_1, \ldots, a_m) \in \mathbb{C}^{1\times m}$ such that $PA_jP = a_jP$ for a rank-k orthogonal projection $P \in \mathbf{M}_n$. Evidently, there is a quantum error correction code V of dimension k for the quantum channel L described above if and only if $\Lambda_k(A_1, \ldots, A_m)$ is non-empty for $(A_1, \ldots, A_m) = (E_1^*E_1, E_1^*E_2, \ldots, E_r^*E_r)$. It is easy to check that $(a_1, \ldots, a_m) \in \Lambda_k(A_1, \ldots, A_m)$ if and only if any one of the following conditions holds.

- There is a unitary $U \in \mathbf{M}_n$ such that the leading $k \times k$ principal submatrix of U^*A_jU is a_jI_k for $j = 1, \ldots, m$.
- There is an $n \times k$ matrix X such that $X^*X = I_k$ and $X^*A_jX = a_jI_k$ for $j = 1, \ldots, m$.

It is also clear that if $(a_1, \ldots, a_m) \in \Lambda_k(A_1, \ldots, A_m)$ then $a_j \in \Lambda_k(A_j)$ for $j = 1, \ldots, m$.

Even for a single matrix $A \in \mathbf{M}_n$, the study of $\Lambda_k(A)$ is highly non-trivial. Recently, interesting results have been obtained for the rank-k numerical range and the joint rank-k numerical range; see [2, 3, 4, 5, 7, 14, 15, 16, 17, 19, 24]. In particular, an explicit description of the rank-k numerical range of $A \in \mathbf{M}_n$ is given in [19], namely,

$$\Lambda_k(A) = \bigcap_{\xi\in[0,2\pi)} \{\mu \in \mathbb{C} : e^{-i\xi}\mu + e^{i\xi}\overline{\mu} \le \lambda_k(e^{-i\xi}A + e^{i\xi}A^*)\}, \tag{1}$$

where $\lambda_k(X)$ is the kth largest eigenvalue of a Hermitian matrix X. For a normal matrix $A \in \mathbf{M}_n$ with eigenvalues $a_1, \ldots, a_n$, we have

$$\Lambda_k(A) = \bigcap_{1\le j_1<\cdots<j_{n-k+1}\le n} \operatorname{conv}\{a_{j_1}, \ldots, a_{j_{n-k+1}}\}, \tag{2}$$

where "$\operatorname{conv} S$" denotes the convex hull of the set S. In [17], a complete description of $\Lambda_k(A)$ for quadratic operators A is given.

When $k = 1$, $\Lambda_k(A)$ reduces to the *classical numerical range* defined and denoted by

$$W(A) = \{x^*Ax \in \mathbb{C} : x \in \mathbb{C}^n \text{ with } x^*x = 1\},$$

which is a useful concept in studying matrices and operators; see [9]. In the study of the classical numerical range and its generalizations, researchers are interested in studying their *preservers*, i.e., maps ϕ on matrices such that A and $\phi(A)$ always have the same (generalized) numerical range; see [1, 8, 12]. For example, a linear map $\phi : \mathbf{M}_n \to \mathbf{M}_n$ satisfies $W(\phi(A)) = W(A)$ for all $A \in \mathbf{M}_n$ if and only if there is a unitary $U \in \mathbf{M}_n$ such that ϕ has the form

$$A \mapsto U^*AU \qquad \text{or} \qquad A \mapsto U^*A^tU. \tag{3}$$

Define the *numerical radius* of $A \in \mathbf{M}_n$ by

$$r(A) = \max\{|\mu| : \mu \in W(A)\}.$$

It is known that a linear map $\phi : \mathbf{M}_n \to \mathbf{M}_n$ satisfies $r(\phi(A)) = r(A)$ for all $A \in \mathbf{M}_n$ if and only if there are $\xi \in \mathbb{C}$ with $|\xi| = 1$ and a unitary $U \in \mathbf{M}_n$ such that ϕ has the form

$$A \mapsto \xi U^*AU \qquad \text{or} \qquad A \mapsto \xi U^*A^tU. \tag{4}$$

In particular, a linear preserver of the numerical radius must be a scalar multiple of a linear preserver of the numerical range.

In [6], linear preservers of the rank-k numerical range are characterized. In particular, it is shown that a linear map $\phi : \mathbf{M}_n \to \mathbf{M}_n$ satisfies

$$\Lambda_k(\phi(A)) = \Lambda_k(A) \qquad \text{for all } A \in \mathbf{M}_n$$

if and only if there is a unitary $U \in \mathbf{M}_n$ such that ϕ has the form (3). Define the *rank-k numerical radius* of $A \in \mathbf{M}_n$ by

$$r_k(A) = \sup\{|\mu| : \mu \in \Lambda_k(A)\}.$$

If $\Lambda_k(A) = \emptyset$, we use the convention that $r_k(A) = -\infty$. [In our discussion, we do not need to perform any arithmetic involving $-\infty$. Our results and proofs are valid as long as $\Lambda_k(A) = \emptyset$ if and only if $\Lambda_k(\phi(A)) = \emptyset$. So, we may actually let $r_k(A)$ to be any quantity not in $[0, \infty)$.) It is shown in [6] that a linear map $\phi : \mathbf{M}_n \to \mathbf{M}_n$ satisfies

$$r_k(\phi(A)) = r_k(A) \qquad \text{for all } A \in \mathbf{M}_n$$

if and only if there are $\xi \in \mathbb{C}$ with $|\xi| = 1$ and a unitary $U \in \mathbf{M}_n$ such that ϕ has the form (4). Once again, a linear preserver of the rank-k numerical radius must be a scalar multiple of a linear preserver of the rank-k numerical range.

Let $\mathcal{S}$ be a semigroup of matrices in $\mathbf{M}_n$. A map $\phi : \mathcal{S} \to \mathbf{M}_n$ is *multiplicative* if

$$\phi(AB) = \phi(A)\phi(B) \quad \text{for all } A, B \in \mathcal{S}.$$

In this paper, we determine the structure of multiplicative preservers of the rank-k numerical range(radius). In the context of quantum error correction, one needs to consider the rank-k numerical range of matrices of the form $A = E_i^* E_j$. In some quantum channels such as the randomized unitary channels and the Pauli channels, the error operators $E_1, \ldots, E_r$ actually come from a certain (semi)group of matrices in $\mathbf{M}_n$; see [21]. Moreover, if the quantum states go through two channels with operator sum representations $L(A) = \sum_{j=1}^r E_j A E_j^*$ and $\tilde{L}(A) = \sum_{j=1}^{\tilde{r}} \tilde{E}_j A \tilde{E}_j^*$, then the combined effect will be a quantum channel of the form $\tilde{L} \circ L(A) = \sum_{i=1}^{\tilde{r}} \sum_{j=1}^r \tilde{E}_i E_j A E_j^* \tilde{E}_i^*$. Thus, it is natural to consider multiplicative maps $\phi : \mathcal{S} \to \mathbf{M}_n$ which preserve the rank-k numerical radius or the rank-k numerical range. In the following, we denote by

$\mathbf{GL}_n$: the group of invertible matrices in $\mathbf{M}_n$;
$\mathbf{SL}_n$: the group of matrices in $\mathbf{GL}_n$ of determinant 1;
$\mathbf{U}_n$: the group of unitary matrices in $\mathbf{M}_n$;
$\mathbf{SU}_n$: the group of matrices in $\mathbf{U}_n$ of determinant 1;
$\mathbf{M}_n^{(m)}$: the semigroup of matrices in $\mathbf{M}_n$ with rank at most m.

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| \le 1\}$ and $\partial\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}$. Here are our main theorems.

Theorem 1.1. *Let $k \in \{1, \ldots, n-1\}$ with $n > 1$ and $\mathcal{S} \in \{\mathbf{U}_n, \mathbf{SU}_n, \mathbf{GL}_n, \mathbf{SL}_n, \mathbf{M}_n^{(m)}\}$ with $m \in \{k, \ldots, n\}$. A multiplicative map $\phi : \mathcal{S} \to \mathbf{M}_n$ satisfies*

$$r_k(\phi(A)) = r_k(A) \quad \textit{for all } A \in \mathcal{S}$$

if and only if there exists a multiplicative map $f : \mathbb{C} \to \partial\mathbb{D}$ such that one of the following holds.

(a) *There exists $U \in \mathbf{U}_n$ such that ϕ has the form*

$$A \mapsto f(\det A) U^* A U \qquad \textit{or} \qquad A \mapsto f(\det A) U^* \overline{A} U.$$

(b) $k = 1$, $\mathcal{S} \in \{\mathbf{SU}_n, \mathbf{U}_n\}$, *and there is a non-zero Hermitian idempotent* $P \in \mathbf{M}_n$ *such that* ϕ *has the form*

$$A \mapsto f(\det A)P.$$

(c) $\mathcal{S} \in \{\mathbf{U}_2, \mathbf{SU}_2\}$, *and* $\phi(\mathcal{S})$ *is a subgroup of* $\mathbf{U}_2$.

Theorem 1.2. *Let* $k \in \{1, \ldots, n-1\}$ *with* $n > 1$ *and* $\mathcal{S} \in \{\mathbf{U}_n, \mathbf{SU}_n, \mathbf{GL}_n, \mathbf{SL}_n, \mathbf{M}_n^{(m)}\}$ *with* $m \in \{k, \ldots, n\}$. *A multiplicative map* $\phi : \mathcal{S} \to \mathbf{M}_n$ *satisfies*

$$\Lambda_k(A) = \Lambda_k(\phi(A)) \quad \textit{for all } A \in \mathcal{S}$$

if and only if there exists $U \in \mathbf{U}_n$ *such that* ϕ *has the form*

$$A \mapsto U^*AU.$$

Note that $\Lambda_k(A) \subseteq \{0\}$ if A has rank smaller than k. Thus, we assume $m \in \{k, \ldots, n\}$ if $\mathcal{S} = \mathbf{M}_n^{(m)}$ to avoid trivial consideration in the above theorems.

It is easy to deduce from Theorem 1.2 that an anti-multiplicative map $\phi : \mathcal{S} \to \mathbf{M}_n$ satisfies $\Lambda_k(A) = \Lambda_k(\phi(A))$ if and only if there exists a unitary matrix U such that ϕ has the form $A \mapsto U^*A^{\mathrm{t}}U$.

It is clear that a linear preserver of the rank-k numerical range (radius) on $\mathbf{M}_n$ is either a multiplicative preserver or an anti-multiplicative preserver of the rank-k numerical range (radius).

We will present some preliminary results on multiplicative maps on matrix (semi)groups in Section 2, and then prove the theorems in Sections 3 and 4. To avoid trivial consideration, we always assume that $n \geq 2$.

2 Preliminary results

In [25] the authors define an almost homomorphism $g : \mathbb{D} \to \mathbb{C}$ as a nonzero map such that $g(a + b) = g(a) + g(b)$ for all $a, b \in \mathbb{D}$ with $a + b \in \mathbb{D}$, and $g(ab) = g(a)g(b)$ for all $a, b \in \mathbb{D}$. We have the following observation.

Lemma 2.1. *An almost homomorphism* $g : \mathbb{D} \to \mathbb{C}$ *can be extended to a field homomorphism on* $\mathbb{C}$.

Proof. Suppose $g : \mathbb{D} \to \mathbb{C}$ is an almost homomorphism. Notice that $g(1) = 1$ and it can be checked that $g(r) = r$ for all $r \in \mathbb{Q} \cap \mathbb{D}$.

For any $z \in \mathbb{C}$, there is a nonzero $r \in \mathbb{Q} \cap \mathbb{D}$ such that $rz \in \mathbb{D}$. Define $h : \mathbb{C} \to \mathbb{C}$ by

$$h(z) = r^{-1}g(rz).$$

We claim that the map h is well defined. To see this, suppose there are nonzero $r, s \in \mathbb{Q} \cap \mathbb{D}$ such that $rz, sz \in \mathbb{D}$. Without loss of generality, we assume $|r| \leq |s|$. Then $r/s \in \mathbb{Q} \cap \mathbb{D}$ and $g(r/s) = r/s$. Thus,

$$(r/s)g(sz) = g(r/s)g(sz) = g(rz) \quad \Rightarrow \quad s^{-1}g(sz) = r^{-1}g(rz).$$

Now for any $z_1, z_2 \in \mathbb{C}$, there is a nonzero $r \in \mathbb{Q} \cap \mathbb{D}$ such that $rz_1, rz_2, r(z_1 + z_2) \in \mathbb{D}$. Then

$$h(z_1 + z_2) = r^{-1}g(r(z_1 + z_2)) = r^{-1}g(rz_1 + rz_2) = r^{-1}g(rz_1) + r^{-1}g(rz_2) = h(z_1) + h(z_2)$$

and as $r^2 z_1 z_2 = (rz_1)(rz_2) \in \mathbb{D}$,

$$h(z_1 z_2) = r^{-2} g(r^2 z_1 z_2) = r^{-2} g((rz_1)(rz_2)) = (r^{-1} g(rz_1))(r^{-1} g(rz_2)) = h(z_1)h(z_2).$$

Thus, h is a homomorphism on $\mathbb{C}$. Furthermore, we see that $h(z) = g(z)$ for all $z \in \mathbb{D}$. □

Lemma 2.2. *Let $\tau : \mathbb{C} \to \mathbb{C}$ be a field homomorphism. The following are equivalent.*

(a) *τ is either the identity map or the conjugate map.*

(b) *$|\tau(z)| = 1$ whenever $|z| = 1$.*

(c) *For any $r, s \in \mathbb{Q}$ with $s \neq 0$ and $z \in \mathbb{C}$ such that $|r + sz| = 1$, we have $|r + s\tau(z)| = 1$.*

Proof. The implications (a) ⇒ (b) ⇒ (c) are clear. The implication (c) ⇒ (a) follows from [8, Lemma 3.1]. □

Let $A_\tau = [\tau(a_{ij})]$. In view of Lemma 2.1, we may restate [25, Theorem 3].

Theorem 2.3. *Suppose $n \geq 3$. A multiplicative map $\phi : \mathbf{U}_n \to \mathbf{M}_n$ has one of the following forms:*

(a) *There are $S \in \mathbf{GL}_n$, a multiplicative map $f : \partial\mathbb{D} \to \mathbb{C}$, and a nonzero field endomorphism τ on $\mathbb{C}$ such that ϕ has the form*

$$A \mapsto f(\det A) S A_\tau S^{-1}.$$

(b) *There are $S \in \mathbf{GL}_n$ and a multiplicative map $g : \partial\mathbb{D} \to \mathbf{GL}_r$ for some $r \in \{0, \ldots, n\}$ such that ϕ has the form*

$$A \mapsto S(g(\det A) \oplus 0_{n-r}) S^{-1}.$$

Recall that a nonzero field endomorphism is always as a field monomorphism. Theorem 2.3 can also be extended to show that multiplicative maps on $\mathbf{SU}_n$ are simply the restrictions of multiplicative maps on $\mathbf{U}_n$.

Theorem 2.4. *Suppose $n \geq 3$. A multiplicative map $\phi : \mathbf{SU}_n \to \mathbf{M}_n$ has one of the following forms:*

(a) *There are $S \in \mathbf{GL}_n$ and a nonzero field endomorphism τ on $\mathbb{C}$ such that ϕ has the form*

$$A \mapsto S A_\tau S^{-1}.$$

(b) *There are $S \in \mathbf{GL}_n$ and $r \in \{0, \ldots, n\}$ such that $\phi(A) = S(I_r \oplus 0_{n-r}) S^{-1}$ for all $A \in \mathbf{SU}_n$.*

Proof. We will extend the map ϕ to a multiplicative map $\psi : \mathbf{U}_n \to \mathbf{M}_n$ so that Theorem 2.3 is applicable. To this end, let $\omega = e^{2\pi i/n}$. Since $(\phi(\omega I_n))^{n+1} = \phi(\omega I_n)$, the minimal polynomial $p(\lambda)$ of the matrix $\phi(\omega I_n)$ is a factor of $\lambda^{n+1} - \lambda$. Thus, the minimal polynomial of $\phi(\omega I_n)$ has linear factors, and therefore $\phi(\omega I_n)$ is diagonalizable. Hence, there exist an invertible $S \in \mathbf{M}_n$, positive integers $n_1, \ldots, n_r$ with $n_1 + \cdots + n_r = n$, and $1 \leq p_1 < \cdots < p_{r-1} \leq n$ such that

$$\phi(\omega I_n) = S(\omega^{p_1} I_{n_1} \oplus \cdots \oplus \omega^{p_{r-1}} I_{n_{r-1}} \oplus 0_{n_r}) S^{-1}.$$

For any $A \in \mathbf{SU}_n$, $\phi(A)$ and $\phi(\omega I_n)$ commute and therefore $\phi(A)$ must have the form

$$S(A_1 \oplus \cdots \oplus A_r) S^{-1}$$

with $A_j \in \mathbf{M}_{n_j}$. We define a map $\psi : \mathbf{U}_n \to \mathbf{M}_n$ as follows. For any $\mu \in \partial\mathbb{D}$, take

$$\psi(\mu I_n) = S(\mu^{p_1} I_{n_1} \oplus \cdots \oplus \mu^{p_{r-1}} I_{n_{r-1}} \oplus 0_{n_r})S^{-1}.$$

For each non-scalar matrix $A \in \mathbf{U}_n$, there exists $\mu \in \partial\mathbb{D}$ such that $\mu A \in \mathbf{SU}_n$. We define

$$\psi(A) = \psi(\mu^{-1} I_n)\phi(\mu A).$$

Clearly, $\psi(\mu\nu I_n) = \psi(\mu I_n)\psi(\nu I_n)$ for all $\mu, \nu \in \partial\mathbb{D}$ and $\psi(\mu I_n)\phi(A) = \phi(A)\psi(\mu I_n)$ for all $\mu \in \partial\mathbb{D}$ and $A \in \mathbf{SU}_n$. Now suppose there are $\mu, \nu \in \partial\mathbb{D}$ such that both μA and νA are in $\mathbf{SU}_n$. Then $\mu\nu^{-1} I_n \in \mathbf{SU}_n$ and

$$\begin{aligned} \psi(\mu^{-1} I_n)\phi(\mu A) &= \psi(\mu^{-1} I_n)\phi(\mu\nu^{-1} I_n)\phi(\nu A) \\ &= \psi(\mu^{-1} I_n)\psi(\mu\nu^{-1} I_n)\phi(\nu A) = \psi(\nu^{-1} I_n)\phi(\nu A). \end{aligned}$$

Thus, ψ is well-defined. In particular, we have $\psi(A) = \phi(A)$ for all $A \in \mathbf{SU}_n$. Now for any $A, B \in \mathbf{U}_n$, there are $\mu, \nu \in \partial\mathbb{D}$ such that $\mu A, \nu B \in \mathbf{SU}_n$. Then $\mu\nu AB \in \mathbf{SU}_n$ and

$$\begin{aligned} \psi(AB) &= \psi(\mu^{-1}\nu^{-1} I_n)\phi(\mu\nu AB) = \psi(\mu^{-1} I_n)\psi(\nu^{-1} I_n)\phi(\mu A)\phi(\nu B) \\ &= \phi(\mu^{-1} I_n)\phi(\mu A)\psi(\nu^{-1} I_n)\phi(\nu B) = \psi(A)\psi(B). \end{aligned}$$

Therefore, ψ is a multiplicative map form $\mathbf{U}_n$ to $\mathbf{M}_n$ and $\psi(A) = \phi(A)$ for all $A \in \mathbf{SU}_n$. Then the result follows from Theorem 2.3. □

Multiplicative maps $\phi : \mathcal{S} \to \mathbf{M}_n$ for $\mathcal{S} \in \{\mathbf{SL}_n, \mathbf{GL}_n, \mathbf{M}_n^{(m)}\}$ have been studied by many authors. We have the following result; for example, see [8, Theorems 2.5 & 2.7], [1, Remark 3.1], [26, Theorems 1 & 2] and their references.

Theorem 2.5. *Suppose $\phi : \mathcal{S} \to \mathbf{M}_n$ is a multiplicative map, where $\mathcal{S} \in \{\mathbf{GL}_n, \mathbf{SL}_n, \mathbf{M}_n^{(m)}\}$. Then there exist $S \in \mathbf{GL}_n$, a multiplicative map $f : \mathbb{C} \to \mathbb{C}$, and a field endomorphism $\tau : \mathbb{C} \to \mathbb{C}$ such that ϕ has one of the following forms.*

(a) $A \mapsto f(\det A) S A_\tau S^{-1}$.

(b) $A \mapsto f(\det A) S((\operatorname{adj} A)^t)_\tau S^{-1}$, *where* adj A *denotes the adjoint matrix of* A.

(c) $A \mapsto S(I_r \oplus g(\det A) \oplus 0_{n-r-s})S^{-1}$, *where* $r \in \{0, \ldots, n\}$, $s \in \{0, \ldots, n-r\}$, *and* $g : \mathbb{C} \to \mathbf{M}_s$ *is a multiplicative map such that* $(g(0), g(1)) = (0_s, I_s)$.

Note that we may assume that $f(1) = 1$ if $\mathcal{S} = \mathbf{SL}_n$, and $f(0) = 1$ if $\mathcal{S} = \mathbf{M}_n^{(m)}$ with $m < n$. Also, the map g in (c) is vacuous when $\mathcal{S} \in \{\mathbf{SL}_n, \mathbf{M}_n^{(m)}\}$. Further, if $\mathcal{S} = \mathbf{M}_n^{(m)}$ with $m < n-1$, then the map in (b) becomes the zero map.

The following results on the classical numerical range of $A \in \mathbf{M}_2$ will be used; see [9, Chapter 1].

- Let $A \in \mathbf{M}_2$. Then $A = U^* R U$ for unitary U and $R = \begin{bmatrix} \lambda_1 & \gamma \\ 0 & \lambda_2 \end{bmatrix}$, and $W(A)$ is an elliptical disk with foci λ_1, λ_2 and minor radius $|\gamma|$.
- Let $A, B \in \mathbf{M}_2$. Then $W(A) = W(B)$ if and only if there exists a unitary U such that $A = U^* B U$.

3 Proof of Theorem 1.1

The sufficiency of the theorem is clear. We focus on the necessity part. Suppose $\phi : \mathcal{S} \to \mathbf{M}_n$ is a multiplicative map satisfying $r_k(\phi(A)) = r_k(A)$ for all $A \in \mathcal{S}$.

3.1 The case when $\mathcal{S} \in \{\mathbf{SU}_n, \mathbf{U}_n\}$

Case 1 Assume that $k > 1$ so that $n > 2$. Then ϕ has the form in Theorems 2.3 or 2.4. First, we show that a map of the form in Theorem 2.3 (b) or 2.4 (b) cannot preserve the rank-k numerical radius. Assume that it is not true and ϕ has such a form and preserves the rank-k numerical radius. Consider the identity matrix I_n and the special unitary diagonal matrix $W = \operatorname{diag}(w, \ldots, w^n)$, where w is the $\frac{n(n+1)}{2}$th root of unity. Then $\Lambda_k(W)$ belongs to the interior of $\mathbb{D}$ by (2), and hence $r_k(I_n) > r_k(W)$. However, we have $\phi(I_n) = \phi(W)$ so that $r_k(\phi(I_n)) = r_k(\phi(W))$, which is a contradiction.

Suppose ϕ has the form in Theorem 2.3 (a) or 2.4 (a), i.e., $\phi(A) = f(\det A) S A_\tau S^{-1}$ for all $A \in \mathcal{S}$ such that $f(\det A) = f(1) = 1$ for all $A \in \mathbf{SU}_n$.

Write $S = QR$ with unitary Q and upper triangular R. Now for each $\mu \in \partial\mathbb{D}$, take $X = [\mu^{1-n}] \oplus \mu I_{n-1} \in \mathbf{SU}_n$. Then

$$\phi(X) = QR \begin{bmatrix} \tau(\mu^{1-n}) & 0 \\ 0 & \tau(\mu) I_{n-1} \end{bmatrix} R^{-1} Q^* = Q \begin{bmatrix} \tau(\mu^{1-n}) & * \\ 0 & \tau(\mu) I_{n-1} \end{bmatrix} Q^*.$$

Notice that when $k > 1$, $\Lambda_k(X) = \{\mu\}$ and $\Lambda_k(\phi(X)) = \{\tau(\mu)\}$. Then

$$|\tau(\mu)| = r_k(\phi(X)) = r_k(X) = 1.$$

Therefore, $|\tau(\mu)| = 1$ for all $\mu \in \partial\mathbb{D}$. By Lemma 2.2, τ is either the identity map or the conjugate map on $\mathbb{D}$.

Next, we show that S is a multiple of a unitary matrix. By replacing ϕ with $A \mapsto \phi(\bar{A})$, if necessary, we may assume that τ is the identity map. Now write $S = UDV$ for unitary U and V and diagonal $D = \operatorname{diag}(d_1, \ldots, d_n)$ with positive diagonal entries. We claim that D is a scalar matrix. Suppose not, without loss of generality, we assume that $d_1 \neq d_2$. Let $B = \begin{bmatrix} 0 & d_1/d_2 \\ d_2/d_1 & 0 \end{bmatrix}$. Then $\Lambda_1(B)$ is an non-degenerate elliptical disk with foci 1 and -1, and hence $\Lambda_1(B) \cap (\partial\mathbb{D} \setminus \{1, -1\})$ is nonempty. Take $w \in \Lambda_1(B) \cap (\partial\mathbb{D} \setminus \{1, -1\})$. Choose $\alpha \in \partial\mathbb{D}$ and distinct $w_{k+2}, \ldots, w_n \in \partial\mathbb{D} \setminus \{1, -1, w\}$ so that $-\alpha^n w^{k-1} w_{k+2} \cdots w_n = 1$. Let

$$X = \alpha V^* \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus w I_{k-1} \oplus W \right) V \quad \text{with } W = \operatorname{diag}(w_{k+2}, \ldots, w_n).$$

Then $X \in \mathbf{SU}_n$. By (2), $\Lambda_k(X)$ lies in the interior of $\mathbb{D}$ and hence $r_k(X) < 1$. On the other hand,

$$\phi(X) = \alpha U (B \oplus w I_{k-1} \oplus W) U^*.$$

Then $\alpha w \in \Lambda_k(\phi(X))$ and hence $r_k(\phi(X)) \geq |\alpha w| = 1$, which is a contradiction. Therefore, S is a multiple of some unitary matrix. Replacing (S, S^{-1}) by $(\gamma S, (\gamma S)^{-1})$ for a suitable $\gamma > 0$, we may assume that S is unitary. Thus condition (a) of Theorem 1.1 follows for $\mathcal{S} = \mathbf{SU}_n$.

In the case when $\mathcal{S} = \mathbf{U}_n$, for any $A \in \mathbf{U}_n$,

$$r_k(A) = r_k(f(\det A)SAS^{-1}) = |f(\det A)|r_k(A).$$

Thus, f is a multiplicative map on $\partial\mathbb{D}$. Finally f can be extended to a multiplicative map from $\mathbb{C}$ to $\partial\mathbb{D}$ by setting $f(0) = 0$ and $f(z) = f(z/|z|)$ for all $z \in \mathbb{C} \setminus \partial\mathbb{D}$. Then condition (a) of Theorem 1.1 holds for $\mathcal{S} = \mathbf{U}_n$.

Case 2 Assume that $k = 1$ and $n > 2$. Recall that $r_1(A)$ reduces to the classical numerical radius $r(A)$.

Let $\mathcal{S} = \mathbf{SU}_n$. If Theorem 2.4 (b) holds, then $\phi(I_n)$ is unitarily similar to $Y = \begin{bmatrix} I_r & Y_{12} \\ 0 & 0_{n-r} \end{bmatrix}$. If Y_{12} is nonzero, then Y have a principal submatrix $B = \begin{bmatrix} 1 & \gamma \\ 0 & 0 \end{bmatrix}$ so that $W(B)$ is an elliptical disk with 1 as an interior point and hence $r(Y) \geq r(B) > 1$, which is a contradiction. So, Y_{12} is zero and hence $\phi(I_n)$ is a Hermitian idempotent. Thus, Theorem 1.1 (b) holds.

Next, suppose Theorem 2.4 (a) holds. Then for any $\mu \in \partial\mathbb{D}$ and $X = [\mu^{1-n}] \oplus \mu I_{n-1}$, we have $\phi(X) = SX_\tau S^{-1}$. Denote by $\rho(Y)$ the spectral radius of $Y \in \mathbf{M}_n$. Then

$$1 = r(X) = r(\phi(X)) \geq \rho(\phi(X)) = \max\{|\tau(\mu)|, |\tau(\mu)|^{1-n}\}.$$

Thus, $|\tau(\mu)| = 1$ for all $\mu \in \partial\mathbb{D}$. By Lemma 2.2, τ has the form $\mu \mapsto \mu$ or $\mu \mapsto \bar{\mu}$. Now using an argument similar to those in Case 1, we see that S is a multiple of some unitary matrix. Hence Theorem 1.1 (a) holds.

Suppose $\mathcal{S} = \mathbf{U}_n$. Considering the restriction of ϕ on $\mathbf{SU}_n$, the restriction map on $\mathbf{SU}_n$ has the form $A \mapsto UAU^*$ or $A \mapsto U\overline{A}U^*$ for some unitary matrix U. We can then get the desired conclusion using the argument in the last paragraph in Case 1.

Case 3 Suppose $(k, n) = (1, 2)$. Let $\mathcal{S} \in \{\mathbf{SU}_2, \mathbf{U}_2\}$. Since $\phi(I_2)^2 = \phi(I_2)$, we see that $\phi(I_2)$ is idempotent, which may have rank $0, 1$ or 2. If $\phi(I_2) = 0$, then $1 = r(I_2) = r(\phi(I_2)) = r(0) = 0$, which is a contradiction. Now, suppose $\phi(I_2) = I_2$. For any $A \in \mathcal{S}$, $\phi(A)\phi(A^{-1}) = \phi(I_2) = I_2$, and $r(\phi(A)) = r(\phi(A^{-1})) = r(\phi(A)^{-1}) = 1$. It follows that $\rho(\phi(A)) = \rho((\phi(A))^{-1}) = 1$ and $\phi(A)$ is normal. Thus, $\phi(A) \in \mathbf{U}_2$. Then $\phi(\mathcal{S})$ is a subgroup $\mathbf{U}_2$, and condition (c) of Theorem 1.1 holds.

Finally, if $\phi(I_2)$ has rank 1, then $\phi(I_2) = U^* \begin{bmatrix} 1 & a \\ 0 & 0 \end{bmatrix} U$ for some unitary matrix U so that $W(\phi(I_2))$ is an elliptical disk with foci $0, 1$ and minor axis with length $|a|$. Since $r(\phi(I_2)) = r(I_2) = 1$, we see that $a = 0$. Replacing ϕ by the map $X \mapsto U\phi(X)U^*$, we may assume that $\phi(I_2) = E_{11}$. Now, $\phi(A) = \phi(I_2AI_2) = \phi(I_2)\phi(A)\phi(I_2)$, we see that $\phi(A) = g(A)E_{11}$ for some multiplicative map $g : \mathcal{S} \to \partial\mathbb{D}$. Note that $\partial\mathbb{D}$ is an Abelian group. So, $\mathrm{Ker}(g)$ contains the commutator subgroup of $\mathcal{S}$. Clearly, $\mathrm{Ker}(g)$ is a subgroup of $\mathbf{SU}_2$. Note that every $A \in \mathbf{SU}_2$ can be written as $V^*\mathrm{diag}\,(a, \bar{a})V$ for some $V \in \mathbf{SU}_2$ and $a \in \partial\mathbb{D}$. Let $b \in \partial\mathbb{D}$ be such that $b^2 = a$. Then $D = \mathrm{diag}\,(a, \bar{a}) = BXB^{-1}X^{-1}$ with

$$B = B^{-1} = \begin{bmatrix} 0 & b \\ \bar{b} & 0 \end{bmatrix} \quad \text{and} \quad X = X^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Thus, $A = V^*DVD^{-1}BXB^{-1}X^{-1}$ belongs to the commutator subgroup. Hence, $\mathbf{SU}_2$ is the commutator subgroup and $\mathrm{Ker}(g) = \mathbf{SU}_2$. As a result, $g(A) = 1$ for every $A \in \mathbf{SU}_2$. When $\mathcal{S} = \mathbf{U}_2$, for any $X, Y \in \mathbf{U}_2$ with $\det(X) = \det(Y)$. Then $XY^{-1} \in \mathbf{SU}_2$ and

$$g(X)g(Y)^{-1}E_{11} = g(X)g(Y^{-1})E_{11} = \phi(X)\phi(Y^{-1}) = \phi(XY^{-1}) = g(XY^{-1})E_{11} = E_{11}.$$

Thus, $g(X) = g(Y)$ and hence $g(A)$ is function of determinant of A. □

3.2 The case when $\mathcal{S} \in \{\mathbf{GL}_n, \mathbf{SL}_n, \mathbf{M}_n^{(m)}\}$

Suppose $k = 1$. If $\mathcal{S} = \mathbf{M}_n^{(m)}$, the result is proved in [1, Proposition 3.10]. If $\mathcal{S} \in \{\mathbf{SL}_n, \mathbf{GL}_n\}$, the result follows from [8, Theorem 3.8].

Assume $k > 1$. Then ϕ has one of the form (a) – (c) in Theorem 2.5. Since there is $A \in \mathcal{S}$ such that $0 < r_k(A) = r_k(\phi(A))$, we see that ϕ is not the zero map. Thus, $f(0) = 1$.

First, we show that ϕ cannot have the form in Theorem 2.5 (c). If $\mathcal{S} = \mathbf{M}_n^{(m)}$ with $m < n$, let $X = I_k \oplus 0_{n-k}$ and $Y = \operatorname{diag}(1, w, \ldots, w^{k-1}) \oplus 0_{n-k}$ such that $w = e^{2\pi i/k}$; if $\mathcal{S} \in \{\mathbf{SL}_n, \mathbf{GL}_n, \mathbf{M}_n\}$, let $X = I_n$ and $Y = \operatorname{diag}(1, w, \ldots, w^{n-1})$ such that $w = e^{4\pi i/n(n-1)}$. In this case, $\det(Y) = 1$. By (2), $1 = r_k(X) > r_k(Y)$. If ϕ has the form (c), then $\phi(X) = \phi(Y)$ so that $r_k(X) = r_k(\phi(X)) = r_k(\phi(Y)) = r_k(Y)$, which is a contradiction.

Second, we show that ϕ cannot have the form in Theorem 2.5 (b). Suppose $\mathcal{S} = \mathbf{M}_n^{(m)}$ with $m \in \{k, \ldots, n\}$. Then for $A = I_k \oplus 0$, we have $r_k(\phi(A)) = 0$ and $r_k(A) = 1$, which is a contradiction. Suppose $\mathcal{S} \in \{\mathbf{GL}_n, \mathbf{SL}_n\}$, and ϕ has the form in Theorem 2.5 (b). Since $f(1)^p = f(1)$ for all positive integer p, we have $f(1) \in \{0, 1\}$. Since ϕ is not the zero map, we have $f(1) = 1$. Let $A = (1/2)I_{n-1} \oplus [2^{n-1}]$. Then $r_k(A) = 1/2$ and $r_k(\phi(A)) = 2$, which is a contradiction.

Now, suppose ϕ has the form in Theorem 2.5 (a). If $\mathcal{S} = \mathbf{M}_n^{(m)}$ with $m < n$, then $f(0) = 1$. For $A_\mu = \mu I_k \oplus 0_{n-k}$ with $\mu \in \partial\mathbb{D}$, we have

$$1 = r_k(A_\mu) = r_k(\phi(A_\mu)) = r_k(\tau(\mu)\phi(A_1)) = |\tau(\mu)| r_k(A_1) = |\tau(\mu)|.$$

Thus, $|\tau(\mu)| = 1$ for all $\mu \in \partial\mathbb{D}$. By Lemma 2.2, τ is the identity map or the conjugation map. Next, we show that all the singular values of S are the same. If it is not true, assume that $S = UDV$ such that U, V are unitary, and $D = \operatorname{diag}(d_1, d_2, \ldots, d_n)$ such that $d_1/d_2 = d > 1$. Let $B = \begin{bmatrix} 1 & d_1/d_2 \\ d_2/d_1 & 1 \end{bmatrix}$. Then $\Lambda_1(B)$ is an non-degenerate elliptical disk with foci 2 and 0, and hence $\Lambda_1(B) \cap (\partial\mathbb{D} \setminus \{1\})$ is nonempty. Take $w \in \Lambda_1(B) \cap (\partial\mathbb{D} \setminus \{1\})$ and let

$$X = V^* \left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \oplus wI_{k-1} \oplus 0_{n-k-1} \right) V$$

Then $X \in \mathbf{M}_n^{(k)} \subseteq \mathbf{M}_n^{(m)}$. By (2), $\Lambda_k(X) \subseteq \{0\}$ and hence $r_k(X) < 1$. On the other hand,

$$\phi(X) = U\left(B \oplus wI_{k-1} \oplus 0_{n-k-1}\right) U^*.$$

Then $w \in \Lambda_k(\phi(X))$ and hence $r_k(\phi(X)) \geq |w| = 1$, which is a contradiction.

If $\mathcal{S} \in \{\mathbf{GL}_n, \mathbf{SL}_n, \mathbf{M}_n\}$, we may consider $\phi(A)$ for $A \in \mathbf{SU}_n$ to conclude that S is unitary and τ is either the identity map or the conjugate map using the argument in Section 3.1. Further, in the case when $\mathcal{S} = \mathbf{GL}_n$ or $\mathbf{M}_n$, for any $A \in \mathcal{S}$,

$$r_k(A) = r_k(f(\det A)SAS^{-1}) = |f(\det A)| r_k(A).$$

Thus, f is a multiplicative map form $\mathbb{C}$ to $\partial\mathbb{D}$ and condition (a) of Theorem 1.1 holds. □

4 Proof of Theorem 1.2

Again, the sufficiency is clear. We prove the necessity part. Suppose $\phi : \mathcal{S} \to \mathbf{M}_n$ is a multiplicative map satisfying $\Lambda_k(\phi(A)) = \Lambda_k(A)$ for all $A \in \mathcal{S}$.

Case 1 Suppose $\mathcal{S} \in \{\mathbf{SU}_n, \mathbf{U}_n\}$ and $n \geq 3$. Then $r_k(\phi(A)) = r_k(A)$, so by Theorem 1.1 ϕ is of the prescribed form. Suppose ϕ is of the form 1.1 (b). Then in particular $\phi(A) = \phi(B)$ and so $\Lambda_k(A) = \Lambda_k(B)$ for all $A, B \in \mathbf{SU}_n$. However, if $A = I_n$ and $B = \omega I_n$ with $\omega = e^{2\pi i/n}$, then $\Lambda_k(A) \neq \Lambda_k(B)$. This is a contradiction, so ϕ must be of the form in Theorem 1.1 (a).

Suppose there exists $U \in \mathbf{U}_n$ such that $\phi(A) = f(\det A)U^*\bar{A}U$ for all $A \in \mathcal{S}$. Choose $A = \omega I_n$ with $\omega = e^{2\pi i/n}$. Then $\Lambda_k(A) = \{\omega\} \neq \{\overline{\omega}\} = \Lambda_k(\overline{A}) = \Lambda_k(\phi(A))$, and hence a contradiction.

Finally suppose there exists $U \in \mathbf{U}_n$ such that $\phi(A) = f(\det A)U^*AU$ for all $A \in \mathcal{S}$. Then for any $\mu \in \partial\mathbb{D}$, $\mu = e^{i\theta}$ for some $\theta \in [0, 2\pi)$. Then

$$\{e^{i\theta/n}\} = \Lambda_k(e^{i\theta/n}I_n) = \Lambda_k(f(\mu)e^{i\theta/n}I_n) = \{f(\mu)e^{i\theta/n}\}.$$

Then $f(\mu) = 1$ for all $\mu \in \partial\mathbb{D}$ and the result follows.

Case 2 Suppose $\mathcal{S} \in \{\mathbf{U}_2, \mathbf{SU}_2\}$. For any $A \in \mathbf{SU}_2$, since $W(\phi(A)) = W(A)$ is always a line segment joining two points (can be the same) in the unit circle, $\phi(A) \subseteq \mathbf{SU}_2$ and hence $\phi(\mathbf{SU}_2) \subseteq \mathbf{SU}_2$. Let $X = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$. Then $W(\phi(X)) = W(X) = \mathrm{conv}\,\{i, -i\}$. Hence, $\phi(X) = U^*XU$ for some $U \in \mathbf{U}_2$. Replacing ϕ by the map $A \mapsto U\phi(A)U^*$, we may and we will assume that $\phi(X) = X$.

Note that for any $A \in \mathcal{S}$, A satisfies $-XAX = A$ if and only if A is diagonal. Thus for any diagonal matrix $A = \mathrm{diag}\,(a_1, a_2) \in \mathcal{S}$, we have $\phi(A) = \mathrm{diag}\,(b_1, b_2)$. Since $W(\phi(Z)) = W(Z)$ for $Z = A$ and XA, we see that $\{a_1, a_2\} = \{b_1, b_2\}$ and $\{ia_1, -ia_2\} = \{ib_1, -ib_2\}$. It follows that $(a_1, a_2) = (b_1, b_2)$. i.e., $\phi(A) = A$ for all diagonal matrices $A \in \mathcal{S}$.

Next, observe that for any $A \in \mathbf{SU}_2$, A satisfies $XAX = A$ if and only if $A = \begin{bmatrix} 0 & \alpha \\ -\bar{\alpha} & 0 \end{bmatrix}$ for some $\alpha \in \partial\mathbb{D}$. As a result, if $Y = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, then there exists $|\beta| = 1$ such that $\phi(Y) = \begin{bmatrix} 0 & \beta \\ -\bar{\beta} & 0 \end{bmatrix}$. Now, replacing ϕ by the map $A \mapsto D^*\phi(A)D$ with $D = \mathrm{diag}\,(\beta, 1)$, we may assume that $\phi(A) = A$ for $A = Y$ and any diagonal matrix $A \in \mathcal{S}$.

For any $\theta \in [0, 2\pi)$, let $R_\theta = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$. In particular, $R_{\pi/2} = Y$. Then $W(R_\theta) = \mathrm{conv}\,\{e^{i\theta}, e^{-i\theta}\}$. Notice that for any $A \in \mathbf{SU}_2$, $-YAY = A$ if any only if $A = R_\theta$ for some θ. Then for each $\theta \in [0, 2\pi)$, $\phi(R_\theta) \in \{R_\theta, R_{-\theta}\}$. Suppose there is a $\theta \in (0, 2\pi)$ such that $\phi(R_\theta) = R_{-\theta}$. Then

$$R_{-\theta+\pi/2} = R_{-\theta}R_{\pi/2} = \phi(R_\theta)\phi(R_{\pi/2}) = \phi(R_\theta R_{\pi/2}) = \phi(R_{\theta+\pi/2}) \in \{R_{\theta+\pi/2}, R_{-\theta-\pi/2}\},$$

which is is impossible. Therefore, $\phi(R_\theta) = R_\theta$ for all $\theta \in [0, 2\pi)$.

Now, for any $A \in \mathbf{SU}_2$, there exist $\alpha, \beta \in \mathbb{D}$ with $|\alpha|^2 + |\beta|^2 = 1$ such that $A = \begin{bmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{bmatrix}$. Let $\alpha = ae^{i\omega}$ and $\beta = be^{i\varphi}$ such that $\omega, \varphi \in [0, 2\pi)$ and $a, b > 0$. Then $1 = |\alpha|^2 + |\beta|^2 = a^2 + b^2$, so in particular we can choose $\theta \in [0, 2\pi)$ such that $a = \cos\theta, b = \sin\theta$. So

$$A = \begin{bmatrix} e^{i\omega}\cos\theta & e^{i\varphi}\sin\theta \\ -e^{-i\varphi}\sin\theta & e^{-i\omega}\cos\theta \end{bmatrix} = \begin{bmatrix} e^{i(\omega+\varphi)/2} & 0 \\ 0 & e^{-i(\omega+\varphi)/2} \end{bmatrix} R_\theta \begin{bmatrix} e^{i(\omega-\varphi)/2} & 0 \\ 0 & e^{-i(\omega-\varphi)/2} \end{bmatrix}.$$

Then, we see that $\phi(A) = A$. If $\mathcal{S} = \mathbf{U}_2$, and $B \in \mathbf{U}_2 \setminus \mathbf{SU}_2$, then $B = \mu A$ with some $\mu \in \partial\mathbb{D}$ and $A \in \mathbf{SU}_2$. Since $\phi(\mu I_2) = \mu I_2$ and $\phi(A) = A$, we can conclude that $\phi(B) = B$ as well.

Case 3 Suppose $\mathcal{S} \in \{\mathbf{SL}_n, \mathbf{GL}_n, \mathbf{M}_n^{(m)}\}$ with $m \in \{k, \ldots, n\}$ and $\phi : \mathcal{S} \to \mathbf{M}_n$ preserves the rank-k numerical range. Then it also preserves the rank-k numerical radius, and has the form described in Theorem 1.1. We may consider $\phi(X)$ for $X \in \mathbf{SU}_n$ and conclude that ϕ on $\mathcal{S}$ has the form $A \mapsto f(\det A)U^*AU$. For $\mathcal{S} \in \{\mathbf{SL}_n, \mathbf{M}_n^{(m)}\}$ with $m < n$, the result follows. Suppose $\mathcal{S} \in \{\mathbf{GL}_n, \mathbf{M}_n\}$. For any $z = re^{i\theta}$ with $r > 0$ and $\theta \in [0, 2\pi)$, let $A = r^{1/n}e^{i\theta/n}I_n$ where $r^{1/n}$ is the positive real nth root of r. Then

$$\{r^{1/n}e^{i\theta/n}\} = \Lambda_k(A) = \Lambda_k(\phi(A)) = \Lambda_k(f(z)A) = \{f(z)r^{1/n}e^{i\theta/n}\}.$$

Hence $f(z) = 1$ and the result follows. □

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