Ranks and determinants of the sum of matrices from unitary orbits

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Dedicated to Professor Yik-Hoi Au-Yeung on the occasion of his 70th birthday.

Abstract

The unitary orbit $\mathcal{U}(A)$ of an $n \times n$ complex matrix A is the set consisting of matrices unitarily similar to A. Given two $n \times n$ complex matrices A and B, ranks and determinants of matrices of the form $X + Y$ with $(X, Y) \in \mathcal{U}(A) \times \mathcal{U}(B)$ are studied. In particular, a lower bound and the best upper bound of the set $R(A, B) = \{\, \mathrm{rank}\,(X + Y) : X \in \mathcal{U}(A), Y \in \mathcal{U}(B)\}$ are determined. It is shown that $\Delta(A, B) = \{\det(X + Y) : X \in \mathcal{U}(A), Y \in \mathcal{U}(B)\}$ has empty interior if and only if the set is a line segment or a point; the algebraic structure of matrix pairs (A, B) with such properties are described. Other properties of the sets $R(A, B)$ and $\Delta(A, B)$ are obtained. The results generalize those of other authors, and answer some open problems. Extensions of the results to the sum of three or more matrices from given unitary orbits are also considered.

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1 Introduction

Let $A \in M_n$. The *unitary orbit* of A is denoted by

$$\mathcal{U}(A) = \{UAU^* : U^*U = I_n\}.$$

Evidently, if A is regarded as a linear operator acting on $\mathbb{C}^n$, then $\mathcal{U}(A)$ consists of the matrix representations of the same linear operator under different orthonormal bases. Naturally, $\mathcal{U}(A)$ captures many important features of the operator A. For instance, A is normal if and only if $\mathcal{U}(A)$ has a diagonal matrix; A is Hermitian (positive semi-definite) if and only if $\mathcal{U}(A)$ contains a (nonnegative) real diagonal matrix; A is unitary if and only if $\mathcal{U}(A)$ has a diagonal matrix with unimodular diagonal entries. There are also results on the characterization of diagonal entries and submatrices of matrices in $\mathcal{U}(A)$; see [14, 20, 23, 30] and their references. In addition, the unitary orbit of A has a lot of interesting geometrical and algebraic properties, see [11].

Motivated by theory as well as applied problems, there has been a great deal of interest in studying the sum of two matrices from specific unitary orbits. For example, eigenvalues of $UAU^* +$

VBV^* for Hermitian matrices A and B were completely determined in terms of those of A and B (see [16] and its references); the optimal norm estimate of $UAU^* + VBV^*$ was obtained (see [10] and its references); the range of values of $\det(UAU^* + VBV^*)$ for Hermitian matrices $A, B \in M_n$ was described, see [15]. Later, Marcus and Oliveira [26, 29] conjectured that if $A, B \in M_n$ are normal matrices with eigenvalues $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$, respectively, then for any unitary $U, V \in M_n$, $\det(UAU^* + VBV^*)$ always lies in the convex hull of

$$P(A, B) = \left\{ \prod_{j=1}^{n} (a_j + b_{\sigma(j)}) : \sigma \text{ is a permutation of } \{1, \ldots, n\} \right\}. \qquad (1.1)$$

In connection to this conjecture, researchers [2, 3, 4, 5, 7, 8, 12, 13] considered the determinantal range of $A, B \in M_n$ defined by

$$\Delta(A, B) = \{\det(A + UBU^*) : U \text{ is unitary}\}.$$

This can be viewed as an analog of the generalized numerical range of A and B defined by

$$W(A, B) = \{\operatorname{tr}(AUBU^*) : U \text{ is unitary}\},$$

which is a useful concept in pure and applied areas; see [18, 21, 25] and their references.

In this paper, we study some basic properties of the matrices in

$$\mathcal{U}(A) + \mathcal{U}(B) = \{X + Y : (X, Y) \in \mathcal{U}(A) \times \mathcal{U}(B)\}.$$

The focus will be on the rank and the determinant of these matrices.

Our paper is organized as follows. In Section 2, we obtain a lower bound and the best upper bound for the set

$$R(A, B) = \{\operatorname{rank}(X + Y) : X \in \mathcal{U}(A), Y \in \mathcal{U}(B)\};$$

moreover, we characterize those matrix pairs (A, B) such that $R(A, B)$ is a singleton, and show that the set $R(A, B)$ has the form $\{k, k+1, \ldots, n\}$ if A and $-B$ have no common eigenvalues. On the contrary if A and $-B$ are orthogonal projections, then the rank values of matrices in $\mathcal{U}(A) + \mathcal{U}(B)$ will either be all even or all odd. In Section 3, we characterize matrix pairs (A, B) such that $\Delta(A, B)$ has empty interior, which is possible only when $\Delta(A, B)$ is a singleton or a nondegenerate line segment. This extends the results of other researchers who treated the case when A and B are normal; see [2, 3, 4, 9]. In particular, our result shows that it is possible to have normal matrices A and B such that $\Delta(A, B)$ is a subset of a line, which does not pass through the origin. This disproves a conjecture in [9]. In [3], the authors showed that if $A, B \in M_n$ are normal matrices such that the union of the spectra of A and $-B$ consists of $2n$ distinct elements, then every *nonzero* sharp point of $\Delta(A, B)$ is an element in $P(A, B)$. (See the definition of sharp point in Section 3.) We showed that every (zero or nonzero) sharp point of $\Delta(A, B)$ belongs to $P(A, B)$ for arbitrary matrices $A, B \in M_n$. In Section 4, we consider the sum of three or more matrices from given unitary orbits, and matrix orbits corresponding to other equivalence relations.

In the literature, some authors considered the set

$$D(A, B) = \{\det(X - Y) : X \in \mathcal{U}(A), Y \in \mathcal{U}(B)\}$$

instead of $\Delta(A, B)$. Evidently, we have $D(A, B) = \Delta(A, -B)$. It is easy to translate results on $D(A, B)$ to those on $\Delta(A, B)$, and vice versa. Indeed, for certain results and proofs, it is more convenient to use the formulation of $D(A, B)$. We will do that in Section 3. On the other hand, it is more natural to use the summation formulation to discuss the extension of the results to matrices from three or more unitary orbits.

2 Ranks

2.1 Maximum and Minimum Rank

In [10], the authors obtained optimal norm bounds for matrices in $\mathcal{U}(A) + \mathcal{U}(B)$ for two given matrices $A, B \in M_n$. By the triangle inequality, we have

$$\max\{\|UAU^* + VBV^*\| : U, V \;\; \text{unitary}\} \le \min\{\|A - \mu I_n\| + \|B + \mu I_n\| : \mu \in \mathbb{C}\}.$$

It was shown in [10] that the inequality is actually an equality. For the rank functions, we have

$$\max\{\,\text{rank}\,(UAU^* + VBV^*) : U, V \;\; \text{unitary}\} \le \min\{\,\text{rank}\,(A - \mu I_n) + \,\text{rank}\,(B + \mu I_n) : \mu \in \mathbb{C}\}.$$

Of course, the right side may be strictly larger than n, and thus equality may not hold in general. It turns out that this obstacle can be overcome easily as shown in the following.

Theorem 2.1 *Let $A, B \in M_n$ and*

$$\begin{aligned}
m &= \min\{\,\text{rank}\,(A - \mu I_n) + \,\text{rank}\,(B + \mu I_n) : \mu \in \mathbb{C}\} \\
&= \min\{\,\text{rank}\,(A - \mu I_n) + \,\text{rank}\,(B + \mu I_n) : \mu \text{ is an eigenvalue of } A \oplus -B\}.
\end{aligned}$$

Then

$$\max\{\,\text{rank}\,(UAU^* + VBV^*) : U, V \text{ unitary}\} = \min\{m, n\}.$$

Proof. If μ is an eigenvalue of $A \oplus -B$, then $\,\text{rank}\,(A - \mu I_n) + \,\text{rank}\,(B + \mu I_n) \le 2n - 1$; if μ is not an eigenvalue of $A \oplus -B$, then $\,\text{rank}\,(A - \mu I_n) + \,\text{rank}\,(B + \mu I_n) = 2n$. As a result, $\,\text{rank}\,(A - \mu I_n) + \,\text{rank}\,(B + \mu I_n)$ will attain its minimum at an eigenvalue μ of the matrix $A \oplus -B$.

It is clear that

$$\max\{\,\text{rank}\,(UAU^* + VBV^*) : U, V \text{ unitary}\} \le \min\{m, n\}.$$

It remains to show that there are U, V such that $UAU^* + VBV^*$ has rank equal to $\min\{m, n\}$.

Suppose $m \le n$ and there is μ such that $\,\text{rank}\,(A - \mu I_n) = k$ and $\,\text{rank}\,(B + \mu I_n) = m - k$. We may replace (A, B) by $(A - \mu I_n, B + \mu I_n)$ and assume that $\mu = 0$. Furthermore, we may assume that $k \le m - k$; otherwise, interchange A and B.

Let $A = XDY$ be such that $X, Y \in M_n$ are unitary, and $D = D_1 \oplus 0_{n-k}$ with invertible diagonal D_1. Replace A by YAY^*, we may assume that

$$A = UD = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} D_1 & 0 \\ 0 & 0_{n-k} \end{pmatrix}$$

with $U = YX$. Similarly, we may assume that

$$B = VE = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix} \begin{pmatrix} 0_k & 0 \\ 0 & E_2 \end{pmatrix},$$

where V is a unitary matrix and E_2 is a diagonal matrix with rank $m - k$. Let W be a unitary matrix such that the first k columns of WU together with the last $n - k$ columns of V are linearly

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independent. That is, if $W = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix}$, the matrix $\begin{pmatrix} W_{11}U_{11} + W_{12}U_{21} & V_{12} \\ W_{21}U_{11} + W_{22}U_{21} & V_{22} \end{pmatrix}$ is invertible. If W_{11} is invertible, then $D_1 W_{11}^*$ has rank k and so

$$WAW^* + B = \begin{pmatrix} W_{11}U_{11} + W_{12}U_{21} & V_{12} \\ W_{21}U_{11} + W_{22}U_{21} & V_{22} \end{pmatrix} \begin{pmatrix} D_1 W_{11}^* & D_1 W_{21}^* \\ 0 & E_2 \end{pmatrix}$$

has rank m. If W_{11} is not invertible, we will replace W by $\tilde{W}$ obtained as follows. By the CS decomposition, there are unitary matrices $P_1, Q_1 \in M_k$ and $P_2, Q_2 \in M_{n-k}$ such that

$$(P_1 \oplus P_2)W(Q_1 \oplus Q_2) = \begin{pmatrix} C & S \\ -S & C \end{pmatrix} \oplus I_{n-2k},$$

where $C = \text{diag}\,(c_1, \ldots, c_k)$ with $1 \geq c_1 \geq \cdots \geq c_k \geq 0$ and $S = \text{diag}\,\left(\sqrt{1 - c_1^2}, \ldots, \sqrt{1 - c_k^2}\right)$. Then perturb the zero diagonal entries of C slightly to $\tilde{C} = \text{diag}\,(\tilde{c}_1, \ldots, \tilde{c}_k)$ so that $\tilde{C}$ is invertible, and set $\tilde{S} = \text{diag}\,\left(\sqrt{1 - \tilde{c}_1^2}, \ldots, \sqrt{1 - \tilde{c}_k^2}\right)$. Then

$$\tilde{W} = (P_1 \oplus P_2)^* \left[\begin{pmatrix} \tilde{C} & \tilde{S} \\ -\tilde{S} & \tilde{C} \end{pmatrix} \oplus I_{n-2k} \right] (Q_1 \oplus Q_2)^*$$

will be a slight perturbation of W with invertible $\tilde{W}_{11} = P_1^* \tilde{C} Q_1^*$, which can be chosen such that the matrix $\begin{pmatrix} \tilde{W}_{11}U_{11} + \tilde{W}_{12}U_{21} & V_{12} \\ W_{21}U_{11} + W_{22}U_{21} & V_{22} \end{pmatrix}$ is still invertible. Then $\tilde{W}A\tilde{W}^* + B$ has rank m.

Now, assume that $\text{rank}\,(A - \mu I_n) + \text{rank}\,(B + \mu I_n) \geq n + 1$ for every $\mu \in \mathbb{C}$. Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be the eigenvalues of A and B. We consider two cases. First, suppose $a_i + b_j \neq 0$ for some i, j. We may assume that $i = j = 1$. Applying suitable unitary similarity transforms, we may assume that A and B are unitarily similar to matrices in upper triangular form

$$\begin{pmatrix} a_1 & * \\ 0 & A_1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} b_1 & * \\ 0 & B_1 \end{pmatrix}.$$

Since $\text{rank}\,(A - \mu I_n) + \text{rank}\,(B + \mu I_n) \geq n + 1$ for every $\mu \in \mathbb{C}$, it follows that $\text{rank}\,(A_1 - \mu I_{n-1}) + \text{rank}\,(B_1 + \mu I_{n-1}) \geq n - 1$ for every $\mu \in \mathbb{C}$. By induction assumption, there is a unitary V_1 such that $\det(A_1 + V_1 B_1 V_1^*) \neq 0$. Let $V = [1] \oplus V_1$. Then $\det(A + VBV^*) = (a_1 + b_1)\det(A_1 + V_1 B_1 V_1^*) \neq 0$.

Suppose $a_i + b_j = 0$ for all $i, j \in \{1, \ldots, n\}$. Replacing (A, B) by $(A - a_1 I_n, B - b_1 I_n)$, we may assume that A and B are nilpotents. If A or B is normal, then it will be the zero matrix. Then $\text{rank}\,(A) + \text{rank}\,(B) < n$, which contradicts our assumption. Suppose neither A nor B is normal and $\text{rank}\,A \leq \text{rank}\,B$.

If $n = 3$, then $\text{rank}\,A = \text{rank}\,B = 2$. We may assume that

$$A = \begin{pmatrix} 0 & \alpha_1 & \alpha_2 \\ 0 & 0 & \alpha_3 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 & 0 \\ \beta_1 & 0 & 0 \\ \beta_2 & \beta_3 & 0 \end{pmatrix}$$

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such that $\alpha_1, \alpha_3, \beta_1, \beta_3$ are nonzero. Interchange the last two rows and the last two columns of A to obtain $\hat{A}$. For $U_\xi = \mathrm{diag}\,(1, 1, e^{i\xi})$, we have

$$U_\xi \hat{A} U_\xi^* = \begin{pmatrix} 0 & \alpha_2 & \alpha_1 e^{-i\xi} \\ 0 & 0 & 0 \\ 0 & \alpha_3 e^{i\xi} & 0 \end{pmatrix}.$$

Evidently, there is $\xi \in [0, 2\pi)$ such that $\det(U_\xi \hat{A} U_\xi^* + B) \neq 0$.

Suppose $n \geq 4$. Applying suitable unitary similarity transforms, we may assume that both A and B are in upper triangular form with nonzero $(1, 2)$ entries; see [27, Lemma 1]. Modify B by interchanging its first two rows and columns. Then, A and B have the form $\begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$ and $\begin{pmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{pmatrix}$ so that $A_{11} = \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix}$ and $B_{11} = \begin{pmatrix} 0 & 0 \\ \beta & 0 \end{pmatrix}$ with $\alpha\beta \neq 0$, and A_{22}, B_{22} are upper triangular nilpotent matrices. If $\mathrm{rank}\,A + \mathrm{rank}\,B \geq n + 2$, then

$$\mathrm{rank}\,(A_{22} - \mu I_{n-2}) + \mathrm{rank}\,(B_{22} + \mu I_{n-2}) \geq \mathrm{rank}\,A_{22} + \mathrm{rank}\,B_{22} \geq \mathrm{rank}\,A + \mathrm{rank}\,B - 4 \geq n - 2.$$

If $\mathrm{rank}\,A + \mathrm{rank}\,B = n + 1$, we **claim** that by choosing a suitable unitary similarity transform, we can further assume that $\mathrm{rank}\,A_{22} = \mathrm{rank}\,A - 1$. Then

$$\mathrm{rank}\,(A_{22} - \mu I_{n-2}) + \mathrm{rank}\,(B_{22} + \mu I_{n-2}) \geq \mathrm{rank}\,A_{22} + \mathrm{rank}\,B_{22} \geq \mathrm{rank}\,A + \mathrm{rank}\,B - 3 = n - 2.$$

In both cases, by induction assumption, there is V_2 such that $\det(A_{22} + V_2 B_{22} V_2^*) \neq 0$. Let $V = I_2 \oplus V_2$. Then
$$\det(A + V B V^*) = -\alpha\beta \det(A_{22} + V_2 B_{22} V_2^*) \neq 0.$$

Now it remains to verify our **claim**. Suppose A has rank k and $\mathrm{rank}\,A + \mathrm{rank}\,B = n + 1$. Then $k \leq (n+1)/2$. Let S be an invertible matrix such that $S^{-1}AS = J$ is the Jordan form of A. If J has a 2×2 Jordan block, then we can always permute J so that $J = \begin{pmatrix} J_{11} & 0 \\ 0 & J_{22} \end{pmatrix}$ with $J_{11} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\mathrm{rank}\,(J_{22}) = p - 1$. By QR factorization, write $S = U^* T$ for some unitary matrix U and invertible upper triangular matrix $T = \begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix}$. Then A is unitary similar to

$$T J T^{-1} = \begin{pmatrix} T_{11} J_{11} T_{11}^{-1} & * \\ 0 & T_{22} J_{22} T_{22}^{-1} \end{pmatrix},$$

which has the described property.

Suppose J does not contain any 2×2 block, then J must have an 1×1 Jordan block. Otherwise, $k = \mathrm{rank}\,A \geq 2n/3$ and hence

$$\mathrm{rank}\,A + \mathrm{rank}\,B \geq 2k \geq 4n/3 = n + n/3 > n + 1.$$

Now we may assume that $J = \begin{pmatrix} 0_2 & J_{12} \\ 0 & J_{22} \end{pmatrix}$ is strictly upper triangular matrix such that J_{12} has only a nonzero entry in the $(1, 1)$-th position and $\mathrm{rank}\,J_{22} = k - 1$. Let $\hat{S}$ be obtained from I_n by

replacing the $(3, 2)$-th entries with one, then

$$\hat{S}^{-1} J \hat{S} = \hat{J} = \begin{pmatrix} \hat{J}_{11} & J_{12} \\ 0 & J_{22} \end{pmatrix}$$

with $\hat{J}_{11} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Applying QR factorization on $S\hat{S} = U^*T$ with unitary U and invertible upper

triangular T, then A is unitary similar to $T\hat{J}T^{-1}$, which has the described form. □

The value m in Theorem 2.1 is easy to determine as we need only to focus on $\operatorname{rank}(A - \mu I_n) + \operatorname{rank}(B + \mu I_n)$ for each eigenvalue μ of $A \oplus -B$. In particular, if μ is an eigenvalue of A, then $\operatorname{rank}(A - \mu I_n) = n - k$, where k is the geometric multiplicity of μ; otherwise, $\operatorname{rank}(A - \mu I_n) = n$. Similarly, one can determine $\operatorname{rank}(B + \mu I_n)$. The situation for normal matrices is even better as shown in the following.

Corollary 2.2 *Suppose A and B are normal matrices such that ℓ is the maximum multiplicity of an eigenvalue of $A \oplus -B$. Then $\min\{\operatorname{rank}(A - \mu I_n) + \operatorname{rank}(B + \mu I_n) : \mu \in \mathbb{C}\}$ equals $2n - \ell$. Consequently,*

$$\max\{\operatorname{rank}(UAU^* + VBV^*) : U, V \text{ unitary}\} = \min\{2n - \ell, n\}.$$

Here are other some consequences of Theorem 2.1.

Corollary 2.3 *Let $A, B \in M_n$. Then $UAU^* + VBV^*$ is singular for all unitary U, V if and only if there is $\mu \in \mathbb{C}$ such that $\operatorname{rank}(A - \mu I_n) + \operatorname{rank}(B + \mu I_n) < n$.*

Corollary 2.4 *Let $A \in M_n$, and*

$$k = \min\{\operatorname{rank}(A - \mu I_n) : \mu \text{ is an eigenvalue of } A\}.$$

Then

$$\max\{\operatorname{rank}(UAU^* - VAV^*) : U, V \text{ unitary}\} = \min\{n, 2k\}.$$

If $k < n/2$, then $UAU^* - VAV^*$ is singular for any unitary $U, V \in M_n$. In case A is normal, then $n - k$ is the maximum multiplicity of the eigenvalues of A.

Partition M_n as the disjoint union of unitary orbits. We can define a metric on the set of unitary orbits by

$$d(\mathcal{U}(A), \mathcal{U}(B)) = \min\{\operatorname{rank}(X - Y) : X \in \mathcal{U}(A), Y \in \mathcal{U}(B)\}.$$

For example, if A and B are two orthogonal projections of rank p and q, respectively, then $d(\mathcal{U}(A), \mathcal{U}(B)) = |p - q|$; see Proposition 2.8. So, the minimum rank of the sum or difference of matrices from two different unitary orbits has a geometrical meaning. However, it is not so easy to determine the minimum rank for matrices in $\mathcal{U}(A) + \mathcal{U}(B)$ in general. We have the following observation.

Proposition 2.5 *Let $A, B \in M_n$ and $\mu \in \mathbb{C}$ be such that $\operatorname{rank}(A - \mu I_n) = p$ and $\operatorname{rank}(B + \mu I_n) = q$. Then*

$$\min\{\operatorname{rank}(UAU^* + VBV^*) : U, V \text{ unitary}\} \leq \max\{p, q\}.$$

The inequality becomes equality if $A - \mu I_n$ and $B + \mu I_n$ are positive semi-definite.

Proof. There exist unitary U, V such that the last $n - p$ columns of $U(A - \mu I_n)U^*$ are zero, and the last $n - q$ columns of $V(B + \mu I_n)V^*$ are zero. Then $\operatorname{rank}(UAU^* + VBV^*) \leq \max\{p, q\}$. $\quad\square$

The upper bound in the above proposition is rather weak. For example, we may have A and B such that

$$\max\{\operatorname{rank}(A - \mu I_n), \operatorname{rank}(B + \mu I_n) : \mu \in \mathbb{C}\} = n - 1 \tag{2.1}$$

and $\operatorname{rank}(UAU^* + VBV^*) = 1$.

Example 2.6 Let $A = \operatorname{diag}(1, 2, \ldots, n)$ and $B = -J - A$, where $J \in M_n$ is the matrix having all entries equal to $1/n$. Then $-B$ has distinct eigenvalues $b_1 > \cdots > b_n$ such that $b_1 > n > b_2 > n - 1 > b_3 > \cdots > b_1 > 1$. Then (2.1) clearly holds and $\operatorname{rank}(A + B) = 1$. In fact, by Theorem 2.7 below, we know that for any $m \in \{1, \ldots, n\}$, there are unitary $U, V \in M_n$ such that $\operatorname{rank}(UAU^* + VBV^*) = m$.

2.2 Additional results

Here we study other possible rank values of matrices in $\mathcal{U}(A) + \mathcal{U}(B)$. The following result shows that if A and $-B$ have disjoint spectra, then one can get every possible rank values from the minimum to the maximum value, which is n.

Theorem 2.7 *Suppose $A, B \in M_n$ such that A and $-B$ have disjoint spectra, and $A + B$ has rank $k < n$. Then for any $m \in \{k + 1, \ldots, n\}$, there is a unitary U such that $UAU^* + B$ has rank m.*

Proof. Let $A, B \in M_n$ satisfy the hypotheses of the theorem. We need only to show that there is a unitary U such that $UAU^* + B$ has rank $k + 1$. Then we can apply the argument again to get a unitary $\hat{U}$ such that $\hat{U}UAU^*\hat{U}^* + B$ has rank $k + 2$. Repeating this procedure, we will get the desired conclusion.

If $k = n - 1$. Then assume VAV^* and WBW^* are in upper triangular form. For $U = W^*V$, we have $UAU^* + B = W^*(VAV^* + WBW^*)W$ is invertible.

For $1 \leq k < n - 1$. We may assume that

$$A + B = C = \begin{pmatrix} C_{11} & 0 \\ C_{21} & 0_{n-k} \end{pmatrix}.$$

Let

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \quad \text{and} \quad -B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

with $A_{11}, B_{11} \in M_k$. Note that $A_{12} \neq 0$. Otherwise, A and $-B$ have common eigenvalues since $A_{22} = B_{22}$.

Assume $C_{21} \neq 0$. We may replace $A + B$ by $V(A + B)V^*$ for some permutation matrix $V \in M_k$ of the form $V_1 \oplus I_{n-k}$ so that the matrix obtained by removing the first row of $V(A + B)V^*$ still has rank k. For notational simplicity, we may assume that $V = I_n$. Since $A_{12} \neq 0$, we may assume that the first row of $A_{12} \neq 0$. Otherwise, replace (A, B) by (VAV^*, VBV^*) for some unitary $V = V_1 \oplus I_{n-2}$. Here we still assume that removing the first row of $A + B$ results in a rank k matrix. Then there exists a small $\xi > 0$ such that for $U = \operatorname{diag}(e^{i\xi}, 1, \ldots, 1)$ the matrix $UAU^* + B$ has rank $k + 1$ because removing its first row has rank k, and adding the first row back will increase the rank by 1.

Now, suppose $C_{21} = 0$. Then $A_{21} \neq 0$. Otherwise, A and $-B$ have common eigenvalues since $A_{22} = B_{22}$. Now, C_{11} is invertible. Assume that the matrix obtained by removing the first row and first column of C_{11} has rank $k - 1$. Otherwise, replace (A, B) by (VAV^*, VBV^*) by some unitary matrix V of the form $V_1 \oplus I_{n-k}$. Since A_{12} and A_{21} are nonzero, we may further assume that $v^t = [a_{1,k+1}, \ldots, a_{1n}] \neq 0$ and $u = [a_{k+1,1}, \ldots, a_{n1}]^t \neq 0$. Then there exists a small $\xi > 0$ such that for $U = \mathrm{diag}\,(e^{i\xi}, 1, \ldots, 1)$ the matrix $UAU^* + B$ has the form

$$\begin{pmatrix} \hat{C}_{11} & \hat{C}_{12} \\ \hat{C}_{21} & 0_{n-2} \end{pmatrix},$$

where $\hat{C}_{11}$ is invertible such that removing its first row and first column results in a rank $k - 1$ matrix, only the first row of $\hat{C}_{12}$ is nonzero and equal to $(e^{i\xi} - 1)v^t$, only the first column of $\hat{C}_{21}$ is nonzero and equal to $(e^{-i\xi} - 1)u$. Now, removing the first row and first column of $UAU^* + B$ has rank $k - 1$; adding the column $(e^{-i\xi} - 1)u$ (to the left) will increase the rank by 1, and then adding the first row back will increase the rank by 1 more. So, $UAU^* + B$ has rank $k + 1$. □

Note that the assumption that A and $-B$ have disjoint spectra is essential. For example, if $A, B \in M_4$ such that A and $-B$ are rank 2 orthogonal projections, then $UAU^* + VBV^*$ can only have ranks 0, 2, 4. More generally, we have the following.

Proposition 2.8 *Suppose $A, B \in M_n$ are such that A and $-B$ are orthogonal projection of rank p and q. Then $k = \mathrm{rank}\,(UAU^* + B)$ for a unitary matrix $U \in M_n$ if and only if $k = |p - q| + 2j$ with $j \geq 0$ and $k \leq \min\{p + q, 2n - p - q\}$.*

Proof. Suppose $UAU^* = I_p \oplus 0_{n-p}$ and $VBV^* = 0_j \oplus -I_q \oplus 0_{n-j-q}$. Then $UAU^* + VBV^*$ has rank $k = |p - q| + 2j \leq \min\{p + q, 2n - p - q\}$. Thus, $V^*UAU^*V + B$ has rank k as well.

Conversely, consider $UAU^* + B$ for a given unitary U. There is a unitary V such that

$$VUAU^*V^* = I_p \oplus 0_{n-p} \quad \text{and} \quad VBV^* = -\left\{ I_r \oplus 0_s \oplus \begin{pmatrix} C^2 & CS \\ CS & S^2 \end{pmatrix} \oplus I_u \oplus 0_v \right\},$$

where C and S are invertible diagonal matrices with positive diagonal entries such that $C^2 + S^2 = I_t$, $r + s + t = p$ and $r + t + u = q$. (Evidently, the first r columns of V^* span the intersection of the range spaces of UAU^* and B, the next s columns of V^* span the intersection of the range space of UAU^* and the null space of B, the last v columns of V^* span the intersections of the null space of UAU^* and B, the u columns preceding those span the intersection of the range space of B and the null space of UAU^*.) So, $UAU^* + B$ has the asserted rank value. □

The following result was proved in [24].

Theorem 2.9 *Let $A, B \in M_n$ Then $UAU^* + VBV^*$ is invertible for all unitary $U, V \in M_n$ if and only if there is $\xi \in \mathbb{C}$ such that the singular values of $A - \xi I_n$ and $B + \xi I_n$ lie in two disjoint closed intervals in $[0, \infty)$.*

Using this result, we can deduce the following.

Theorem 2.10 *Let $A, B \in M_n$ and $k \in \{0, \ldots, n\}$. Then $\mathrm{rank}\,(UAU^* + VBV^*) = k$ for all unitary $U, V \in M_n$ if and only if one of the following holds.*

(a) One of the matrices A or B is scalar, and $\mathrm{rank}\,(A + B) = k$.

(b) $k = n$ and there is $\xi \in \mathbb{C}$ such that the singular values of $A - \xi I_n$ and $B + \xi I_n$ lie in two disjoint closed intervals in $[0, \infty)$.

Proof. If (a) holds, say, $B = \xi I_n$, then $\operatorname{rank}(UAU^* + VBV^*) = \operatorname{rank}(A - \xi I_n) = k$ for all unitary $U, V \in M_n$.

If (b) holds, then $\|(A - \xi I_n)x\| > \|(B + \xi I_n)y\|$ for all unit vectors $x, y \in \mathbb{C}^n$, or $\|(A - \xi I_n)x\| < \|(B + \xi I_n)y\|$ for all unit vectors $x, y \in \mathbb{C}^n$. Thus, $(UAU^* + VBV^*)x \neq 0$ for all unit vector $x \in \mathbb{C}^n$. So, $\operatorname{rank}(UAU^* + VBV^*) = n$ for all unitary $U, V \in M_n$.

Conversely, suppose $\operatorname{rank}(UAU^* + VBV^*) = k$ for all unitary $U, V \in M_n$. Assume that neither A nor B is scalar. If $k < n$ then by Theorem 2.1, there is μ such that $\operatorname{rank}(A - \mu I_n) + \operatorname{rank}(B + \mu I_n) = k$. Since neither A nor B is a scalar, $\operatorname{rank}(A - \mu I_n) < k$ and $\operatorname{rank}(B + \mu I_n) < k$. By Proposition 2.5, there are unitary matrices $U, V \in M_n$ such that $\operatorname{rank}(UAU^* + VBV^*) < k$, which is a contradiction. Thus, $n = k$. By Theorem 2.9, condition (b) holds. $\qquad\square$

3 Determinants

Let $A, B \in M_n$ with eigenvalues $a_1, \ldots, a_n$, and $b_1, \ldots, b_n$, respectively. In this section we study the properties of $\Delta(A, B)$ and $P(A, B)$. For notational convenience and easy description of the results and proofs, we consider the sets

$$D(A, B) = \Delta(A, -B) = \{\det(X - Y) : X \in \mathcal{U}(A), \ Y \in \mathcal{U}(B)\}$$

and

$$Q(A, B) = P(A, -B) = \left\{ \prod_{j=1}^{n}(a_j - b_{\sigma(j)}) : \sigma \text{ is a permutation of } \{1, \ldots, n\} \right\}.$$

It is easy to translate the results on $D(A, B)$ and $Q(A, B)$ to those on $\Delta(A, B)$ and $P(A, B)$, and vice versa.

For any permutation $(\sigma(1), \ldots, \sigma(n))$ of $(1, \ldots, n)$, there are unitary matrices U and V such that UAU^* and VBV^* are upper triangular matrices with diagonal entries $a_1, \ldots, a_n$ and $b_{\sigma(1)}, \ldots, b_{\sigma(n)}$, respectively. It follows that

$$Q(A, B) \subseteq D(A, B).$$

The elements in $Q(A, B)$ are called σ-points.

Note also that if we replace (A, B) by $(UAU^* - \mu I_n, VBV^* - \mu I_n)$ for any $\mu \in \mathbb{C}$ and unitary $U, V \in M_n$, the sets $Q(A, B)$ and $D(A, B)$ will be the same. Moreover, $D(B, A) = (-1)^n D(A, B)$ and $Q(B, A) = (-1)^n Q(A, B)$.

The following result can be found in [9].

Theorem 3.1 *Suppose $A, B \in M_2$ have eigenvalues α_1, α_2 and β_1, β_2, respectively, and suppose $A - (\operatorname{tr} A/2)I_2$ and $B - (\operatorname{tr} B/2)I_2$ have singular values $a \geq b \geq 0$ and $c \geq d \geq 0$. Then $D(A, B)$ is an elliptical disk with foci $(\alpha_1 - \beta_1)(\alpha_2 - \beta_2)$ and $(\alpha_1 - \beta_2)(\alpha_2 - \beta_1)$ with length of minor axis equal to $2(ac - bd)$. Consequently, $D(A, B)$ is a singleton if and only if A or B is a scalar matrix; $D(A, B)$ is a nondegenerate line segment if and only if A and B are non-scalar normal matrices.*

In the subsequent discussion, let

$$W(A) = \{x^* A x : x \in \mathbb{C}^n, \ x^* x = 1\}$$

be the *numerical range* of $A \in M_n$.

3.1 Matrices whose determinantal ranges have empty interior

Theorem 3.2 *Let $A, B \in M_n$ with $n \geq 3$. Then $D(A, B) = \{\delta\}$ if and only if one of the following holds.*

(a) $\delta = 0$, and there is $\mu \in \mathbb{C}$ such that $\operatorname{rank}(A - \mu I_n) + \operatorname{rank}(B - \mu I_n) < n$.

(b) $\delta \neq 0$, one of the matrices A or B is a scalar matrix, and $\det(A - B) = \delta$.

Proof. If (a) or (b) holds, then clearly $D(A, B)$ is a singleton. If $D(A, B) = \{0\}$, then condition (a) holds by Corollary 2.3.

Suppose $D(A, B) = \{\delta\}$ with $\delta \neq 0$. We claim that A or B is a scalar matrix. Suppose A and B have eigenvalues $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$, respectively. Assume that A has at least two distinct eigenvalues a_1, a_2 and B also has two distinct eigenvalues b_1, b_2. Then $\prod_{j=1}^n (a_j - b_j)$ and $(a_1 - b_2)(a_2 - b_1)\prod_{j=3}^n (a_j - b_j)$ will be two distinct σ-points, which is a contradiction because $Q(A, B) \subseteq D(A, B)$ is also a singleton.

So, we have $a_1 = \cdots = a_n$ or $b_1 = \cdots = b_n$. We may assume that the latter case holds; otherwise, interchange the roles of A and B. Suppose neither A nor B is a scalar matrix. Applying a suitable unitary similarity transform to B, we may assume that B is in upper triangular form with nonzero $(1, 2)$ entries. Also, we may assume that A is in upper triangular form so that the leading two-by-two matrix is not a scalar matrix. If $A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$ and $B = \begin{pmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{pmatrix}$ with $A_{11}, B_{11} \in M_2$, then $D(A_{11}, B_{11})$ is a non-degenerate circular disk by Theorem 3.1. Since

$$\{\det(A_{22} - B_{22})\delta : \delta \in D(A_{11}, B_{11})\} \subseteq D(A, B),$$

we see that $D(A, B)$ cannot be a non-zero singleton. $\square$

Theorem 3.3 *Suppose $A, B \in M_n$ are such that $D(A, B)$ is not a singleton. The following conditions are equivalent.*

(a) $D(A, B)$ *has empty interior.*

(b) $D(A, B)$ *is a non-degenerate line segment.*

(c) $Q(A, B)$ *is not a singleton, i.e., there are at least two distinct σ-points, and one of the following conditions holds.*

    (c.1) *A and B are normal matrices with eigenvalues lying on the same straight line or the same circle.*

    (c.2) *There is $\mu \in \mathbb{C}$ such that one of the matrices $A - \mu I_n$ or $B - \mu I_n$ is rank one normal, and the other one is invertible normal so that the inverse matrix has collinear eigenvalues.*

    (c.3) *There is $\mu \in \mathbb{C}$ such that $A - \mu I_n$ is unitarily similar to $\tilde{A} \oplus 0_{n-k}$ and $B - \mu I_n$ is unitarily similar to $0_k \oplus \tilde{B}$ so that $\tilde{A} \in M_k$ and $\tilde{B} \in M_{n-k}$ are invertible.*

In [3], the authors conjectured that for normal matrices $A, B \in M_n$, if $D(A, B)$ is contained in a line $\mathcal{L}$, then $\mathcal{L}$ must pass through the origin. Using the above result, we see that the conjecture is not true. For example, if $A = \operatorname{diag}(1, 1 + i, 1 - i)^{-1}$ and $B = \operatorname{diag}(-1, 0, 0)$, then $D(A, B)$ is a straight line segment joining the points $1 - i/2$ and $1 + i/2$; see Corollary 3.12. This shows that

$D(A, B)$ can be a subset of a straight line not passing through the origin. Of course, Theorem 3.3 covers more general situations. In (c.1), the line segment $D(A, B)$ and the origin are collinear; in (c.3) the line segment $D(A, B)$ has endpoints 0 and $(-1)^{n-k} \det(\tilde{A}) \det(\tilde{B})$; in (c.2) the line segment and the origin may or may not be collinear.

Since, $D(A, B) = (-1)^n D(B, A)$, $D(A, B)$ has empty interior (is a line segment) if and only if $D(B, A)$ has empty interior (is a line segment). This symmetry will be used in the following discussion. We establish several lemmas to prove the theorem.

Given $a, b, c, d \in \mathbb{C}$, with $ad - bc \neq 0$, let $f(z) = (az + b)/(cz + d)$ be the *fractional linear transform* on $\mathbb{C} \setminus \{-d/c\}$ ($\mathbb{C}$, if $c = 0$). If $A \in M_n$ is such that $cA + dI_n$ is invertible, one can define $f(A) = (aA + bI_n)(cA + dI_n)^{-1}$. The following is easy to verify.

Lemma 3.4 *Suppose $A, B \in M_n$, and $f(z) = (az + b)/(cz + d)$ is a fractional linear transform such that $f(A)$ and $f(B)$ are well defined. Then*

$$D(f(A), f(B)) = \det((cA + dI_n)(cB + dI_n))^{-1}(ad - bc)^n D(A, B).$$

Lemma 3.5 *Let $A, B \in M_n$ with eigenvalues $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$, respectively. If $D(A, B)$ has empty interior, then*

$$D(A, B) \;\; = D(A, \operatorname{diag}(b_1, \ldots, b_n)) = D(\operatorname{diag}(a_1, \ldots, a_n), B)$$

$$= D(\operatorname{diag}(a_1, \ldots, a_n), \operatorname{diag}(b_1, \ldots, b_n)).$$

Proof. Assume that $D(A, B)$ has empty interior. Applying a unitary similarity transform we assume that $A = (a_{rs})$ and $B = (b_{rs})$ are an upper triangular matrices. For any unitary matrix $V \in M_n$, let $D = (d_{rs}) = VBV^*$. For any $\xi \in [0, 2\pi)$, let $U_\xi = [e^{i\xi}] \oplus I_{n-1}$. Denote by X_{rs} the $(n-1) \times (n-1)$ matrices obtained from $X \in M_n$ by deleting its rth row and the sth column. Expanding the determinant $\det(U_\xi A U_\xi^* - D)$ along the first row yields

$$\det(U_\xi A U_\xi^* - D)$$

$$= \; (a_{11} - d_{11}) \det(A_{11} - D_{11}) + \sum_{j=2}^{n} (-1)^{j+1}(e^{i\xi} a_{1j} - d_{1j}) \det(A_{1j} - D_{1j})$$

$$= \; \left[(a_{11} - d_{11}) \det(A_{11} - D_{11}) - \sum_{j=2}^{n} (-1)^{j+1} d_{1j} \det(A_{1j} - D_{1j}) \right] + e^{i\xi} \gamma,$$

where $\gamma = \sum_{j=2}^{n} (-1)^{j+1} a_{1j} \det(A_{1j} - D_{1j})$. Thus,

$$C(A, VBV^*) = \{\det(U_\xi A U_\xi^* - VBV^*) : \xi \in [0, 2\pi)\}$$

is a circle with radius $|\gamma|$. If $|\gamma| \neq 0$, i.e., $C(A, VBV^*)$ is a non-degenerate circle. Repeating the construction in the previous paragraph on $V = I_n$, we get a degenerate circle

$$C(A, B) = \{\det(A - B)\}.$$

Since the unitary group is path connected, there is a continuous function $t \mapsto V_t$ for $t \in [0, 1]$ so that $V_0 = V$ and $V_1 = I_n$. Thus, we have a family of circles $C(A, V_t B V_t^*)$ in $D(A, B)$ transforming $C(A, V B V^*)$ to $C(A, B)$. Hence, all the points inside $C(A, V B V^*)$ belong to $D(A, B)$. Thus, $D(A, B)$ has non-empty interior. As a result,

$$\gamma = \sum_{j=2}^{n} (-1)^{j+1} a_{1j} \det(A_{1j} - D_{1j}) = 0,$$

and

$$\det(A - V B V^*)$$
$$= \det(A - D) = (a_{11} - d_{11}) \det(A_{11} - D_{11}) - \sum_{j=2}^{n} (-1)^{j+1} d_{1j} \det(A_{1j} - D_{1j})$$
$$= \det(A_1 - D) = \det(A_1 - V B V^*),$$

where A_1 is the matrix obtained from A by changing all the non-diagonal entries in the first row to zero. It follows that $D(A, B) = D(A_1, B)$. Inductively, by expanding the determinant $\det(U_j A_j U_j^* - D)$ along the $(j+1)$th row with $U_j = I_j \oplus [e^{i\xi}] \oplus I_{n-j-1}$, we conclude that $D(A_j, B) = D(A_{j+1}, B)$ where A_{j+1} is the matrix obtained from A_j by changing all the non-diagonal entries in the $(j+1)$-th row to zero. Therefore,

$$D(A, B) = D(A_1, B) = D(A_2, B) = \cdots = D(A_{n-1}, B) = D(\operatorname{diag}(a_{11}, \ldots, a_{nn}), B).$$

Note that $a_{11}, \ldots, a_{nn}$ are the eigenvalues of A as A is in the upper triangular form. Similarly, we can argue that $D(A, B) = D(A, \operatorname{diag}(b_1, \ldots, b_n))$. Now, apply the argument to $D(\operatorname{diag}(a_1, \ldots, a_n), B)$ to get the last set equality. $\qquad \square$

Lemma 3.6 *Let $A = \hat{A} \oplus 0_{n-k}$, where $\hat{A} \in M_k$ with $k \in \{1, \ldots, n-1\}$ is upper triangular invertible. If $B \in M_n$ has rank $n - k$, then*

$$D(A, B) = \{(-1)^{n-k} \det(\hat{A}) \det(X^* B X) : X \text{ is } n \times (n-k), \ X^* X = I_{n-k}\}.$$

If $B = 0_k \oplus \hat{B}$ so that $\hat{B} \in M_{n-k}$ is invertible, then $D(A, B)$ is the line segment joining 0 and $(-1)^{n-k} \det(\hat{A}) \det(\hat{B})$.

Proof. Suppose $A = (a_{rs})$ has columns $A_1, \ldots, A_n$, and $U^* B U$ has columns $B_1, \ldots, B_n$. Let C be obtained from $A - U^* B U$ by removing the first column, and let B_{22} be obtained from C by removing the first row. By linearity of the determinant function on the first column,

$$\det(A - U^* B U) = \det([A_1 | C]) - \det([B_1 | C]) = -a_{11} \det(B_{22}) + 0,$$

because $[B_1 | C]$ has rank at most $n - 1$. Inductively, we see that

$$\det(A - U^* B U) = (-1)^{n-k} \det(\hat{A}) \det(Y)$$

where Y is obtained from $U^* B U$ by removing its first k rows and first k columns.

Now if $B = 0_k \oplus \hat{B}$ so that $\hat{B} \in M_{n-k}$ is invertible, then the set $\{\det(X^* B X) : X^* X = I_{n-k}\}$ is a line segment joining 0 and $\det(\hat{B})$; e.g., see [6]. Thus, the last assertion follows. $\qquad \square$

Lemma 3.7 *Suppose A and B are not both normal such that $A \oplus B$ has exactly n nonzero eigenvalues. If $D(A, B)$ has no interior point, then there exist $\mu \in \mathbb{C}$ and $0 \leq k \leq n$ such that $A - \mu I_n$ and $B - \mu I_n$ are unitarily similar to matrices of the form $\tilde{A} \oplus 0_{n-k}$ and $0_k \oplus \tilde{B}$ for some invertible matrices $\tilde{A} \in M_k$ and $\tilde{B} \in M_{n-k}$.*

Proof. Suppose A or B is a scalar matrix, say $B = \mu I_n$. Under the given hypothesis, $A - \mu I_n$ is invertible and $B - \mu I_n = 0$. Thus, the result holds for $k = n$. In the rest of the proof, assume that neither A nor B is a scalar matrix.

We may assume by Theorem 3.2 (b) that

$$A = (a_{rs}) = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} \quad \text{and} \quad B = (b_{rs}) = \begin{pmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{pmatrix} \tag{3.1}$$

such that $A_{11}, B_{11} \in M_m$ and $A_{22}, B_{22} \in M_{n-m}$ are upper triangular matrices so that A_{11}, B_{22} are invertible, and A_{22}, B_{11} are nilpotent.

If $m = 0$, then $A = A_{11}$ is nilpotent and $B = B_{22}$ is invertible. We are going to show that $A = 0$. Hence, the lemma is satisfied with $k = 0$.

Suppose $A \neq 0$. We may assume that $a_{12} \neq 0$. Let $X = \begin{pmatrix} 0 & a_{12} \\ 0 & 0 \end{pmatrix}$ and $Y = \begin{pmatrix} b_{11} & b_{12} \\ 0 & b_{22} \end{pmatrix}$. Since B is not a scalar matrix, we may assume that Y is not a scalar matrix either. Then $D(X, Y)$ is a non-degenerate elliptical disk and

$$\left\{ \frac{(-1)^n \mu \det(B)}{b_{11} b_{22}} : \mu \in D(X, Y) \right\} \subseteq D(A, B).$$

Therefore, $D(A, B)$ has non-empty interior, a contradiction. Similarly, if $m = n$, then $B = 0$. Hence, we may assume that $1 \leq m < n$ in the following.

We are going to show that $A_{12} = 0 = B_{12}$ in (3.1). To this end, let $X, Y \in M_2$ be the principal submatrices of A and B lying in rows and columns m and $m + 1$. If $a_{m,m+1} \neq 0$ or $b_{m,m+1} \neq 0$, then

$$-(a_{m,m} \, b_{m+1,m+1})^{-1} \det(A - B) D(X, Y)$$

is an elliptical disk in $D(A, B)$, which is impossible. Next, we show that $a_{m-1,m+1} = 0 = b_{m-1,m+1}$. If it is not true, let $X, Y \in M_2$ be the principal submatrices of A and B lying in rows and columns $m - 1$ and $m + 1$. For any unitary $U, V \in M_2$, let $\gamma = \det(UXU^* - VYV^*)$. Construct $\hat{U}$ (respectively, $\hat{V}$) from I_n by changing the principal submatrix at rows and columns $m - 1$ and $m + 1$ by U (respectively, V). Then $\hat{U} A \hat{U}^*$ is still in upper triangular block form so that its leading $(m-2) \times (m-2)$ principal submatrix and its trailing $(n-m-1) \times (n-m-1)$ principal submatrix are the same as A. Moreover, since we have shown that $a_{m,m+1} = 0 = b_{m,m+1}$, the principal submatrix of $\hat{U} A \hat{U}^*$ lying in rows $m - 1, m, m + 1$ has the form

$$\begin{pmatrix} * & * & * \\ 0 & a_{mm} & 0 \\ * & * & * \end{pmatrix}.$$

A similar result is true for $\hat{V} B \hat{V}^*$. Hence,

$$\det(\hat{U} A \hat{U}^* - \hat{V} B \hat{V}^*) = -\det(A - B)\gamma/(a_{m-1,m-1} \, b_{m+1,m+1}).$$

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As a result, $D(A, B)$ contains the set

$$-(a_{m-1,m-1} \, b_{m+1,m+1})^{-1} \det(A - B) D(X, Y),$$

which is an elliptical disk. This is a contradiction.

Next, we can show that $a_{m-2,m+1} = 0 = b_{m-2,m+1}$ and so forth, until we show that $a_{1,m+1} = 0 = b_{1,m+1}$. Note that it is important to show that $a_{j,m+1} = 0 = b_{j,m+1}$ in the order of $j = m, m-1, \ldots, 1$. Remove the $(m+1)$th row and column from B and A to get $\hat{B}$ and $\hat{A}$. Then $a_{m+1,m+1} D(\hat{A}, \hat{B})$ is a subset of $D(A, B)$ and has no interior point. An inductive argument shows that the $(1, 2)$ blocks of A and B are zero. Thus, $A_{12} = 0 = B_{12}$.

If B_{11} and A_{22} are both equal to zero, then the desired conclusion holds. Suppose B_{11} or A_{22} is non-zero. By symmetry, we may assume that $B_{11} \neq 0$.

Claim   (1) $A_{11} = \mu I_m$ and   (2) $B_{22} = \mu I_{n-m}$ for some $\mu \neq 0$.

If this claim is proved, then $A - \mu I_n = 0_k \oplus (A_{22} - \mu I_{n-m})$ and $B - \mu I_n = (B_{11} - \mu I) \oplus 0_{n-k}$. Thus, the result holds with $k = n - m$.

To prove our claim, suppose $B_{11} \neq 0$. Then $m \geq 2$ and we may assume that its leading 2×2 submatrix B_0 is a nonzero strictly upper triangular matrix. If A_{11} is non-scalar, then we may assume that its leading 2×2 submatrix A_0 is non-scalar. But then $D(A_0, B_0)$ will generate an elliptical disk in $D(A, B)$, which is impossible. So, $A_{11} = \mu I_m$ for some $\mu \neq 0$. This proves (1).

Now we prove (2). Suppose $B_{22} \neq \mu I_{m-n}$ Thus, $n - m \geq 2$ and we may assume that 4×4 submatrices of A and B lying at rows and columns labeled by $m-1, m, m+1, m+2$ have the forms

$$A' = \mu I_2 \oplus \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad B' = B_1' \oplus B_2'$$

with $\alpha \in \mathbb{C}$, a nonzero 2×2 nilpotent matrix B_1' and a 2×2 matrix B_2' such that $B_2' \neq \mu I_2$. Let $P = [1] \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus [1]$, $V = V_1 \oplus V_2$ with unitary $V_1, V_2 \in M_2$. Then $\det(PA'P^t - VB'V^*) = \delta_1 \delta_2$ with

$$\delta_1 = \det(\operatorname{diag}(\mu, 0) - V_1 B_1' V_1^*) \qquad \text{and} \qquad \delta_2 = \det(\operatorname{diag}(\mu, 0) - V_2 B_2' V_2^*).$$

Since $B_2' \neq \mu I_2$, by Theorem 3.1, we can choose some unitary V_2 such that $\delta_2 \neq 0$. Also as B_1' is nonzero nilpotent, by Theorem 3.1, one can vary the unitary matrices V_1 to get all values δ_1 in the non-degenerate circular disks $D(\operatorname{diag}(\mu, 0), B_1')$. Hence,

$$(\mu^2 b_{m+1,m+1} \, b_{m+2,m+2})^{-1} \delta_2 \det(A - B) \, D(\operatorname{diag}(\mu, 0), B_1') \subseteq D(A, B)$$

so that $D(A, B)$ also has non-empty interior, which is the desired contradiction. □

Lemma 3.8 *Let $A, B \in M_n$ be normal matrices. Then $Q(A, B) = \{\delta\}$ if and only if one of the following holds.*

(a) *$\delta = 0$, and $A \oplus B$ has an eigenvalue with multiplicity at least $n + 1$.*

(b) *$\delta \neq 0$, and one of the matrix A or B is a scalar matrix, and $\det(A - B) = \delta$.*

Proof. Clearly if (a) or (b) holds, then $Q(A, B)$ is a singleton. Let A and B have eigenvalues $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$, respectively.

Suppose $Q(A, B) = \{\delta\}$. If both A and B are not scalar matrices, then A has at least two distinct eigenvalues, say a_1, a_2 and B also has two distinct eigenvalues, say b_1, b_2. Then $\delta = \prod_{i=1}^{n}(a_i - b_i) = (a_1 - b_2)(a_2 - b_1) \prod_{i=3}^{n}(a_i - b_i)$ implies that $\delta = 0$.

Now we claim that condition (a) holds. Suppose not, then every eigenvalue of $A \oplus B$ has multiplicity at most n. For $k = 1, 2, \ldots, n$, let $S_k = \{i : b_i \neq a_k\}$. Suppose $1 \leq k_1 < k_2 < \cdots < k_m \leq n$. Then $i \notin \cup_{j=1}^{m} S_{k_j}$ if and only if $b_i = a_{k_1} = a_{k_2} = \cdots = a_{k_m}$. Therefore, there are at most $n - m$ i not in $\cup_{j=1}^{m} S_{k_j}$. Hence, $\cup_{j=1}^{m} S_{k_j}$ contains at least m elements. By the theorem of P. Hall [19], there exist $i_k \in S_k$, $k = 1, \ldots, n$, with $i_k \neq i_{k'}$ for $k \neq k'$. Thus, $\prod_{k=1}^{n}(a_k - b_{i_k}) \neq 0$, which contradicts the fact that $\delta = 0$. $\square$

Lemma 3.9 *Suppose $A, B \in M_n$ are normal matrices such that A has at least three distinct eigenvalues, each eigenvalue of B has multiplicity at most $n - 2$, and each eigenvalue of $A \oplus B$ has multiplicity at most $n - 1$. Then there are three distinct eigenvalues a_1, a_2, a_3 of A satisfying the following condition.*

> *For any eigenvalue b of B with $b \notin \{a_1, a_2, a_3\}$, there exist eigenvalues b_1, b_2, b_3 of B with $b_1 \notin \{b_2, b_3\}$, $b_3 = b$, and the remaining eigenvalues can be labeled so that $\prod_{j=4}^{n}(a_j - b_j) \neq 0$.*

Moreover, if A has more than three distinct eigenvalues, and B has exactly two distinct eigenvalues, then we can replace a_3 by any eigenvalue of A different from a_1, a_2, a_3, and get the same conclusion.

Proof. Let A and B have k distinct common eigenvalues $\gamma_1, \gamma_2, \ldots, \gamma_k$ so that γ_j has multiplicity m_j in the matrix $A \oplus B$ for $j = 1, \ldots, k$, with $m_1 \geq \cdots \geq m_k$. By our assumption, $n - 1 \geq m_1$.

The choices for a_i and b_i depend on k. We illustrate the different cases in the following table.

	$k = 0$	$k = 1$	$k = 2$	$k \geq 3$
a_1	$*$	γ_1	γ_1	γ_1
a_2	$*$	$*$	γ_2	γ_2
a_3	$*$	$*$	$*$	γ_3
b_1	$\neq b$	γ_1	γ_1	γ_1
b_2	$\neq b_1$	$\neq b_1$	γ_2	γ_2
b_3	b	b	b	b

where $*$ denotes any choice subject to the condition that a_1, a_2, a_3 are distinct eigenvalues of A. For any eigenvalue b of B with $b \notin \{a_1, a_2, a_3\}$, set $b_3 = b$ and choose $b_1 = \gamma_1$ if $k \geq 1$ and b_1 to be any eigenvalue of B not equal to b. Since the multiplicity of b_1 is $\leq n - 2$, there is always a third eigenvalue b_2 of B, with $b_2 \neq b_1$. Furthermore, we can choose $b_2 = \gamma_2$ if $k \geq 2$.

Use the remaining eigenvalues of A and B to construct the matrices $\hat{A} = \text{diag}(a_4, \ldots, a_n)$ and $\hat{B} = \text{diag}(b_4, \ldots, b_n)$. By Lemma 3.8, the proof will be completed if we can prove the following:

Claim *If μ is a common eignevalue of $\hat{A}$ and $\hat{B}$ then the multiplicity of μ in the matrix $\hat{A} \oplus \hat{B}$ is at most $n - 3$.*

To verify our claim, let μ be a common eigenvalue of $\hat{A}$ and $\hat{B}$. Then $\mu = \gamma_r$ with $r \in \{1, \ldots, k\}$.

If $r \in \{1, 2\}$, then two of the entries $(a_1, a_2, a_3, b_1, b_2, b_3)$ equals γ_r by our construction. Since $n - 1 \geq m_r$, the multiplicity of γ_r in $\hat{A} \oplus \hat{B}$ equals $m_r - 2 \leq n - 3$.

If $r = 3$, then $b_3 \neq \gamma_i$ for $i = 1, 2, 3$. Thus, $m_3 \leq \dfrac{m_1 + m_2 + m_3}{3} \leq \left[\dfrac{2n - 1}{3} \right] \leq n - 2$, where $[t]$ is the integral part of the real number t. Since one of the entries in $(a_1, a_2, a_3, b_1, b_2, b_3)$ equals γ_3, we see that the multiplicity of γ_3 in $\hat{A} \oplus \hat{B}$ equals $m_3 - 1 \leq n - 3$.

Suppose $r = 4$. If $n = 4$ then $(a_1, a_2) = (b_1, b_2) = (\gamma_1, \gamma_2)$, $a_3 = \gamma_3$ and $b_3 = \gamma_4$ by our construction. Thus, $a_4 - b_4 = \gamma_4 - \gamma_3 \neq 0$. If $n \geq 5$, then the multiplicity of γ_r in $\hat{A} \oplus \hat{B}$ is at most $m_4 \leq \left[\dfrac{2n}{4} \right] \leq n - 3$.

If $r \geq 5$, then $n \geq r \geq 5$ and the multiplicity of γ_r in $\hat{A} \oplus \hat{B}$ is at most $m_r \leq \dfrac{2n}{5} \leq n - 3$.

By the above arguments, the claim holds.

Note that if B has exactly two distinct eigenvalues, then $k \leq 2$ and a_3 can be chosen to be any eigenvalue different from a_1, a_2 in our construction. Thus, the last assertion of the lemma follows. $\square$

Lemma 3.10 *Let $A = \operatorname{diag}(a_1, a_2, a_3)$ and $B = \operatorname{diag}(b_1, b_2, b_3)$ with $a_j \neq a_k$ for $1 \leq j < k \leq 3$ and $b_1 \neq b_2$. Suppose $D(A, B)$ has empty interior. Then a_1, a_2, a_3, b_3 are either concyclic or collinear.*

Proof. By Lemma 3.4, we may apply a suitable fractional linear transform and assume that $(a_1, a_2, a_3) = (a, 1, 0)$ with $a \in \mathbb{R} \setminus \{0, 1\}$. By the result in [7], if $U = (u_{rs}) \in M_3$ is unitary and $S_U = (|u_{rs}|^2)$, then

$$\det(UAU^* - B) = \det(A) + (-1)^3 \det(B) - (b_1, b_2, b_3) S_U (0, 0, a)^t + (b_2 b_3, b_1 b_3, b_1 b_2) S_U (a, 1, 0)^t.$$

Let

$$C = (a, 1, 0)^t (b_2 b_3, b_1 b_3, b_1 b_2) - (0, 0, a)^t (b_1, b_2, b_3).$$

Then

$$\det(UAU^* - B) = \det(A) + (-1)^3 \det(B) + \operatorname{tr}(CS_U).$$

It follows that the set

$$\mathcal{R} = \{\operatorname{tr}(C(|u_{rs}|^2)) : (u_{rs}) \text{ is unitary}\}$$

has empty interior. Let S_0 be the 3×3 matrix with all entries equal to $1/3$. For $\alpha, \beta \in [0, 1/5]$, let

$$S = S(\alpha, \beta) = \begin{pmatrix} \frac{1}{3} - \alpha & \frac{1}{3} + \beta & \frac{1}{3} + (\alpha - \beta) \\ \frac{1}{3} + \alpha & \frac{1}{3} - \beta & \frac{1}{3} - (\alpha - \beta) \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} = S_0 + \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} (-\alpha \quad \beta \quad \alpha - \beta).$$

Since

$$\frac{4}{15} \leq \sqrt{\frac{1}{9} - \alpha^2}, \ \sqrt{\frac{1}{9} - \beta^2}, \ \sqrt{\frac{1}{9} - (\alpha - \beta)^2} \leq \frac{1}{3},$$

by the result in [1], there is a unitary (u_{rs}) such that $(|u_{rs}|^2) = S$. Direct calculation shows that

$$\operatorname{tr}(CS) = \operatorname{tr}(CS_0) + (b_1 - b_2)[\alpha(ab_3 + a) - \beta(b_3 + a)].$$

The set $\mathcal{R}$ having empty interior implies that $(ab_3 + a)$ and $(b_3 + a)$ are linearly independent over reals, which is possible only when b_3 is real. Thus, $\{a_1, a_2, a_3, b_3\} \subseteq \mathbb{R}$ and the result follows. $\qquad \square$

Proof of Theorem 3.3. The implication (b) $\Rightarrow$ (a) is clear.

Suppose (c) holds. If (c.1) holds, then $D(A, B)$ is a line segment on a line passing through origin as shown in [3].

If (c.2) holds, then we can assume that $A - \mu I_n = \mathrm{diag}\,(a, 0, \dots, 0)$, and $B - \mu I_n$ has full rank and the eigenvalues of $(B - \mu I_n)^{-1}$ are collinear. We may replace (A, B) by $(A - \mu I_n, B - \mu I_n)$ and assume that $\mu = 0$. Since B^{-1} is normal with collinear eigenvalues, the numerical range $W(B^{-1})$ of B^{-1} is a line segment.

Let $U \in M_n$ be unitary, and $U^* B U = \begin{pmatrix} b_{11} & * \\ * & B_{22} \end{pmatrix}$ with $B_{22} \in M_{n-1}$. Then $\det(B_{22})$ is the $(1, 1)$ entry of $\det(B) U^* B^{-1} U$. Hence, $\det(B_{22})/\det(B) \in W(B^{-1})$. Thus,

$$\det(UAU^* - B) = \det(A - U^* BU) = a(-1)^{n-1} \det(B_{22}) + (-1)^n \det(U^* BU)$$

$$\in \quad a(-1)^{n-1} \det(B) W(B^{-1}) + (-1)^n \det(B).$$

If (c.3) holds, then $D(A, B)$ is the line segment joining 0 and $(-1)^{n-k} \det(\tilde{A}) \det(\tilde{B})$ by Lemma 3.6. Thus, we have (c) $\Rightarrow$ (b).

Finally, suppose (a) holds, i.e., $D(A, B)$ has empty interior. Since $D(A, B)$ is not a singleton, neither A nor B is a scalar matrix. Suppose A and B have eigenvalues $a_1, \dots, a_n$ and $b_1, \dots, b_n$. Let $A' = \mathrm{diag}\,(a_1, \dots, a_n)$ and $B' = \mathrm{diag}\,(b_1, \dots, b_n)$. By Lemma 3.5, $D(A', B') = D(A, B)$ and hence $D(A', B')$ is not a singleton. It then follows from Corollary 2.2 and Lemma 3.8 that $Q(A', B')$ is not a singleton, so as $Q(A, B)$. Now we show that one of (c.1) – (c.3) holds.

Suppose A and B have k distinct common eigenvalues $\gamma_1, \gamma_2, \dots, \gamma_k$ such that γ_j has multiplicity m_j in the matrix $A \oplus B$ for $j = 1, \dots, k$, with $m_1 \geq \cdots \geq m_k$. Since $Q(A, B) = Q(A', B') \neq \{0\}$, we have $m_1 \leq n$.

If $m_1 = n$, then $(A - \gamma_1 I) \oplus (B - \gamma_1 I)$ has exactly n nonzero eigenvalues, and hence (c.3) follows from Lemma 3.7.

Suppose $k = 0$ or $m_j \leq n - 1$ for all $1 \leq j \leq k$. We claim that both A and B are normal. If it is not true, we may assume that A is not normal. Otherwise, interchange the roles of A and B. Then we may assume that A is in upper triangular form with nonzero $(1, 2)$ entry by the result in [27]. Suppose A has diagonal entries $a_1, \dots, a_n$. Let A_1 be the leading 2×2 principal submatrix of A. We can also assume that B is upper triangular with diagonal $b_1, \dots, b_n$, where $b_1 \neq b_2$ satisfies the following additional assumptions:

(1) If $\{a_1, a_2\} \cap \{\gamma_1, \dots, \gamma_k\} = \emptyset$, then b_1 and b_2 are chosen so that $\gamma_j \in \{b_1, b_2\}$ for $1 \leq j \leq \min\{k, 2\}$.

(2) If $\{a_1, a_2\} \cap \{\gamma_1, \dots, \gamma_k\} \neq \emptyset$, then b_1 and b_2 are chosen so that $\gamma_j \in \{a_1, a_2, b_1, b_2\}$ for $1 \leq j \leq \min\{k, 3\}$.

Then $b_3, \dots, b_n$ can be arranged so that $p = \prod_{j=3}^n (a_j - b_j) \neq 0$. It follows from Theorem 3.1 that $\{p\delta : \delta \in D(A_1, \mathrm{diag}\,(b_1, b_2))\}$ is a nondegenerate elliptical disk in $D(A, B)$, which is a contradiction.

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Now, suppose both A and B are normal, and assume that $k = 0$ or $m_j \le n-1$ for all $1 \le j \le k$.

Case 1 *Suppose A or B has an eigenvalue with multiplicity $n-1$.*

Interchanging the role of A and B, if necessary, we may assume that $A = \text{diag}\,(a, 0, \dots, 0) + a_2 I_n$. We can further set $a_2 = 0$; otherwise, replace (A, B) by $(A - a_2 I_n, B - a_2 I_n)$. Since $m_j \le n-1$, we see that B is invertible. Moreover, for any unitary matrix U, let u be the first column of U, and let $\tilde{U}$ be obtained from U by removing u. Then

$$\det(A - U^* B U) = (-1)^n \left(\det(B) - a \det(\tilde{U}^* B \tilde{U}) \right).$$

Note that $\det(\tilde{U}^* B \tilde{U}) / \det(B)$ is the $(1,1)$ entry of $(U^* B U)^{-1}$, and equals $u^* B^{-1} u$. So,

$$D(A, B) = \{ (-1)^n \left(\det(B) - a \det(B) u^* B^{-1} u \right) : u \in \mathbb{C}^n, \ u^* u = 1 \}.$$

Since $D(A, B)$ is a set with empty interior and so is the numerical range $W(B^{-1})$ of B^{-1}. Thus, B^{-1} has collinear eigenvalues; see [21]. Hence condition (c.2) holds.

Case 2 *Suppose both A and B have two distinct eigenvalues and each eigenvalue of A and B has multiplicity at most $n-2$.*

Let A and B have two distinct eigenvalues, say, a_1, a_2 and b_1, b_2, respectively. We claim that a_1, a_2, b_1, b_2 are on the same straight line or circle, i.e., condition (c.1) holds. Suppose it is not true. Assume that a_1, a_2 and b_1 are not collinear and b_2 is not on the circle passing through a_1, a_2 and b_1. Then there is a factional linear transform $f(z)$ such that $f(A)$ and $f(B)$ has eigenvalues $1, 0$ and a, b, respectively, where $a \in \mathbb{R} \setminus \{1, 0\}$ and $b \notin \mathbb{R}$. By Lemma 3.4, $D(A, B)$ has empty interior if and only if $D(f(A), f(B))$ has empty interior. We may replace (A, B) by $(f(A), f(B))$ and assume that $A = \text{diag}\,(1, 0, 1, 0) \oplus A_2$, $B = \text{diag}\,(a, b, a, b) \oplus B_2$ with $\det(A_2 - B_2) \ne 0$. By Theorem 3.1,

$$D(\text{diag}\,(1,0), \text{diag}\,(a,b)) = \{ (1-s)a(b-1) + sb(a-1) : s \in [0,1] \} = \{ a(b-1) + s(a-b) : s \in [0,1] \}.$$

Hence, $D(A, B)$ contains the set

$$\begin{aligned}
\mathcal{R} &= \{ \det(A_2 - B_2)(a(b-1) + s(a-b))(a(b-1) + t(a-b)) : s, t \in [0,1] \} \\
&= \left\{ \det(A_2 + B_2)(a(b-1))^2 \left(1 + (s+t)\frac{a-b}{a(b-1)} + st \left(\frac{a-b}{a(b-1)} \right)^2 \right) : s, t \in [0,1] \right\}.
\end{aligned}$$

Note that $\{ (st, s+t) : s, t \in [0,1] \}$ has non-empty interior. Let $r = \dfrac{a-b}{a(b-1)}$. Then $(ar+1)b = a(1+r)$ and so r cannot be real. Therefore, the complex numbers r and r^2 are linearly independent over reals. Hence the mapping $(u, v) \mapsto 1 + ur + vr^2$ is an invertible map from $\mathbb{R}^2$ to $\mathbb{C}$. Thus, the set $\mathcal{R} \subseteq D(A, B)$ has nonempty interior, which is a contradiction.

Case 3 *Suppose each eigenvalue of A and B has multiplicity at most $n-2$ and one of the matrices has at least three distinct eigenvalues.*

Assume that A has at least three distinct eigenvalues. Otherwise, interchange the roles of A and B. By Lemma 3.9, there are three distinct eigenvalues of A, say, a_1, a_2, a_3, such that the conclusion of the lemma holds. Applying a fractional linear transformation, if necessary, we may assume that

a_1, a_2, a_3 are collinear. For any eigenvalue b of B with $b \notin \{a_1, a_2, a_3\}$ we can get b_1, b_2 and $b_3 = b$ satisfying the conclusion of Lemma 3.9. Therefore, $D(A, B)$ contains the set

$$\left\{ \delta \prod_{j=4}^{n} (a_j - b_j) : \delta \in D(\operatorname{diag}(a_1, a_2, a_3), \operatorname{diag}(b_1, b_2, b_3)) \right\}.$$

Since $D(A, B)$ has empty interior, Lemma 3.10 ensures that a_1, a_2, a_3 and b are collinear. Therefore, all eigenvalues of B lie on the line L passing through a_1, a_2, a_3.

Suppose B has three distinct eigenvalues. We can interchange the roles of A and B and conclude that the eigenvalues of A lie on the same straight line L. Suppose B has exactly two eigenvalues, and a is an eigenvalue of A with $a \notin \{a_1, a_2, a_3\}$ such that a is not an eigenvalue of B. By Lemma 3.9, we may replace a_3 by a and show that a_1, a_2, a and the two eigenvalues of B belong to the same straight line. Hence, all eigenvalues of A and B are collinear and (c.1) holds in this case. $\quad\square$

3.2 Sharp points

A boundary point μ of a compact set $\mathcal{S}$ in $\mathbb{C}$ is a sharp point if there exists $d > 0$ and $0 \le t_1 < t_2 < t_1 + \pi$ such that

$$\mathcal{S} \cap \{z \in \mathbb{C} : |z - \mu| \le d\} \subseteq \{\mu + \rho e^{i\xi} : \rho \in [0, d], \xi \in [t_1, t_2]\}.$$

It was shown [3, Theorem 2] that for two normal matrices $A, B \in M_n$ such that the union of the spectra of A and B has $2n$ distinct elements, a nonzero sharp point of $D(A, B)$ is a σ-point, that is, an element in $Q(A, B)$. More generally, we have the following.

Theorem 3.11 *Let $A, B \in M_n$. Every sharp point of $D(A, B)$ is a σ-point.*

Proof. Using the idea in [2], we can show that a nonzero sharp point $\det(UAU^* - B)$ is a σ-point as follows. For simplicity, assume $U = I_n$ so that $\det(A - B)$ is a sharp point of $D(A, B)$. For each Hermitian $H \in M_n$, consider the following one parameter curve in $D(A, B)$:

$$\xi \mapsto \det\left(A - e^{-i\xi H} B e^{i\xi H}\right) = \det(A - B)\left\{1 + i\xi \operatorname{tr}\left((A - B)^{-1}[H, B]\right) + O(\xi^2)\right\},$$

where $[X, Y] = XY - YX$. Since $\det(A - B)$ is a sharp point,

$$0 = \operatorname{tr}(A - B)^{-1}[H, B] = \operatorname{tr} H[B, (A - B)^{-1}] \qquad \text{for all Hermitian } H,$$

and hence
$$0 = [B, (A - B)^{-1}] = B(A - B)^{-1} - (A - B)^{-1}B.$$

Consequently, $0 = (A - B)B - B(A - B)$, equivalently, $AB = BA$. Thus, there exists a unitary V such that both VAV^* and VBV^* are in triangular form. As a result, $\det(A - B) = \det(V(A - B)V^*)$ is a σ-point.

Next, we refine the previous argument to treat the case when $\det(A - B) = 0$ is a sharp point. If the spectra of A and B overlap, then 0 is a σ-point. So, we assume that A and B has disjoint spectra. By Theorem 2.7, there is unitary U such that $UAU^* - B$ has rank $n - 1$. Assume $U = I_n$

so that $A - B$ has rank $n - 1$ and $\det(A - B) = 0$ is a sharp point. Then for any Hermitian H and $1 \le k \le n$,

$$\det(A - e^{-i\xi H} B e^{i\xi H}) = \det(A - B + i\xi(HB - BH) + \xi^2 M)$$

$$= \det(A - B) + i\xi \left(\sum_{j=1}^{n} r_{kj} s_{jk} \right) + O(\xi^2),$$

where $\mathrm{adj}(A - B) = R = (r_{pq})$ and $HB - BH = S = (s_{pq})$. Thus, for $k = 1, \ldots, n$ we have $\sum_{j=1}^{n} r_{kj} s_{jk} = 0$, and hence

$$0 = \mathrm{tr}\, RS = \mathrm{tr}\,(\mathrm{adj}(A - B)(HB - BH)) = \mathrm{tr}\, H[B\,\mathrm{adj}(A - B) - \mathrm{adj}(A - B)B]$$

for every Hermitian H. So, B and $\mathrm{adj}(A - B)$ commute. Since $A - B$ has rank $n - 1$, the matrix $\mathrm{adj}(A - B)$ has rank 1, and equals uv^t for some nonzero vectors u, v. Comparing the columns of the matrices on left and right sides of the equality $Buv^t = uv^t B$, we see that $Bu = bu$ for some $b \in \mathbb{C}$. Similarly, we have $Auv^t = uv^t A$ and hence $Au = au$ for some $a \in \mathbb{C}$. Consequently, $0 = (A - B)\mathrm{adj}(A - B) = (A - B)uv^t = (a - b)uv^t$. Thus, $a - b = 0$, i.e., the spectra of A and B overlap, which is a contradiction. $\qquad\square$

Clearly, if $D(A, B)$ is a line segment, then the end points are sharp points. By Theorem 3.3 and the above theorem, we have the following corollary showing that Marcus-Oliveira conjecture holds if $D(A, B)$ has empty interior.

Corollary 3.12 Let $A, B \in M_n$. If $D(A, B)$ has empty interior, then $D(A, B)$ equals the convex hull of $Q(A, B)$.

By Theorem 2.9, $0 \in D(A, B)$ if for every $\xi \in \mathbb{C}$, the singular values of $A - \xi I_n$ and $B - \xi I_n$ do not lie in two disjoint closed intervals in $[0, \infty)$. Following is a sufficient condition for $A, B \in M_n$ to have 0 as a sharp point of $D(A, B)$ in terms of $W(A)$ and $W(B)$.

Proposition 3.13 Let $A, B \in M_n$ be such that $0 \in D(A, B)$ and

$$W(A) \cup W(-B) \subseteq \{re^{i\xi} : r \ge 0, \ \xi \in (-\pi/(2n), \pi/(2n))\}.$$

Then

$$D(A, B) \subseteq \{re^{i\xi} : r \ge 0, \ \xi \in (-\pi/2, \pi/2)\}.$$

Proof. Note that for any unitary U and V, there is a unitary matrix R such that

$$R(UAU^* - VBV^*)R^* = (a_{pq}) - (b_{pq})$$

is in upper triangular form. Hence, $a_{pp} - b_{qq} = r_p e^{i\xi_p}$ with $r_p \ge 0$ and $\xi_p \in (-\pi/(2n), \pi/(2n))$ for $p = 1, \ldots, n$. So,

$\det(UAU^* - VBV^*) = \prod_{p=1}^{n}(a_{pp} - b_{pp}) = re^{i\xi}$ with $r \ge 0$ and $\xi \in (-\pi/2, \pi/2)$. $\qquad\square$

4 Further extensions

There are many related topics and problems which deserve further investigation.

One may ask whether the results can be extended to the sum of k matrices from k different unitary orbits for $k > 2$.

For the norm problem, in an unpublished manuscript Li and Choi have extended the norm bound result to k matrices $A_1, \ldots, A_k \in M_n$ for $k \geq 2$ to

$$\max\{\|X_1 + \cdots + X_k\| : X_j \in \mathcal{U}(A_j), j = 1, \ldots, k\}$$

$$= \min \left\{ \sum_{j=1}^{k} \|A_j - \mu_j I_n\| : \mu_j \in \mathbb{C}, j = 1, \ldots, k, \; \sum_{j=1}^{k} \mu_j = 0 \right\}.$$

However, we are not able to extend the maximum rank result in Section 2 to k matrices with $k > 2$ at this point. In any event, it is easy to show that for any $\mu_1, \ldots, \mu_k \in \mathbb{C}$ satisfying $\sum_{j=1}^{k} \mu_j = 0$,

$$\min\{\operatorname{rank}(X_1 + \cdots + X_k) : X_j \in \mathcal{U}(A_j), j = 1, \ldots, k\} \leq \max\{\operatorname{rank}(A_j - \mu_j I_n) : j = 1, \ldots, k\}.$$

It is challenging to determine all the possible rank values of matrices in $\mathcal{U}(A_1) + \cdots + \mathcal{U}(A_k)$.

For Hermitian matrices $A_1, \ldots, A_k$, there is a complete description of the eigenvalues of the matrices in $\mathcal{U}(A_1) + \cdots + \mathcal{U}(A_k)$; see [16]. Evidently, the set

$$\Delta(A_1, \ldots, A_k) = \left\{ \det\left(\sum_{j=1}^{k} X_j \right) : X_j \in \mathcal{U}(A_j), j = 1, \ldots, k \right\}$$

is a real line segment. When $k = 2$, the end points of the line segment have the form $\det(X_1 + X_2)$ for some diagonal matrices $X_1 \in \mathcal{U}(A_1)$ and $X_2 \in \mathcal{U}(A_2)$; see [15]. However, this is not true if $k > 2$ as shown in the following example.

Example 4.1 Let

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & -1 \\ -1 & 5 \end{bmatrix}.$$

Then for any unitary $U, V, W \in M_2$, the matrix $UAU^* + VBV^* + WCW^*$ is positive definite with eigenvalues $7 + d$ and $7 - d$ with $d \in [0, 7)$. Hence

$$\det(UAU^* + VBV^* + WCW^*) \leq 7^2 = \det(A + B + C).$$

Thus, the right end point of the set $\Delta(A, B, C)$ is not of the form $(a_1 + b_{\sigma(1)} + c_{\tau(1)})(a_2 + b_{\sigma(2)} + c_{\tau(2)})$ for permutations σ and τ of $(1, 2)$.

It is interesting to determine the set $\Delta(A_1, \ldots, A_k)$ for Hermitian, normal, or general matrices $A_1, \ldots, A_k \in M_n$. Inspired by the Example 4.1, we have the following observations.

1. Suppose $A_1, \ldots, A_k$ are positive semi-definite matrices. If there are unitary $U_1, \ldots, U_k$ such that

$$\sum_{j=1}^{k} U_j A U_j^* = \alpha I_n \tag{4.2}$$

   for some scalar α, then $\max \Delta(A_1, \ldots, A_k) = \alpha^n$. The necessary and sufficient conditions for (4.2) to hold can be found in [16].

2. Let A_1, A_2, A_3 be Hermitian matrices such that $\det(A_1 + A_2 + A_3) = \max \Delta(A_1, A_2, A_3)$. Then there exist unitary U and V such that $U A_1 U^*$ and $V(A_2 + A_3)V^*$ are diagonal and

$$\det(U A_1 U^* + V(A_2 + A_3)V^*) = \det(A_1 + A_2 + A_3).$$

   Proof. Let U be unitary matrix such that $U A_1 U^* = D_1$ is diagonal. Suppose $A_2 + A_3$ has eigenvalues $\lambda_1, \ldots, \lambda_n$. By the result of [15], there exists a permutation matrix P such that

$$\det(D_1 + P \mathrm{diag}\,(\lambda_1, \ldots, \lambda_n)P^*) \geq \det(A_1 + (A_2 + A_3)).$$

So if V is unitary such that $V(A_2 + A_3)V^* = P \mathrm{diag}\,(\lambda_1, \ldots, \lambda_n)P^*$, then

$$\det(A_1 + A_2 + A_3) = \max \Delta(A_1, A_2, A_3) \geq \det(U A_1 U^* + V(A_2 + A_3)V^*) \geq \det(A_1 + A_2 + A_3)$$

and hence the above inequalities become equalities. $\qquad\square$

Besides the unitary orbits, one may consider orbits of matrices under other group actions. For example, we can consider the usual similarity orbit of $A \in M_n$

$$\mathcal{S}(A) = \{SAS^{-1} : S \in M_n \text{ is invertible}\};$$

the unitary equivalence orbit of $A \in M_n$

$$\mathcal{V}(A) = \{UAV : U, V \in M_n \text{ are unitary}\};$$

the unitary congruence orbit of $A \in M_n$

$$\mathcal{U}^t(A) = \{UAU^t : U \in M_n \text{ is unitary}\}.$$

It is interesting to note that for any $A, B \in M_n$,

$$\begin{aligned} & \max\{\,\mathrm{rank}\,(U A U^* + V B V^*) : U, V \in M_n \text{ are unitary}\} \\ \leq\; & \max\{\,\mathrm{rank}\,(S A S^{-1} + T B T^{-1}) : S, T \in M_n \text{ are invertible}\} \\ \leq\; & \min\{\,\mathrm{rank}\,(A + \mu I_n) + \mathrm{rank}\,(B - \mu I_n) : \mu \in \mathbb{C}\}. \end{aligned}$$

By our result in Section 2, the inequalities are equalities.

One may consider the ranks, determinants, eigenvalues, and norms of the sum, the product, the Lie product, the Jordan product of matrices from different orbits; [17, 22, 28]. One may also consider similar problems for matrices over arbitrary fields or rings. Some problems are relatively easy. For example, the set $\{\det(S A S^{-1} + T B T^{-1}) : S, T \text{ are invertible}\}$ is either a singleton or $\mathbb{C}$. But some of them seem very hard. For example, it is difficult to determine when

$$0 \in \{\det(S_1 A_1 S_1^{-1} + S_2 A_2 S_2^{-1} + S_3 A_3 S_3^{-1}) : S_1, S_2, S_3 \text{ are invertible}\}.$$

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