MAPS PRESERVING MATRIX PAIRS WITH ZERO JORDAN PRODUCT

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ABSTRACT. We study maps on complex Hermitian and complex symmetric matrices which preserve zeros of Jordan product.

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1. PRELIMINARIES

One of the most active subjects in matrix theory during the past one hundred years is the linear preserver problem which concerns characterization of linear operators on matrix spaces that leave certain functions, subsets, relations, etc., invariant. For surveys of the topic we refer to the papers [18, 17, 26]. Such problems arise in most parts of mathematics. This is not surprising since in many cases the corresponding results provide important information on the automorphisms of the underlying structures. In the last few decades a lot of results on linear preservers on matrix algebras as well as on more general rings and operator algebras have been obtained (see [23]). Besides linear preservers also a more general problem of characterizing additive preservers and related problem of characterizing multiplicative preservers on matrix algebras were studied a lot. It is surprising that in some cases we can get nice structural results for preservers without any algebraic assumption like linearity, additivity or multiplicativity. Probably the first fundamental attempt to attack non-linear preserver problems on matrices and their subspaces was made by Hua [16, 15]. Later, Baribeau and Ransford in [3] studied spectrum preserving non-linear maps of matrix algebras under some mild differentiability condition. We refer to Guterman and Mikhalev [12] for a nice survey of the methods and historical remarks about preserver problems.

One among the most basic preserver problems is classifying maps that preserve zeros of various products. For example, Botta, Pierce, and Watkins [4] classified linear maps that preserve nilpotents on $n \times n$ matrices, i.e. zeros of the product $X \mapsto X^n$. Howard [14] (see also Li and Pierce [19]) extended this result and classified linear bijections on matrix algebras preserving zeros of any fixed polynomial of degree at least two. Linear maps that preserve commutativity on matrices, i.e. zeros of a Lie product $(X, Y) \mapsto XY - YX$, were classified by Watkins [31], while general ones were given by Šemrl [29]. Maps on matrices that preserve zeros of a fixed homogeneous multilinear polynomial in k noncommuting variables were studied by Guterman and the second author [11]. We refer to Chebotar, Ke, Lee and Wong [8] for linear preservers of zero product on quite general algebras, and to Zhao and Hou [33] for additive preservers of zeros of Jordan product on certain operator algebras.

The aim of this article is to study maps that preserve zeros of a Jordan product on complex Hermitian and complex symmetric matrices without additional assumptions like linearity/additivity/.... In Hermitian case we only use the assumption that the preserver does not annihilate nonzero matrices. In symmetric case we assume more: that the map is injective and continuous. We emphasize that in symmetric case without imposing some additional regularity conditions, like continuity, we cannot hope for a nice structural result (consider for example an injective map with the image contained in square-zero nilpotents). We remark that our results on symmetric matrices are in the spirit of Šemrl [29] who studied injective continuous maps which preserve zeros of a Lie product.

Let us list some mostly standard notation. Throughout, $n \geq 3$ will be an integer and $\mathcal{M}_n$ will be the algebra of all $n \times n$ matrices over the field of complex numbers and $\mathcal{M}_{m,n}$ will be the space of $m \times n$ complex matrices. Let E_{ij} be the standard basis of $\mathcal{M}_n$. We denote by $\mathcal{H}_n \subseteq \mathcal{M}_n$ the real space of $n \times n$ Hermitian matrices (i.e. $A^* = A$, where $A^* = \overline{A}^t$) and by $\mathcal{S}_n \subseteq \mathcal{M}_n$ the space of $n \times n$ complex symmetric matrices (i.e. $A^t = A$). We will study maps with the property

$$A \circ B = 0 \quad \Longrightarrow \quad \phi(A) \circ \phi(B) = 0,$$

where $A \circ B = AB + BA$ is a Jordan product. In case of symmetric matrices we will further assume that a map is injective and continuous. This will enable us to utilize Brouwer's theorem about the invariance of domain theorem [10, p.344] which states that if U is an open subset of $\mathbb{R}^m$ and $F : U \to \mathbb{R}^m$ is a continuous injective map, then $F(U)$ is open. In particular, there is no injective continuous map from $\mathbb{R}^k$ into $\mathbb{R}^m$ whenever $m < k$.

Every rank one Hermitian matrix can be written as $A = xx^*$ for some column vector $x \in \mathbb{C}^n$. Similarly, every rank one symmetric matrix can be written as $A = xx^t$. Despite apparent similarity, there is a more profound difference between the two classes of matrices. Namely, no Hermitian matrix other than zero is nilpotent. However there are nilpotent symmetric matrices of rank one. A typical example is given by xx^t with $x = e_1 + \mathbf{i}e_2$, where $\mathbf{i}^2 = -1$ and where e_k are column vectors from a standard basis of $\mathbb{C}^n$. Actually, every complex matrix is similar to a symmetric matrix [13, Theorem 4.4.9].

We remark that complex Hermitian and complex symmetric matrices are used to describe different geometries on $\mathbb{C}^n$. While the former are used to describe all possible unitary geometries, the latter describe all possible orthogonal geometries on $\mathbb{C}^n$. More on different geometries can be found in a book by Artin [2].

2. Maps on Hermitian matrices

Before stating our first main result on Hermitian matrices we recall the following facts. Given $A \in \mathcal{H}_n$, there exists a unitary matrix U (i.e. $U^*U = \mathrm{Id}$) such that

$$U^*AU = \mathrm{diag}(a_1, a_2, \ldots, a_n) \tag{1}$$

where $a_1, a_2, \ldots, a_n$ are real scalars. Define a Jordan commutant of A as

$$A^\# = \{X \in \mathcal{H}_n \ : \ A \circ X = 0\}.$$

Note that this is always a real vector space. It is easy to see that $E_{nn}^\# = \mathcal{H}_{n-1} \oplus 0$, so $X \circ E_{nn} = 0$ if and only if $XE_{nn} = 0 = E_{nn}X$. From here, one can easy deduce that rank one Hermitian matrices are linearly dependent if and only if they have the same Jordan commutant. Note also that $A = 0$ if and only if $A^\# = \mathcal{H}_n$.

We can now state our first result.

Theorem 2.1. *Let $n \geq 3$. Assume a map $\phi : \mathcal{H}_n \to \mathcal{H}_n$ preserves zeros of Jordan product in one direction only, and assume $\phi(X) = 0$ is possible only if $X = 0$. Then $\phi(0) = 0$ and there exists a unitary matrix U such that either*

$$\phi(A) = t_A UAU^* \tag{i}$$

for every rank one matrix $A \in \mathcal{H}_n$, or

$$\phi(A) = t_A UA^tU^* \tag{ii}$$

for every rank one matrix $A \in \mathcal{H}_n$. Here, t_A is a nonzero real number determined by A.

It is necessary to impose the condition about $\phi(X) = 0$. Without it there are much more possibilities: we could map, say, E_{11} into $E_{11} - E_{22}$, and E_{22} into $E_{12} + E_{21}$ and every other matrix into 0; this map preserves zeros of Jordan product. However, when the zeros of Jordan product are preserved in both directions no additional assumption is needed because if $\phi(X) = 0$ then $0 = \phi(X) \circ \phi(X)$. This yields that $0 = X \circ X$ since zeros are preserved in both directions. Hence, $X^2 = 0$ and in Hermitian matrices this automatically implies $X = 0$. We thus record the following immediate corollary to our Theorem.

Corollary 2.2. *Let $n \geq 3$. Assume $\phi : \mathcal{H}_n \to \mathcal{H}_n$ is a map with the property*

$$AB + BA = 0 \iff \phi(A)\phi(B) + \phi(B)\phi(A) = 0. \tag{2}$$

Then, (i)–(ii) of Theorem 2.1 hold.

We record one more corollary for injective maps, which we will prove at the end of this section.

Corollary 2.3. *Let $n \geq 3$ and let $\phi : \mathcal{H}_n \to \mathcal{H}_n$ be an injective map that preserves zeros of Jordan product in one direction. Then (i)–(ii) from Theorem 2.1 hold.*

Remark 2.4. Without additional assumptions the validity of Theorem 2.1 and Corollaries 2.2 and 2.3 cannot be extended to the whole $\mathcal{H}_n$, see Example 2.11 below. However if ϕ posses more regularity, say if it is additive, then ϕ is nice everywhere. We refer to the last section for a proof.

Let us start to prove Theorem 2.1. We will rely heavily on classifying subsets with maximal possible number of elements in $\mathcal{H}_n$ that consist of pairwise Jordan–orthogonal matrices, see Lemma 2.5. A similar idea of using orthogonality in solving preserver problems was also considered in the paper by Chan, Li, and the last author [7].

Given two subsets $\Omega_1, \Omega_2 \subseteq \mathcal{H}_n$ we write for simplicity $\Omega_1 \sim \Omega_2$ if $\mathbb{R}\Omega_1 = \mathbb{R}\Omega_2$, where $\mathbb{R}\Omega = \{\lambda A \,:\, \lambda \in \mathbb{R},\, A \in \Omega\}$. Observe that this happens if and only if for each $S \in \Omega_1$ and each $T \in \Omega_2$ there exist nonzero real numbers λ_S and λ_T such that $\lambda_S S \in \Omega_2$ and $\lambda_T T \in \Omega_1$. We also write $A \sim B$ for matrices A, B if $\{A\} \sim \{B\}$. Given a real number x, let $\lfloor x \rfloor$ be the largest integer not exceeding x. Define

$$D_{ij} = E_{ii} - E_{jj}, \quad F_{ij} = (E_{ij} + E_{ji}), \quad G_{ij} = \mathbf{i}(E_{ij} - E_{ji}); \qquad (i \neq j),$$

and let

$$\mathcal{F}_n = \bigcup_{i=1}^{n/2} \{D_{(2i-1)(2i)}, F_{(2i-1)(2i)}, G_{(2i-1)(2i)}\}; \qquad n \geq 1 \text{ is even}$$

$$\mathcal{F}_n = \{E_{nn}\} \cup \bigcup_{i=1}^{(n-1)/2} \{D_{(2i-1)(2i)}, F_{(2i-1)(2i)}, G_{(2i-1)(2i)}\}; \qquad n \geq 1 \text{ is odd.}$$

We remark that $\mathcal{F}_1 = \{1\} \subset \mathcal{H}_1 = \mathbb{R}$. Notice that $S \circ T = 0$ for any distinct $S, T \in \mathcal{F}_n$, and notice that the cardinality of $\mathcal{F}_n$ satisfies $|\mathcal{F}_n| = \lfloor \frac{3n}{2} \rfloor$. In the next lemma we prove the converse of this statement.

Lemma 2.5. *Let $n \geq 1$. Given a subset $\mathcal{T}$ of nonzero matrices in $\mathcal{H}_n$, suppose that*

$$S \circ T = 0 \qquad \text{for all distinct} \quad S, T \in \mathcal{T}. \tag{3}$$

Then, $|\mathcal{T}| \leq \lfloor \frac{3n}{2} \rfloor$. The equality holds if and only if there is a unitary matrix U such that

$$\mathcal{T} \sim \{USU^* \,:\, S \in \mathcal{F}_n\}.$$

Proof. We argue by induction on the size of matrices n. The case when $n=1$ can be proved easily. So assume the two assertions hold in all sizes up to $n-1$. Pick any $\mathcal{T}\subseteq\mathcal{H}_n$ which satisfies (3) and consider two cases.

(i) Suppose, to start with, that there is at least one $A\in\mathcal{T}$ with two distinct eigenvalues λ and μ such that $\lambda\neq-\mu$. Then, A is unitarily similar to $A'\oplus A''$ where $\mathrm{Sp}(A')\subseteq\{\lambda,-\lambda\}$ and $\mathrm{Sp}(A'')=\mathrm{Sp}(A)\setminus\{\lambda,-\lambda\}$. So, up to unitary similarity, $A'=\lambda\operatorname{Id}_{n_1}\oplus(-\lambda)\operatorname{Id}_{n_2}$, and $A''=\mu\operatorname{Id}_{n_3}\oplus\mu_4\operatorname{Id}_{n_4}\oplus\cdots\oplus\mu_k\operatorname{Id}_{n_k}$ where we agreed that if $(-\lambda)\notin\mathrm{Sp}(A)$ or if $\lambda=0=-\lambda$ we let $n_2=0$ and omit the summand $(-\lambda)\operatorname{Id}_{n_2}$. Due to $\{\lambda,-\lambda\}\cap\{\mu,\mu_4,\dots,\mu_k\}=\emptyset$, the condition (3) easily implies that every $B\in\mathcal{T}$ takes the form $B=B'\oplus B''\in\mathcal{H}_m\oplus\mathcal{H}_{n-m}$ where $m=n_1+n_2\in\{1,\dots,n-1\}$. By the induction hypothesis, the sets $\mathcal{T}'=\{B'\,:\,B\in\mathcal{T}\}$ and $\mathcal{T}''=\{B''\,:\,B\in\mathcal{T}\}$, which also satisfy (3), have at most $|\mathcal{F}_m|$ and $|\mathcal{F}_{n-m}|$ nonzero elements. A simple argument then gives

$$|\mathcal{T}|\leq|\mathcal{F}_m|+|\mathcal{F}_{n-m}|=\lfloor\tfrac{3m}{2}\rfloor+\lfloor\tfrac{3(n-m)}{2}\rfloor\leq\lfloor\tfrac{3n}{2}\rfloor.$$

Notice that the last inequality is strict unless m or $n-m$ is even. Furthermore, if there is some $B=B'\oplus B''\in\mathcal{T}$ such that $B'\neq0$ and $B''\neq0$ then $|\mathcal{T}|\leq1+(|\mathcal{F}_m|-1)+(|\mathcal{F}_{n-m}|-1)$ so the first inequality, i.e. $|\mathcal{T}|\leq|\mathcal{F}_m|+|\mathcal{F}_{n-m}|$, is strict. Hence, the equality under (i) holds only if $\mathcal{T}=(\mathcal{T}'\oplus0_{n-m})\cup(0_m\oplus\mathcal{T}'')$, and m or $n-m$ is even. With the help of unitary similarity given by permutation matrix we can achieve that m is even. Using the induction hypothesis the equality thus holds if and only if, up to unitary similarity, m is even and $\mathcal{T}\sim(\mathcal{F}_m\oplus0_{n-m})\cup(0_m\oplus\mathcal{F}_{n-m})$. It is easy to see that $(\mathcal{F}_m\oplus0_{n-m})\cup(0_m\oplus\mathcal{F}_{n-m})=\mathcal{F}_n$, which proves the induction step under (i).

(ii) It remains to consider the case when each $A\in\mathcal{T}$ is either a scalar or $\mathrm{Sp}(A)=\lambda_A\{-1,1\}$ for some nonzero λ_A. Now, if $\mathcal{T}$ contains a scalar, then this is its only member, giving $1=|\mathcal{T}|<\lfloor\frac{3n}{2}\rfloor$. So suppose every $A\in\mathcal{T}$ satisfies $\mathrm{Sp}(A)=\lambda_A\{-1,1\}$. Clearly we can assume $\lambda_A=1$, i.e., $\mathcal{T}$ contains only involutions. Fix one A and assume without loss of generality that $A=\operatorname{Id}_m\oplus(-\operatorname{Id}_{n-m})$. If $B\in\mathcal{T}\setminus\{A\}$ then $B\circ A=0$ and it readily follows that $B=\left(\begin{smallmatrix}0&V\\V^*&0\end{smallmatrix}\right)$ for some $m\times(n-m)$ matrix V. By the assumptions, $\mathrm{Sp}(B)=\lambda_B\{-1,1\}=\{-1,1\}$ so B is invertible and $B^2=\operatorname{Id}$ which is possible only if $n=2m$ and if V is unitary.

Fix another $B_1=\left(\begin{smallmatrix}0&V_1\\V_1^*&0\end{smallmatrix}\right)\in\mathcal{T}\setminus\{A\}$. Using unitary similarity $U_1=V_1^*\oplus\operatorname{Id}_m$ we have $U_1AU_1^*=A$ and $U_1B_1U_1^*=\left(\begin{smallmatrix}0&\operatorname{Id}_m\\\operatorname{Id}_m&0\end{smallmatrix}\right)$. We may assume that $U_1=\operatorname{Id}_{2m}$, otherwise we would apply U_1 to every member of $\mathcal{T}$ without affecting (3). Then, given any two $B,\widehat{B}\in\mathcal{T}\setminus\{A\}$ an easy computation reveals that

$$B\circ\widehat{B}=2\Big(\mathrm{Re}(V\widehat{V}^*)\oplus\mathrm{Re}(V^*\widehat{V})\Big),\tag{4}$$

where $\operatorname{Re} X = (X + X^*)/2$ is the real part of a square matrix $X = V\widehat{V}^*$. Thus, for every $B \in \mathcal{T} \setminus \{A, B_1\}$ we have $B_1 \circ B = 0$ which gives

$$\operatorname{Re}(V^*) = 0 = \operatorname{Re}(V) \tag{5}$$

and the unitary V equals $V = \mathbf{i}H$ for some Hermitian H. Hence, every $B \in \mathcal{T} \setminus \{A, B_1\}$ takes the form

$$B = \begin{pmatrix} 0 & \mathbf{i}H \\ -\mathbf{i}H & 0 \end{pmatrix}; \qquad H \in \mathcal{H}_m, \quad \operatorname{Sp}(H) \subseteq \{-1, 1\}.$$

Recall $B \circ \widehat{B} = 0$ for distinct $B, \widehat{B} \in \mathcal{T} \setminus \{A, B_1\}$, so (4) implies that $0 = (\mathbf{i}H)(\mathbf{i}\widehat{H})^* + (\mathbf{i}\widehat{H})(\mathbf{i}H)^* = H \circ \widehat{H}$.

By the induction hypothesis we infer that, together with A and B_1, there exist at most $2 + \lfloor \frac{3m}{2} \rfloor$ matrices inside $\mathcal{T}$ under (ii). Due to $n = 2m$ we easily see that $|\mathcal{T}| \le 2 + \lfloor \frac{3m}{2} \rfloor \le 3m = \lfloor \frac{3n}{2} \rfloor$. The equality holds only when $m = 1$, that is, when $n = 2m = 2$, in which case, up to unitary similarity,

$$\mathcal{T} \sim \left\{ \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & \mathbf{i} \\ -\mathbf{i} & 0 \end{smallmatrix}\right) \right\}.$$

This proves the inductive step also under (ii). □

Lemma 2.6. *Let $\{X_1, X_2, X_3\} \subseteq \mathcal{F}_n$.*

(i) *If $\{X_1, X_2, X_3\} = V\{D_{12}, F_{12}, G_{12}\}V^*$ for some permutation matrix V, then*

$$\bigcap_{H \in \mathcal{F}_n \setminus \{X_1, X_2, X_3\}} H^{\#} = V(\mathcal{H}_2 \oplus 0_{n-2})V^*. \tag{6}$$

(ii) *If $X_1X_2 = 0 = X_2X_1$, then*

$$\bigcap_{H \in \mathcal{F}_n \setminus \{X_1, X_2, X_3\}} H^{\#} = \mathbb{R}X_1 + \mathbb{R}X_2 + \mathbb{R}X_3. \tag{7}$$

Proof. (i) We may assume that $V = \mathrm{Id}$. Now, it is easy to see that $D_{12}^{\#} = \mathrm{Lin}_{\mathbb{R}}\{F_{12}, G_{12}\} + (0_2 \oplus \mathcal{H}_{n-2})$, wherefrom $D_{12}^{\#} \cap F_{12}^{\#} \cap G_{12}^{\#} = 0_2 \oplus \mathcal{H}_{n-2}$. With the help of unitary similarity given by permutation matrix we now find that, for every i, $D_{(2i-1)(2i)}^{\#} \cap F_{(2i-1)(2i)}^{\#} \cap G_{(2i-1)(2i)}^{\#}$ consists of all Hermitian matrices that vanish on $(2i-1)$-th and $(2i)$-th rows and columns. Noting that $\mathcal{F}_n \setminus \{X_1, X_2, X_3\} = \bigcup_{i=2}^{n/2} \{D_{(2i-1)(2i)}, F_{(2i-1)(2i)}, G_{(2i-1)(2i)}\}$ if n is even, and $\mathcal{F}_n \setminus \{X_1, X_2, X_3\} = \{E_{nn}\} \cup \bigcup_{i=2}^{(n-1)/2} \{D_{(2i-1)(2i)}, F_{(2i-1)(2i)}, G_{(2i-1)(2i)}\}$ if n is odd, we easily derive (6).

(ii) After reindexing X_1, X_2, X_3 and applying unitary similarity given by a permutation matrix we are facing four possibilities (a) $(X_1, X_2, X_3) = (D_{12}, D_{34}, D_{56})$, (b) $(X_1, X_2, X_3) = (E_{nn}, D_{34}, D_{56})$, (c) $(X_1, X_2, X_3) = (D_{12}, F_{34}, G_{34})$, and (d) $(X_1, X_2, X_3) = (E_{nn}, F_{34}, G_{34})$. Assume option (d). As in the proof of (i) we see that each matrix from $\Omega = \bigcap_{H \in \mathcal{F}_n \setminus \{X_1, X_2, X_3\}} H^{\#}$ has all rows and columns zero except possibly the 3-rd, 4-th and n-th. We infer that $\Omega \subseteq 0_2 \oplus \mathcal{H}_2 \oplus 0_{n-5} \oplus \mathcal{H}_1$. Since $D_{34} \in \mathcal{F}_n \setminus \{X_1, X_2, X_3\}$ we deduce that $\Omega \subseteq (0_2 \oplus \mathcal{H}_2 \oplus 0_{n-5} \oplus \mathcal{H}_1) \cap D_{34}^{\#} = \mathbb{R}F_{34} + \mathbb{R}G_{34} + \mathbb{R}E_{nn}$, and it is easy to prove the inverse inclusion. This proves (7) under (d). Likewise we argue in each of the remaining opinions (a)–(c). □

We acknowledge that the original idea for the proof of the next lemma used the theory of graph coloring, and came from [25].

Lemma 2.7. *Let $n \geq 3$. Assume a map $\phi : \mathcal{H}_n \to \mathcal{H}_n$ preserves zeros of Jordan product in one direction only, and assume $\phi(X) = 0$ is possible only if $X = 0$. Then, $\phi(0) = 0$ and ϕ preserves the set of rank one Hermitian matrices.*

Proof. Let us start by showing that (i) if n is even then $\operatorname{rk}\phi(A) = 2$ for any Hermitian A with rank two, trace-zero and (ii) if n is odd then $1 \leq \operatorname{rk}\phi(A) \leq 2$ for any Hermitian A with rank two, trace-zero or with rank one. We regard only the very last possibility because the first one in case (ii) and case (i) can be proved likewise. So assume n is odd and $\operatorname{rk} A = 1$. There exists a unitary similarity U and a nonzero $a \in \mathbb{R}$ such that $A = aUE_{nn}U^*$. Consider the set

$$\begin{aligned}\mathcal{T} &= aU\mathcal{F}_nU^* \\ &= a\{UE_{nn}U^*\} \cup \bigcup_{i=1}^{(n-1)/2} a\{UD_{(2i-1)(2i)}U^*, UF_{(2i-1)(2i)}U^*, UG_{(2i-1)(2i)}U^*\}\end{aligned}$$

of $\lfloor \frac{3n}{2} \rfloor$ nonzero matrices, with pairwise zero Jordan products, which contains A. By the assumptions, $\phi(\mathcal{T})$ also consists of nonzero Hermitian matrices with pairwise zero Jordan products. Since no nonzero Hermitian matrix is square-zero, ϕ is injective on $\mathcal{T}$ and $\phi(\mathcal{T})$ contains $\lfloor \frac{3n}{2} \rfloor$ nonzero matrices so, by Lemma 2.5, $\phi(\mathcal{T}) \sim V\mathcal{F}_nV^*$ for some unitary similarity V. This shows that $1 \leq \operatorname{rk}\phi(A) \leq 2$ whenever $\operatorname{rk} A = 1$. If A is rank two, trace-zero and n is either odd or even we would consider $A = aU(E_{11} - E_{22})U^*$ and then repeat the above arguments to show once more $1 \leq \operatorname{rk}\phi(A) \leq 2$ or $\operatorname{rk}\phi(A) = 2$, respectively.

The first part of the claim in Lemma 2.7 is now easy: Since $0 \circ X = 0$ for every $X \in \mathcal{T}$, we have $\phi(0) \circ Y = 0$ for every $Y \in \phi(\mathcal{T}) \sim V\mathcal{F}_nV^*$. Therefore, $\phi(\mathcal{T}) \cup \{\phi(0)\}$ contains $1 + \lfloor \frac{3n}{2} \rfloor$ distinct matrices with pairwise zero Jordan products, which, by Lemma 2.5, is possible only if some of them is zero, giving $\phi(0) = 0$.

The proof of the second part of Lemma 2.7 relies on the following claim:

Claim. For any unitary U, there is a unitary V such that

$$\left\{\phi(UD_{12}U^*)\,,\, \phi(UF_{12}U^*)\,,\, \phi(UG_{12}U^*)\right\} \sim V\{D_{12}, F_{12}, G_{12}\}V^*.$$

Without loss of generality we assume that $U = \operatorname{Id}_n$. Then, there is a unitary V with $\phi(\mathcal{F}_n) \sim V\mathcal{F}_nV^*$. By temporarily replacing ϕ with $\phi : X \mapsto \gamma_X V^*\phi(X)V$ for appropriate scalars γ_X we may also assume $V = \operatorname{Id}_n$ and $\phi(\mathcal{F}_n) = \mathcal{F}_n$. Denote

$$X = \phi(D_{12}), \qquad Y = \phi(F_{12}), \qquad Z = \phi(G_{12}).$$

Since $\{X,Y,Z\}\subseteq\mathcal{F}_n$, the claim will hold if no two matrices from $\{X,Y,Z\}$ are orthogonal. Assume otherwise that, say X,Y are orthogonal. Then, Z must also be orthogonal to at least one of X,Y. Without loss of generality we assume Z is orthogonal to X. By the fact that $\phi(\mathcal{F}_n\setminus\{D_{12},F_{12},G_{12}\})=\mathcal{F}_n\setminus\{X,Y,Z\}$, Lemma 2.6 yields

$$\phi(\mathbb{R}D_{12}+\mathbb{R}F_{12}+\mathbb{R}G_{12})\subseteq\bigcap_{H\in\mathcal{F}_n\setminus\{X,Y,Z\}}H^{\#}=\mathbb{R}X+\mathbb{R}Y+\mathbb{R}Z.$$

However, every nonzero matrix $A\in\mathbb{R}D_{12}+\mathbb{R}F_{12}+\mathbb{R}G_{12}$ is rank two, trace-zero hence $\operatorname{rk}\phi(A)\le 2$ and as X is orthogonal to Y,Z we actually have

$$\phi(\mathbb{R}D_{12}+\mathbb{R}F_{12}+\mathbb{R}G_{12})\subseteq\mathbb{R}X\cup(\mathbb{R}Y+\mathbb{R}Z).\tag{8}$$

Observe that $S\circ G_{12}=0$ for all $S\in\mathbb{R}D_{12}+\mathbb{R}F_{12}$. Therefore, $\phi(S)\circ Z=0$ for all $S\in\mathbb{R}D_{12}+\mathbb{R}F_{12}$. Similarly we get $\phi(T)\circ Y=0$ for all $T\in\mathbb{R}D_{12}+\mathbb{R}G_{12}$. This implies

$$\phi(\mathbb{R}D_{12}+\mathbb{R}F_{12})\in\mathbb{R}X\cup\mathbb{R}Y\qquad\text{and}\qquad\phi(\mathbb{R}D_{12}+\mathbb{R}G_{12})\in\mathbb{R}X\cup\mathbb{R}Z.$$

Consequently, since $(D_{12}+F_{12})\circ(D_{12}-F_{12})=0$, there is $\alpha\in\{-1,1\}$ with

$$\phi(D_{12}-\alpha F_{12})\sim X\qquad\text{and}\qquad\phi(D_{12}+\alpha F_{12})\sim Y.$$

Similarly, there is $\beta\in\{-1,1\}$ with

$$\phi(D_{12}-\beta G_{12})\sim X\qquad\text{and}\qquad\phi(D_{12}+\beta G_{12})\sim Z.$$

Observe that

$$(D_{12}+\alpha F_{12}-\beta G_{12})\in(D_{12}-\alpha F_{12})^{\#}\cap(D_{12}+\beta G_{12})^{\#}$$

and

$$(D_{12}-\alpha F_{12}+\beta G_{12})\in(D_{12}+\alpha F_{12})^{\#}\cap(D_{12}-\beta G_{12})^{\#}.$$

Together with (8) it follows that

$$\phi(D_{12}+\alpha F_{12}-\beta G_{12})\sim Y\qquad\text{and}\qquad\phi(D_{12}-\alpha F_{12}+\beta G_{12})\sim Z.$$

Finally, $(\alpha F_{12}+\beta G_{12})\in(D_{12}+\alpha F_{12}-\beta G_{12})^{\#}\cap(D_{12}-\alpha F_{12}+\beta G_{12})^{\#}$ implies that

$$\phi(\alpha F_{12}+\beta G_{12})\sim X=\phi(D_{12}).$$

But this is impossible as $D_{12}\circ(\alpha F_{12}+\beta G_{12})=0$. Thus, the claim holds.

Back to the proof, we consider two cases separately.

Case n is odd. Consider an arbitrary rank one $A=aUE_{nn}U^*$; we need to show that $\operatorname{rk}\phi(A)=1$. Now, without loss of generality we may assume that already $A=E_{nn}$ and that $\phi(\mathcal{F}_n)\sim\mathcal{F}_n$. Then the claim implies that every rank two, trace-zero matrix in $\mathcal{F}_n$ is mapped into a rank two, trace-zero matrix. Hence, $\phi(\mathcal{F}_n\setminus\{E_{nn}\})\sim\mathcal{F}_n\setminus\{E_{nn}\}$, and so $\phi(A)=\phi(E_{nn})\sim E_{nn}$, as claimed.

Case n is even. Given $A = aUE_{11}U^*$, we may assume that $A = E_{11}$ and that $\phi(\mathcal{F}_n) \sim \mathcal{F}_n$. Now, by the above Claim, and due to $\phi(\mathcal{F}_n) \sim \mathcal{F}_n$, we have $\{\phi(D_{12}), \phi(F_{12}), \phi(G_{12})\} \sim \{D_{12}, F_{12}, G_{12}\}$. Consequently,

$$\phi\big(\mathcal{F}_n \setminus \{D_{12}, F_{12}, G_{12}\}\big) \sim \mathcal{F}_n \setminus \{D_{12}, F_{12}, G_{12}\}. \tag{9}$$

With the help of yet another unitary similarity, given by permutation matrix of the form $\mathrm{Id}_2 \oplus V''$ we may assume that in addition, $\phi(D_{34}) \sim D_{34}$. By the Claim we then have $\{\phi(D_{34}), \phi(F_{34}), \phi(G_{34})\} \sim \{D_{34}, F_{34}, G_{34}\}$.

Now, $A = E_{11}$ has zero Jordan product with every matrix from $\mathcal{F}_n \setminus \{D_{12}, F_{12}, G_{12}\}$, so $\phi(E_{11})$ has zero Jordan product with every matrix from $\phi\big(\mathcal{F}_n \setminus \{D_{12}, F_{12}, G_{12}\}\big) \sim \mathcal{F}_n \setminus \{D_{12}, F_{12}, G_{12}\}$. In particular,

$$\phi(E_{11}) \in \mathcal{H}_2 \oplus 0_{n-2}.$$

Arguing likewise shows that

$$\phi(E_{44}) \in 0_2 \oplus \mathcal{H}_2 \oplus 0_{n-4},$$

so $\phi(E_{11})$ is orthogonal to $\phi(E_{44})$. Furthermore, D_{23}, F_{23}, G_{23} have zero Jordan product with every matrix from $\mathcal{F}_n \setminus \{D_{12}, F_{12}, G_{12}, D_{34}, F_{34}, G_{34}\}$, so $\phi(D_{23}), \phi(F_{23}), \phi(G_{23})$ have zero Jordan product with every matrix from $\phi\big(\mathcal{F}_n \setminus \{D_{12}, F_{12}, G_{12}, D_{34}, F_{34}, G_{34}\}\big) \sim \mathcal{F}_n \setminus \{D_{12}, F_{12}, G_{12}, D_{34}, F_{34}, G_{34}\}$. From here we can infer that

$$\phi(D_{23}), \phi(F_{23}), \phi(G_{23}) \in \mathcal{H}_4 \oplus 0_{n-4},$$

and by the Claim, there is a unitary similarity $V_3 = V''' \oplus \mathrm{Id}_{n-4}$ such that $\{\phi(D_{23}), \phi(F_{23}), \phi(G_{23})\} \sim V_3\{D_{12}, F_{12}, G_{12}\}V_3^*$. Now, $\phi(E_{11}), \phi(E_{44}) \in \mathcal{H}_4 \oplus 0_{n-4} = V_3(\mathcal{H}_4 \oplus 0_{n-4})V_3^*$ have zero Jordan products with the above three matrices. It is easy to see that they must both lie inside $V_3(0_2 \oplus \mathcal{H}_2 \oplus 0_{n-4})V_3^*$. Being orthogonal, their rank is at most one. In particular, $\mathrm{rk}\,\phi(E_{11}) = 1$. □

The well-known Uhlhorn's generalization [30] of Wigner's [32] unitary–antiunitary theorem states that any bijection φ on a projective space $\mathbb{P}(H) = \{[x] = \mathbb{C}x \ : \ x \in H\backslash\{0\}\}$ of a complex Hilbert space H, which preserves orthogonality of points (i.e., of one dimensional subspaces in Hilbert space) in both directions is given by linear or conjugate linear bijective isometry. Recall that orthogonality is defined by $[x]\perp[y]$ if $\langle x, y\rangle = 0$ for some representatives $x \in [x]$, $y \in [y]$, where $\langle \cdot, \cdot \rangle$ is a scalar product on H.

This result was extended by van den Broek [5] and then generalized by Molnár [22] to bijections which preserve orthogonality on indefinite inner product spaces. Later, Šemrl [28] showed that on finite dimensional spaces, bijectivity can be relaxed to injectivity, while still assuming that orthogonality is preserved in both directions. Rodman and Šemrl [27] classified maps on finite dimensional projective spaces over skew-fields which preserve orthogonality in one direction only. Their results also hold in indefinite inner product spaces, however because of the possible presence of isotropic vectors they had to assume that the map is injective when dimension is three and bijective when dimension is greater than three. Our next series of lemmas will show that on the projective space over finite dimensional complex Hilbert space we can drop injectivity – any map which preserves orthogonality in one direction only is automatically bijective.

Lemma 2.8. *Let $n \geq 3$. Suppose a map $\varphi : \mathbb{P}(\mathbb{C}^n) \to \mathbb{P}(\mathbb{C}^n)$ preserves orthogonality. Then, for any set of three orthonormal vectors $\{x_1, x_2, x_3\} \subseteq \mathbb{C}^n$ there exists a set of three orthonormal vectors $\{\hat{x}_1, \hat{x}_2, \hat{x}_3\} \subseteq \mathbb{C}^n$ such that*

$$\varphi\big([\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3]\big) = [\alpha_1 \hat{x}_1 + \alpha_2 \hat{x}_2 + \alpha_3 \hat{x}_3] \tag{10}$$

holds for every $\alpha_1, \alpha_2, \alpha_3 \in \{-1, 0, 1\}$ except $\alpha_1 = \alpha_2 = \alpha_3 = 0$.

Proof. Given a set of orthonormal vectors $\{x_1, x_2, x_3\}$ we first extend it to an orthonormal basis $\{x_1, x_2, x_3, \dots, x_n\}$ of $\mathbb{C}^n$. Let $\hat{x}_j$ be a unit vector such that

$$[\hat{x}_j] = \varphi([x_j]); \qquad (j = 1, \dots, n). \tag{11}$$

Since φ preserves orthogonality, $\{\hat{x}_1, \dots, \hat{x}_n\}$ is another orthonormal basis. Hence, it easily follows that φ maps every line from the subspace $[x_1] + [x_2] + [x_3]$ into a line in $[\hat{x}_1] + [\hat{x}_2] + [\hat{x}_3]$. Therefore, $\varphi([x_1 + x_2 + x_3]) = [\beta_1 \hat{x}_1 + \beta_2 \hat{x}_2 + \beta_3 \hat{x}_3]$. Without loss of generality we assume that $\beta_1 > 0$ and $\beta_2, \beta_3 \geq 0$, otherwise we would temporarily replace φ with an orthogonality preserving map $[x] \mapsto V\varphi([x])$ for a suitably chosen unitary V. So, we have

$$\varphi([x_1 + x_2 + x_3]) = [\beta_1 \hat{x}_1 + \beta_2 \hat{x}_2 + \beta_3 \hat{x}_3] = [\hat{x}_1 + \beta \hat{x}_2 + \gamma \hat{x}_3], \tag{12}$$

where $\beta = \frac{\beta_2}{\beta_1} \geq 0$ and $\gamma = \frac{\beta_3}{\beta_1} \geq 0$. We will verify at the end of the proof that actually $\beta = 1 = \gamma$. Next, by the fact that $[x_2] \perp [x_1 - x_3] \perp [x_1 + x_2 + x_3] \perp [x_1 - x_2] \perp [x_3]$, (11) and (12) imply that

$$\varphi([x_1 - x_3]) = [\gamma \hat{x}_1 - \hat{x}_3] \qquad \text{and} \qquad \varphi([x_1 - x_2]) = [\beta \hat{x}_1 - \hat{x}_2]. \tag{13}$$

Next, with $[x_2] \perp [x_1 + x_3] \perp [x_1 - x_3]$ and $[x_3] \perp [x_1 + x_2] \perp [x_1 - x_2]$, (11) and (13) imply that

$$\varphi([x_1 + x_3]) = [\hat{x}_1 + \gamma \hat{x}_3] \qquad \text{and} \qquad \varphi([x_1 + x_2]) = [\hat{x}_1 + \beta \hat{x}_2]. \tag{14}$$

Further, as $[x_1 - x_3] \perp [x_1 - x_2 + x_3] \perp [x_1 + x_2]$ and $[x_1 - x_2] \perp [x_1 + x_2 - x_3] \perp [x_1 + x_3]$, (13) and (14) give

$$\begin{aligned} &\varphi([x_1 - x_2 + x_3]) = [\beta \hat{x}_1 - \hat{x}_2 + \beta\gamma \hat{x}_3] \quad \text{and} \\ &\varphi([x_1 + x_2 - x_3]) = [\gamma \hat{x}_1 + \beta\gamma \hat{x}_2 - \hat{x}_3]. \end{aligned} \tag{15}$$

Similarly, with $[x_1 + x_2 - x_3] \perp [x_2 + x_3] \perp [x_1] \perp [x_2 - x_3] \perp [x_1 + x_2 + x_3]$, (11), (12), and (15) give

$$\varphi([x_2 + x_3]) = [\hat{x}_2 + \beta\gamma \hat{x}_3] \qquad \text{and} \qquad \varphi([x_2 - x_3]) = [\gamma \hat{x}_2 - \beta \hat{x}_3]. \tag{16}$$

Notice that $[x_2 + x_3] \perp [x_2 - x_3]$. Then, $\gamma - \beta^2 \gamma = 0$. On the other hand, $[x_1 - x_2 + x_3] \perp [x_2 + x_3]$ implies from (15) and (16) that $-1 + (\beta\gamma)^2 = 0$.

The two equations, together with the initial assumption $\beta, \gamma \geq 0$, give $\beta = 1 = \gamma$. Finally, as $[x_1 + x_3] \perp [-x_1 + x_2 + x_3] \perp [x_1 + x_2]$, by (13) we have $\varphi([-x_1 + x_2 + x_3]) = [-\hat{x}_1 + \hat{x}_2 + \hat{x}_3]$. This exhausts all 13 lines inside Eq. (10). □

Lemma 2.9. *For any orthonormal pair $\{w_1, w_2\}$ we have $\varphi([w_1]) \neq \varphi([w_1 + \beta w_2])$ if $\beta > \sqrt{2}$.*

Proof. Suppose not. Take a unit vector w_3 such that $\{w_1, w_2, w_3\}$ forms an orthonormal set. Let $\xi = \sqrt{\frac{2}{3}\big(1 + \frac{1}{\beta^2}\big)} - \frac{1}{\beta}$. Then $\beta > \sqrt{2}$ implies $0 < \xi < 1$. Now define

$$u = w_1, \quad v = w_1 + \beta w_2, \quad \text{and} \quad z = w_1 + \xi w_2 + \sqrt{1 - \xi^2} w_3.$$

Notice that $\|u\| = 1$, $\|v\| = \sqrt{1 + \beta^2}$, $\|z\| = \sqrt{2}$, $u^* z = 1$, and $v^* z = \sqrt{2(\beta^2 + 1)/3}$. Then, there exist two orthonormal sets $\{x_1, x_2, x_3\}$ and $\{y_1, y_2, y_3\}$ such that $[x_1] = [u]$, $[x_1 + x_2] = [z]$, and $[y_1] = [v]$, $[y_1 + y_2 + y_3] = [z]$, respectively. By Lemma 2.8,

$$[\hat{x}_1] = \varphi([u]) = \varphi([v]) = [\hat{y}_1] \quad \text{and} \quad [\hat{x}_1 + \hat{x}_2] = \varphi([z]) = [\hat{y}_1 + \hat{y}_2 + \hat{y}_3]$$

for some orthonormal sets $\{\hat{x}_1, \hat{x}_2, \hat{x}_3\}$ and $\{\hat{y}_1, \hat{y}_2, \hat{y}_3\}$, respectively. It follows that $\hat{x}_1 = \gamma \hat{y}_1$ and $\hat{x}_1 + \hat{x}_2 = \delta(\hat{y}_1 + \hat{y}_2 + \hat{y}_3)$ for some $|\gamma| = 1$ and some $|\delta| = \sqrt{\frac{2}{3}}$. Then, $(\gamma - \delta)\hat{y}_1 = \delta(\hat{y}_2 + \hat{y}_3) - \hat{x}_2$. But this is impossible as $\hat{y}_1$ is orthogonal to $\hat{y}_2, \hat{y}_3$, and also to $\hat{x}_2$ because $\hat{y}_1 = \frac{1}{\gamma}\hat{x}_1$. □

Lemma 2.10. *Let $n \geq 3$. Suppose a map $\varphi : \mathbb{P}(\mathbb{C}^n) \to \mathbb{P}(\mathbb{C}^n)$ preserves orthogonality. Then, there exists a unitary matrix V such that*

(i) $$\varphi([x]) = [Vx], \quad \textit{or}$$

(ii) $$\varphi([x]) = [V\bar{x}].$$

Proof. Let us show that φ is injective. Suppose otherwise. Then, there are lines $[u_1] \neq [u_2]$ with $\varphi([u_1]) = \varphi([u_2])$. Clearly, $[u_1]$ cannot be orthogonal to $[u_2]$. Thus, there exits an orthonormal basis $x_1, \ldots, x_n$ such that $[u_1] = [x_1]$ and $[u_2] = [x_1 + \beta x_2]$ for some complex $\beta = |\beta| e^{\mathbf{i}\varphi} \neq 0$. Replacing x_2 with $e^{\mathbf{i}\varphi} x_2$ we may assume that $\beta = |\beta| > 0$. By Lemma 2.8 there exist orthonormal vectors $\{\hat{x}_1, \hat{x}_2, \hat{x}_3\}$ such that (10) holds. In particular, $\varphi([x_1 + \beta x_2]) = \varphi([x_1]) = [\hat{x}_1]$. Furthermore, $[\beta x_1 - x_2 + \beta x_3] \perp [x_j]$, $(j = 4, \ldots, n)$ implies that $\varphi([\beta x_1 - x_2 + \beta x_3]) \in [\hat{x}_1] + [\hat{x}_2] + [\hat{x}_3]$. Next, with $[x_1 + \beta x_2] \perp [\beta x_1 - x_2 + \beta x_3] \perp [x_1 - x_3]$, combined with (10), we get $\varphi([\beta x_1 - x_2 + \beta x_3]) = [\hat{x}_2]$. Now, the vectors $w_2 = x_2$ and $w_3 = -(x_1 + x_3)/\sqrt{2}$ form an orthonormal pair and

$$\varphi([w_2 + \sqrt{2}\beta w_3]) = \varphi([-x_2 + \beta(x_1 + x_3)]) = [\hat{x}_2] = \varphi([x_2]) = \varphi([w_2]).$$

So, if $\varphi([x_1]) = \varphi([x_1 + \beta x_2])$ we can find another orthonormal pair w_2, w_3 with $w_2 = x_2$ and $\varphi([w_2 + \sqrt{2}\beta w_3]) = \varphi([w_2])$. Recursively, we define a sequence of vectors $w_2, w_3, \ldots$, such that $\{w_j, w_{j+1}\}$ forms an orthonormal pair and

$$\varphi([w_j + \beta_j w_{j+1}]) = \varphi([w_j]); \qquad \beta_j = \sqrt{2}\beta_{j-1} \text{ and } \beta_2 = \sqrt{2}\beta.$$

Then, for sufficiently large m, $\beta_m > \sqrt{2}$. But this contradicts to Lemma 2.9. Hence, φ is injective.

We next show that $[z] \subseteq [x] + [y]$ yields $\varphi([z]) \subseteq \varphi([x]) + \varphi([y])$. There is nothing to prove if $[x] = [y]$. So, assume that x and y are linearly independent. We can find pairwise orthogonal lines $[z_3], \ldots, [z_n]$ which are also orthogonal to $[x]$ and to $[y]$. It follows that $[z_i]$ are orthogonal to $[z]$. Thus, the lines $\varphi([x]), \varphi([y])$, and $\varphi([z])$ are contained in the two-dimensional orthogonal complement of $\varphi([z_3]), \ldots, \varphi([z_n])$. Since $\varphi([x]) \neq \varphi([y])$, we have $\varphi([z]) \subseteq \varphi([x]) + \varphi([y])$, as desired. We can therefore apply [9, Theorem 4.1] to conclude that there exists a unitary or antiunitary (conjugate-linear isometry) operator $V_1 : \mathbb{C}^n \to \mathbb{C}^n$ such that $\varphi([x]) = [V_1 x]$ for every nonzero $x \in \mathbb{C}^n$. If V_1 is conjugate-linear then $V : x \mapsto V_1 \bar{x}$ is a linear isometry, and $\varphi([x]) = [V\bar{x}]$. □

Proof of Theorem 2.1. Let P be a rank one Hermitian matrix. It follows by Lemma 2.7 that $\operatorname{rk}\phi(P) = 1$. Now, a matrix X satisfies $X \circ E_{11} = 0$ if and only if its first row and column vanishes, which is equivalent to $XE_{11} = 0 = E_{11}X$. Hence, ϕ preserves orthogonality among rank one Hermitian matrices.

Write $P = aUE_{11}U^*$ for some unitary U and some scalar $a \neq 0$, choose any $\lambda \in \mathbb{R}$ and note that P and λP are orthogonal to $UE_{22}U^*, \ldots, UE_{nn}U^*$. Hence, $\phi(\lambda P)$ and $\phi(P)$ are orthogonal to $n-1$ pairwise orthogonal rank one Hermitian matrices $\phi(UE_{ii}U^*)$. It easily follows that there exists a uniquely determined rank one projection Q such that $\phi(\mathbb{R}P) \subseteq \mathbb{R}Q$.

Since every projection of rank one can be identified with an element of projective space $\mathbb{P}(\mathbb{C}^n)$ in a natural way, the map ϕ induces an orthogonality preserving map $\varphi : \mathbb{P}(\mathbb{C}^n) \to \mathbb{P}(\mathbb{C}^n)$ defined by $\varphi([x]) = [y]$ if $\phi(xx^*) \in \mathbb{R}\, yy^*$. By Lemma 2.10, $\varphi([x]) = [Vx]$ or $\varphi([x]) = [V\bar{x}]$ for some unitary V. In the first case, for every rank one projection $P = xx^*$ and every nonzero real number r there exists a nonzero $t \in \mathbb{R}$ such that $\phi(rP) = tVPV^*$. In the second case we similarly conclude $\phi(rP) = tV\bar{P}V^* = tVP^tV^*$. □

Proof of Corollary 2.3. We only need to show that $\phi(0) = 0$; injectivity then implies that no other matrix is annihilated, and then Theorem 2.1 applies.

Consider two disjoint sets $\mathcal{F}_n$ and $2\mathcal{F}_n$. Due to injectivity, ϕ annihilates at most one matrix, so either $\phi(\mathcal{F}_n)$ does not contain a zero matrix or else this holds for $\phi(2\mathcal{F}_n)$. Assume the second case holds. Then, $\phi(2\mathcal{F}_n)$ contains $\lfloor \frac{3n}{2} \rfloor$ distinct Hermitian nonzero matrices with pairwise zero Jordan product. By Lemma 2.5, $\phi(2\mathcal{F}_n) \sim V\mathcal{F}_nV^*$ for some unitary V. Since $0 \circ X = 0$ for every $X \in 2\mathcal{F}_n$ we deduce that $\phi(0) \circ Y = 0$ for every $Y \in \phi(2\mathcal{F}_n) \sim V\mathcal{F}_nV^*$. But then, $\{\phi(0)\} \cup \phi(2\mathcal{F}_n)$ contains $1 + \lfloor \frac{3n}{2} \rfloor$ distinct matrices with pairwise zero Jordan product and by Lemma 2.5 one of the members must be zero.

Hence, $\phi(0) = 0$, as claimed. We argue similarly if $\phi(\mathcal{F}_n)$ does not contain a zero matrix. □

Example 2.11. Let $n \geq 3$. There exists a continuous bijective but non-identical map $\phi : \mathcal{H}_n \to \mathcal{H}_n$, which fixes all rank one Hermitian matrices, and preserves zeros of Jordan product in both directions.

To see this we will rely on the following fact: given a matrix A we define an elementary operator $\mathbf{T}_A : \mathcal{M}_n \to \mathcal{M}_n$ via $X \mapsto AX + XA = A \circ X$. Then, by the Lumer–Rosenblum theorem [20], its spectrum equals

$$\mathrm{Sp}(\mathbf{T}_A) = \mathrm{Sp}(A) + \mathrm{Sp}(A) = \{t + r \ : \ t, r \in \mathrm{Sp}(A)\}.$$

In particular, if $0 \notin \mathrm{Sp}(A) + \mathrm{Sp}(A)$ then $\mathbf{T}_A$ is injective, i.e. $A \circ X = 0$ precisely when $X = 0$. Now, let

$$\Xi = \{A \in \mathcal{H}_n \ : \ \mathrm{Sp}(A) + \mathrm{Sp}(A) \not\ni 0\}$$

be the set of all Hermitian matrices such that the sum of any two eigenvalues is always nonzero. Clearly, it contains positive definite Hermitian matrices, so Ξ is nonempty. Moreover, if $\mathcal{H}_n \subseteq \mathbb{C}^{n\times n}$ is given relative topology it inherits from Euclidean space $\mathbb{C}^{n\times n}$, the set Ξ is open, by continuity of the eigenvalues. And lastly, Ξ contains no rank one matrix.

Let $B_r(A) \subseteq \Xi$ be a small open ball of radius $r > 0$, centered at A and such that $\overline{B_r(A)} \subseteq \Xi$. Consider the continuous map $\phi : \overline{B_r(A)} \to \overline{B_r(A)}$, defined by $X \mapsto A + \frac{\|X-A\|}{r}(X - A)$. Clearly, it fixes A as well as the boundary of $B_r(A)$, but it is not identity. Its inverse is given by $\phi^{-1} : Y \mapsto A + \frac{\sqrt{r}}{\sqrt{\|Y-A\|}}(Y - A)$ for $Y \in \overline{B_r(A)} \setminus \{A\}$, and $\phi^{-1}(A) = A$. Then, therefore, the piecewise defined map

$$\phi : X \mapsto \begin{cases} X; & X \in \mathcal{H}_n \setminus B_r(A) \\ \phi(X); & X \in \overline{B_r(A)} \end{cases}$$

is well-defined, continuous bijection. It fixes rank one matrices, but is not identity.

It only remains to show that ϕ preserves zeros of Jordan product in both directions. Now, for $X, Y \notin B_r(A)$ this is clearly the case, since ϕ fixes them. On the other hand, if $Y \in B_r(A) \subseteq \Xi$ and $X \in \mathcal{H}_n$ is arbitrary nonzero, then $X \circ Y$ is never zero, because $\mathrm{Sp}(Y) + \mathrm{Sp}(Y) \not\ni 0$. But $\phi(Y)$ remains in the set Ξ, while bijectivity of ϕ forces $\phi(X) \neq 0$. Therefore, also $\phi(X) \circ \phi(Y)$ cannot be zero. We argue similarly to prove that $\phi(X) \circ \phi(Y) = 0$ implies $X \circ Y = 0$.

3. Maps on symmetric matrices

Before stating the main result of this section we recall the following facts. Using a standard notation, say [13, p.209], we define the $k \times k$ symmetric

Jordan block with an eigenvalue λ as follows

$$S_k(\lambda) = \lambda\,\mathrm{Id} + \begin{bmatrix} 0 & 1 & \dots & 0 \\ 1 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \dots & 1 & 0 \end{bmatrix} + \mathbf{i}\begin{bmatrix} 0 & \dots & -1 & 0 \\ \vdots & \cdot^{\cdot^{\cdot}} & 0 & 1 \\ -1 & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \vdots \\ 0 & 1 & \dots & 0 \end{bmatrix} \in \mathcal{M}_k.$$

Note that $S_1(\lambda) = [\lambda]$ and $S_2(\lambda) = \begin{bmatrix} \lambda - \mathbf{i} & 1 \\ 1 & \lambda + \mathbf{i} \end{bmatrix}$. It is well-known [13] that $S_k(\lambda)$ is similar to an elementary Jordan block with eigenvalue λ, and that given a symmetric $A \in \mathcal{S}_n$ there exists an orthogonal matrix Q (i.e. $QQ^t = \mathrm{Id}$) such that

$$A = Q(S_{n_1}(\lambda_1) \oplus S_{n_2}(\lambda_2) \oplus \cdots \oplus S_{n_k}(\lambda_k))Q^t,$$

where $\lambda_1, \lambda_2, \dots, \lambda_k$ are eigenvalues of A and $n_1+n_2+\cdots+n_k = n$. Following the previous section, we define a Jordan commutant of a symmetric matrix A as

$$A^{\#} = \{X \in \mathcal{S}_n \ : \ A \circ X = 0\}.$$

Note that this is always a complex vector space and we will be in particular interested in its dimension, $\dim_{\mathbb{C}} A^{\#}$. We emphasize that, contrary to the previous section, here $A^{\#}$ contains only symmetric matrices.

Though symmetric and Hermitian matrices look similar, there is a fundamental difference between them. Namely, a symmetric matrix can well be nilpotent. To distinguish rank one symmetric matrices from the others, we will require the following two lemmas.

Lemma 3.1. *Suppose the m vectors $x_1, \dots, x_m \in \mathbb{C}^n$ are linearly independent. Then, $\bigcap_{k=1}^m (x_k x_k^t)^{\#}$ is an $\frac{(n-m)(n-m+1)}{2}$ dimensional subspace of $\mathcal{S}_n$ spanned by rank one symmetric matrices.*

Proof. Pick any nonzero vector x, and consider $A \in (xx^t)^{\#}$. By definition, $0 = Axx^t + xx^tA = (Ax)x^t + x(Ax)^t$. This is possible only when Ax and x are linearly dependent, which further forces $Ax = 0$. Any symmetric matrix can be written as $A = \sum_{k=1}^r \lambda_k z_k z_k^t$, where z_k are linearly independent vectors and $r = \mathrm{rk}\, A$. Then, $Ax = 0$ gives $\sum \lambda_k (z_k^t x) z_k = 0$, which, by linearly independence of z_k further forces $z_k^t x = 0$. Consequently, given $A = \sum_{k=1}^r \lambda_k z_k z_k^t \in \bigcap_{j=1}^m (x_j x_j^t)^{\#}$ we have $z_k^t x_j = 0$ for every k, j. This gives that each $A \in \bigcap_{j=1}^m (x_j x_j^t)^{\#}$ is spanned by rank one symmetric matrices $z_k z_k^t \in \bigcap_{j=1}^m (x_j x_j^t)^{\#}$.

As for the dimension of $\bigcap_{k=1}^m (x_k x_k^t)^{\#}$, a rank one symmetric matrix zz^t belongs to this space if and only if $x_k^t z = 0$ for $k = 1, \dots, m$. Equivalently, if and only if z belongs to the common kernel of functionals $F_{x_k} : z \mapsto x_k^t z$. Clearly, the linear independence of x_k implies linear independence of functionals F_{x_k}. Namely, $F_x = 0$ precisely when $x^t \mathbb{C}^n = 0$, giving $x = 0$. Having m linearly independent functionals, their common kernel is $(n-m)$

dimensional. Since $\bigcap_{k=1}^m (x_k x_k^t)^\#$ is spanned by rank one vectors from this common kernel we have $\dim_{\mathbb{C}} \bigcap_{k=1}^m (x_k x_k^t)^\# = \frac{(n-m)(n-m+1)}{2}$, as claimed. □

Corollary 3.2. *If $A \in \mathcal{S}_n$ is of rank one, then* $\dim A^\# = \frac{n(n-1)}{2}$.

Lemma 3.3. *For nonzero $A \in \mathcal{S}_n$,*

$$\dim A^\# \leq \frac{n(n-1)}{2} = \dim \mathcal{S}_n - n.$$

Further, the equality holds if and only if A is rank one.

Proof. With the help of orthogonal similarity we may assume that

$$A = S_{n_1}(\lambda_1) \oplus S_{n_2}(\lambda_2) \oplus \cdots \oplus S_{n_k}(\lambda_k) \oplus 0_{n_{k+1}}$$

with $n_1 + n_2 + \cdots + n_k + n_{k+1} = n$ and $S_{n_j}(\lambda_j) \neq 0$ for $j = 1, \ldots, k$. Here, n_{k+1} can possibly be zero. First we will prove that our lemma holds true when $k = 1$ with $n_1 = n$. So, let $A = S_n(\lambda)$. To compute $\dim A^\#$ we pick any $B \in A^\# = S_n(\lambda)^\#$. Recall that $S_n(\lambda)$ is similar to an elementary Jordan upper-triangular block $J_n(\lambda) = \lambda \operatorname{Id}_n + N_n$. If T is the similarity among them, then $BA + AB = 0$ is equivalent to

$$(\lambda \operatorname{Id}_n + N_n)(T^{-1}BT) + (T^{-1}BT)(\lambda \operatorname{Id}_n + N_n) = 0.$$

When $\lambda \neq 0$ the Lumer–Rosenblum theorem [20] implies that the only solution is $T^{-1}BT = 0$, giving $A^\# = 0$. Assume now that $\lambda = 0$. Then, an easy computation gives that the solution equals

$$T^{-1}BT = \sum_{\substack{0 \leq i \leq n-1 \\ 1 \leq j \leq n-i}} \alpha_i (-1)^j E_{j(j+i)}$$

for some scalars α_i. Note however that not all choices of α_i will give B symmetric. Therefore, $\dim A^\# \leq n < \frac{n(n-1)}{2}$ if $n > 3$. But if $n = 3$ it is easy to see that $\dim A^\# = \dim \begin{bmatrix} 0 & 1-i & 0 \\ 1-i & 0 & 1+i \\ 0 & 1+i & 0 \end{bmatrix}^\# = \dim \left\{ \begin{bmatrix} y-2ix & 0 & x \\ 0 & ix-y & 0 \\ x & 0 & y \end{bmatrix} \; : \; x, y \in \mathbb{C} \right\} = 2 < \frac{3 \cdot 2}{2}$.

Now, suppose that $k > 1$ or $k = 1$ with $n_1 < n$. Note that $\lambda_j = 0$ implies $n_j \geq 2$ for each $j = 1, \ldots, k$. By the first part, $\dim S_{n_j}(\lambda_j)^\# < \dim \mathcal{S}_{n_j} - n_j$ whenever $n_j \geq 3$. It is easy to compute that $n_j = 2$ gives $\dim S_{n_j}(\lambda_j)^\# < \dim \mathcal{S}_{n_j} - n_j$ unless $\operatorname{rk} S_{n_j}(\lambda_j) = 1$. Next, given $1 \leq i < j \leq k+1$, define

$$\mathcal{T}_{ij} = \{B \in \mathcal{M}_{n_i,n_j} \; : \; S_{n_i}(\lambda_i)B + BS_{n_j}(\lambda_j) = 0\}.$$

Clearly, $\dim \mathcal{T}_{ij} < \dim \mathcal{M}_{n_i,n_j}$. Further when $n_{k+1} > 0$, $\dim \mathcal{T}_{i,k+1} \leq \dim \mathcal{M}_{n_i,n_{k+1}} - n_{k+1}$ and equality holds only if $\operatorname{rk} S(\lambda_i) = 1$. Note that

$A^{\#} = \{[B_{ij}] \in \mathcal{S}_n \ : \ B_{ij} \in \mathcal{T}_{ij}\}$. Then

$$\begin{aligned}
\dim A^{\#} &= \sum_{1\le j\le k} \dim S_{n_j}(\lambda_j)^{\#} + \dim \mathcal{S}_{n_{k+1}} + \sum_{1\le i<j\le k+1} \dim \mathcal{T}_{i,j} \\
&\le \sum_{1\le j\le k} (\dim \mathcal{S}_{n_j} - n_j) + \dim \mathcal{S}_{n_{k+1}} \\
&\quad + \sum_{1\le i<j\le k} \dim \mathcal{M}_{n_i,n_j} + \sum_{1\le i\le k} (\dim \mathcal{M}_{n_i,n_{k+1}} - n_{k+1}) \\
&= \sum_{1\le j\le k+1} \dim \mathcal{S}_{n_j} + \sum_{1\le i<j\le k+1} \dim \mathcal{M}_{n_i,n_j} - (k-1)n_{k+1} - \sum_{1\le j\le k+1} n_j \\
&\le \dim \mathcal{S}_n - 0 - n = \dim \mathcal{S}_n - n.
\end{aligned}$$

The inequality is strict unless (i) $\dim S_{n_j}(\lambda_j)^{\#} = \dim \mathcal{S}_{n_j} - n_j$, and (ii) $\dim \mathcal{T}_{i,j} = \dim \mathcal{M}_{n_i,n_j}$ for all $1 \le i < j \le k$, and (iii) some further conditions. But $\dim \mathcal{T}_{i,j} < \dim \mathcal{M}_{n_i,n_j}$ so the last condition gives $k = 1$. Now the first identity imply that the block $S_{n_1}(\lambda_1)$ is rank one. Hence, the equality holds only if $k = 1$, $n_1 < n$, and $\operatorname{rk} S_{n_1}(\lambda_1) = 1$, i.e., only if $\operatorname{rk} A = 1$. The inverse implication was proven in Corollary 3.2. □

We can now state the main result of this section.

Theorem 3.4. *Let $n \ge 3$ and let $\phi : \mathcal{S}_n \to \mathcal{S}_n$ be an injective continuous map that preserves zeros of Jordan product. Then $\phi(0) = 0$ and there exists an orthogonal matrix Q such that either*

(i) $$\phi(A) = \lambda_A Q A Q^t$$

for every rank one matrix $A \in \mathcal{S}_n$, or

(ii) $$\phi(A) = \lambda_A Q \overline{A} Q^t$$

for every rank one matrix $A \in \mathcal{S}_n$. Here, λ_A is a nonzero complex number determined by A.

Remark 3.5. As in the Hermitian case we cannot expect to get a nice structural result for all symmetric matrices. As a matter of fact, Example 2.11 works also here.

In the proof of above theorem we will require the next lemma.

Lemma 3.6. *Suppose a continuous and injective $\phi : \mathcal{S}_n \to \mathcal{S}_n$ preserves zeros of Jordan product, and also preserves the set of rank one symmetric matrices. If $x_1, \dots, x_k$ are linearly independent vectors and $\lambda_i \in \mathbb{C} \setminus \{0\}$, then $\phi(\lambda_i x_i x_i^t) = \mu_i z_i z_i^t$, where $\mu_i \ne 0$, and $z_1, \dots, z_k$ are also independent.*

Proof. We proceed by induction on k. For $k = 1$ there is nothing to do. Assume Lemma 3.6 holds for some $k \ge 1$. Pick now a linearly independent $k+1$ tuple $x_1, \dots, x_{k+1}$, and denote $P_i = \lambda_i x_i x_i^t$. Therefore, $\phi(P_i) = \mu_i z_i z_i^t$ for some nonzero z_i and nonzero μ_i. Moreover, by the inductive assumption, $z_1, \dots, z_k$ are linearly independent. Assume erroneously that z_{k+1} is their linear combination. Then, if a symmetric A annihilates all of $z_1, \ldots, z_k$ then also $Az_{k+1} = 0$. Since $A \circ zz^t = 0$ precisely when $Az = 0$, this shows that

$$\bigcap_{i=1}^{k} \phi(P_i)^{\#} \subseteq (\mu_{k+1} z_{k+1} z_{k+1}^t)^{\#} = \phi(P_{k+1})^{\#}. \tag{17}$$

Next, due to the assumed linear independence of $z_1, \ldots, z_k$, Lemma 3.1 gives that

$$\Xi = \bigcap_{i=1}^{k} \phi(P_i)^{\#}$$

is an $\frac{(n-k)(n-k+1)}{2}$ dimensional complex subspace of $\mathcal{S}_n$. The same holds for the subspace $\Omega = \bigcap_{i=1}^{k} P_i^{\#}$, which is mapped injectively and continuously into Ξ by a map ϕ. Since both subspaces have the same dimension, $\phi(\Omega)$ is an open subset of Ξ, by the invariance of domain for continuous injective maps [10, p.344], it clearly contains $\phi(0) = 0$. By the same argument, $\phi(P_{k+1}^{\#})$ is an open subset inside $\phi(P_{k+1})^{\#}$ which contains $\phi(0) = 0$. Now, in view of Eq. (17),

$$\phi(\Omega) \cap \phi(P_{k+1}^{\#}) = \big(\phi(\Omega) \cap \Xi\big) \cap \big(\phi(P_{k+1}^{\#}) \cap \Xi\big).$$

But this is the intersection of two open subsets in Ξ, both containing 0, so it is nonempty and open in Ξ.

Now, due to linear independence of $x_1, \ldots, x_{k+1}$, there exists an $R \in \Omega \setminus P_{k+1}^{\#}$. By continuity, $\phi(tR) \in \phi(\Omega) \subseteq \Xi$ tends to $\phi(0) = 0 \in \Xi$ as $t \to 0$. So, for small enough nonzero t, the matrix $\phi(tR)$ belongs to an open neighborhood $\phi(\Omega) \cap \phi(P_{k+1}^{\#})$ of a matrix $0 \in \Xi$. In particular, at least some $B \in P_{k+1}^{\#}$ must be mapped into $\phi(tR)$. However, $tR \notin P_{k+1}^{\#}$, which contradicts injectivity of ϕ. □

Proof of Theorem 3.4. Note that $A = 0$ if and only if $A^{\#} = \mathcal{S}_n$. Now, since ϕ preserves zeros of Jordan product it follows that for every symmetric matrix $A \in \mathcal{S}_n$ we have $\phi(A^{\#}) \subseteq \phi(A)^{\#}$. In particular, $\phi(\mathcal{S}_n) = \phi(0^{\#}) \subseteq \phi(0)^{\#}$. Recall that $\phi(0)^{\#}$ is a subspace of $\mathcal{S}_n$, and since ϕ is injective and continuous, it follows by the invariance of domain theorem that $\phi(0)^{\#}$ cannot be contained inside a proper linear subspace of $\mathcal{S}_n$. This yields that $\phi(0)^{\#} = \mathcal{S}_n$, and consequently, $\phi(0) = 0$.

Let $A \in \mathcal{S}_n \setminus \{0\}$. Then $(\lambda A)^{\#} = A^{\#}$ for every nonzero complex number λ. By Lemma 3.1 (with $m = 1$) we see that the Jordan commutant of every rank one symmetric matrix has dimension $\frac{n(n-1)}{2}$. By Lemma 3.3, $\dim A^{\#} = \frac{n(n-1)}{2}$ if and only if A is a scalar multiple of a rank one symmetric matrix. As a consequence, every rank one $A \in \mathcal{S}_n$ is mapped into a matrix of rank one. Otherwise, ϕ would map $A^{\#}$, which is of dimension $\frac{n(n-1)}{2}$, continuously and injectively into $\phi(A)^{\#}$, whose dimension is strictly smaller than $\frac{n(n-1)}{2}$. But this is impossible by the invariance of domain theorem.

We have proved that for every symmetric matrix A of rank one there exists a rank one symmetric matrix B such that $\phi(A) = B$. Moreover, $\phi(\mathbb{C}A) \subseteq \mathbb{C}B$. Indeed, if there was a nonzero complex number λ such that $\phi(\lambda A) \notin \mathbb{C}B$, then ϕ would map $A^{\#} = (\lambda A)^{\#}$ injectively and continuously into $B^{\#} \cap \phi(\lambda A)^{\#}$, which would be a proper subset of $B^{\#}$. But this is impossible by the invariance of domain theorem. Furthermore, by Lemma 3.6 we also have that whenever A_1, A_2 are linearly independent rank one symmetric matrices, with $\phi(\mathbb{C}A_i) \subseteq \mathbb{C}B_i$, $i = 1, 2$, then B_1, B_2 must be linearly independent rank one symmetric matrices as well.

We claim that ϕ preserves orthogonality on rank one symmetric matrices. Indeed, recall from the proof of Lemma 3.1 that rank one symmetric matrices A_1, A_2 satisfy $A_1 \circ A_2 = 0$ if and only if they are orthogonal, i.e $A_1A_2 = 0 = A_2A_1$. Now, let $A_1, A_2 \in \mathcal{S}_n$ be two orthogonal rank one symmetric matrices with $\phi(\mathbb{C}A_k) \subseteq \mathbb{C}B_k$, $k = 1, 2$. Then, because $A_1 \circ A_2 = 0$, we have $B_1 \circ B_2 = 0$. Thus, B_1 and B_2 are also orthogonal.

We can now invoke the fundamental theorem of projective geometry as follows. Every symmetric matrix of rank one can be written as $A = xx^t$, where a column vector x is uniquely determined by A up to a scalar multiple. So, we can identify rank one symmetric matrices in a natural way with the elements of the projective space $\mathbb{P}(\mathbb{C}^n)$. By what we have proved, the map ϕ induces an injective map $\varphi : \mathbb{P}(\mathbb{C}^n) \to \mathbb{P}(\mathbb{C}^n)$, defined by $\varphi([x]) = [y]$ if and only if $A = xx^t$ and $B = yy^t$ with $\phi(\mathbb{C}A) \subseteq \mathbb{C}B$. We will show that $[z] \subseteq [x] + [y]$ yields $\varphi([z]) \subseteq \varphi([x]) + \varphi([y])$. There is nothing to prove if $[x] = [y]$. So, assume that x and y are linearly independent. As in Lemma 3.1 we denote by $F_x : z \mapsto x^t z$. Since x, y are linearly independent, it is easy to see that F_x, F_y are linearly independent, as well. It follows that $\operatorname{Ker} F_x \cap \operatorname{Ker} F_y$ is an $(n-2)$ dimensional subspace. Let $w_3, \dots, w_n \in \operatorname{Ker} F_x \cap \operatorname{Ker} F_y$ be its basis. Denote by $[\widetilde{x}] = \varphi([x])$, $[\widetilde{y}] = \varphi([y])$, $[\widetilde{z}] = \varphi([z])$, and $[\widetilde{w_k}] = \varphi([w_k])$, $k = 3, \dots, n$. In view of Lemma 3.6, the vectors $\widetilde{w}_3, \dots, \widetilde{w}_n$ are also linearly independent. Note also that $(xx^t) \circ (w_k w_k^t) = 0$ implies $(\widetilde{x}\widetilde{x}^t) \circ (\widetilde{w}_k \widetilde{w}_k^t) = 0$, which further gives $\widetilde{w_k}^t \widetilde{x} = 0$. Similarly we deduce also $\widetilde{w_k}^t \widetilde{y} = \widetilde{w_k}^t \widetilde{z} = 0$, $k = 3, \dots, n$. Thus, the vectors $\widetilde{x}, \widetilde{y}, \widetilde{z}$ all belong to two dimensional subspace $\operatorname{Ker} F_{\widetilde{w}_3} \cap \cdots \cap \operatorname{Ker} F_{\widetilde{w}_n}$ and so they are linearly dependent. Moreover, Lemma 3.6 forces $\widetilde{x}, \widetilde{y}$ to be linearly independent, so we easily conclude that $\varphi([z]) \subseteq \varphi([x]) + \varphi([y])$, as desired.

Denote by $[\widetilde{e}_i] = \varphi([e_i])$, where e_i is the standard basis of $\mathbb{C}^n$. Again, Lemma 3.6 forces that $\widetilde{e}_1, \dots, \widetilde{e}_n$ are n linearly independent vectors. As $n \geq 3$ we see that the image of φ is not contained in a projective line. Moreover, it also follows that φ is an injective function. We can, therefore, apply [9, Theorem 3.1], with $g = \varphi$, $\mathcal{V}_1 = \mathbb{P}(\mathbb{C}^n) = \mathcal{V}_2$, and with the kernel of φ, $\mathcal{P}(W) = \emptyset$, to deduce that $\varphi([x]) = [Ax]$ for some σ-quasilinear map A. It only remains to show that σ is a complex conjugation or an identity homomorphism.

By definition of φ, one has $\phi(\lambda xx^t) = A(\lambda x)(Ax)^t = \sigma(\lambda)\,(Ax)(Ax)^t$. Since A is nonzero, there exists a vector x with $Ax \neq 0$. Now, continuity of ϕ forces continuity of the map $\lambda \mapsto \sigma(\lambda)\,(Ax)(Ax)^t$, which in turn implies continuity of σ. Clearly, the only continuous field homomorphism of $\mathbb{C}$ is either identity or complex conjugation. $\square$

4. Zeros of generalized Jordan product

The results in the previous sections can be extended to other products, say $A \circ_\xi B = AB + \xi BA$ when $\xi \neq \pm 1$. Here we are facing a peculiar phenomenon, namely, given $\xi \neq \pm 1$ then $A, B \in \mathcal{H}_n$ satisfy $AB + \xi BA = 0$ if and only if $AB = 0 = BA$ [6, Theorem 1.1]. As a corollary, if $\phi : \mathcal{H}_n \to \mathcal{H}_n$ preserves the zeros of a polynomial $p(x, y) = xy + \xi yx$ and $\xi \neq \pm 1$, then ϕ preserves orthogonality.

Theorem 4.1. *Let $\xi \neq \pm 1$, let $n \geq 3$, and suppose that an injective map $\phi : \mathcal{H}_n \to \mathcal{H}_n$ satisfies*

$$AB + \xi BA = 0 \Longrightarrow \phi(A)\phi(B) + \xi\phi(B)\phi(A) = 0.$$

Then ϕ takes one of the two forms in Theorem 2.1 on rank one Hermitian matrices.

Proof. Clearly, there exists n pairwise orthogonal nonzero Hermitian matrices and up to unitary similarity every such n-tuple equals $t_1E_{11}, \dots, t_nE_{nn}$ (see [7, Lemma 2.2]). Now, choose any rank one projection P_1. Augment it with rank one projections $P_2, \dots, P_n$ to a maximal pairwise orthogonal set Ξ. Since ϕ is injective, either $\phi(\Xi)$ does not contain a zero matrix or else this holds for $\phi(2\,\Xi)$. Whatever the case does occur, since ϕ preserves orthogonality, $\phi(0)$ is orthogonal to every member from $\phi(\Xi)$ and also from $\phi(2\,\Xi)$ wherefrom $\phi(0) = 0$. By injectivity, no other matrix is annihilated by ϕ. Therefore, $Q_i = \phi(P_i)$ are n distinct, nonzero and pairwise orthogonal Hermitian matrices, so they are all of rank one. Thus, ϕ preserves the set of rank one Hermitian matrices as well as their orthogonality. The result now follows by imitating the proof of Theorem 2.1. $\square$

Remark 4.2. A similar phenomenon holds also for symmetric matrices. Indeed, assume $\xi \neq \pm 1$. If $AB = \xi BA$, then, upon transposing this equality, we get $BA = \xi AB = \xi^2 BA$. Hence, $BA = 0 = AB$. Moreover, introducing $A^{\#_\xi} = \{X \in \mathcal{S}_n \,:\, XA + \xi AX = 0\} = \{X \in \mathcal{S}_n \,:\, XA = 0\}$, it can be shown (c.f. Lemma 3.3) that, for a nonzero A, we have $\dim A^{\#_\xi} \leq \frac{n(n-1)}{2}$ and the equality holds if and only if $\operatorname{rk} A = 1$. So, if $\phi : \mathcal{S}_n \to \mathcal{S}_n$ is a continuous injection which preserves zeros of $A \circ_\xi B$, it also preserves orthogonality on symmetric matrices and it preserves the set of rank one symmetric matrices. Adapting the proof of Theorem 3.4 one can derive the same results on rank one symmetric matrices also in this case.

5. Applications

At the end let us give two applications of our results. In the first application we also refer to Zhao and Hou [34, Theorem 1.3] for additive surjections which preserve zeros of Jordan product in both directions on self adjoint operators on infinite dimensional complex Hilbert space, and we refer to Zhao and Hou [33] for additive surjections which preserve zeros of Jordan product on algebra $B(H)$ of bounded operators on infinite dimensional Hilbert space.

Proposition 5.1. *Let $n \geq 3$. If an additive and injective map $\phi : \mathcal{H}_n \to \mathcal{H}_n$ preserves zeros of Jordan product in one direction only then the results of Theorem 2.1 hold for every Hermitian A, with $t_A = t$ independent of A.*

Our final application of Corollary 2.2 will classify Jordan homomorphisms on $\mathcal{H}_n$. A *Jordan homomorphism* is a (possibly nonadditive) map which satisfies

$$\phi(AB + BA) = \phi(A)\phi(B) + \phi(B)\phi(A).$$

We show that even with such a limited assumptions, linearity comes for free, except when ϕ is a constant. For a similar treatment with bijective ϕ on standard operator algebras we refer to a work by Molnár [21].

Proposition 5.2. *Let $n \geq 3$ and let $\phi : \mathcal{H}_n \to \mathcal{H}_n$ be a Jordan homomorphism. Then it takes one of the following forms*

(i) $$\phi(X) = \tfrac{1}{2}P; \qquad (X \in \mathcal{H}_n),$$

(ii) $$\phi(X) = UXU^*,$$

(iii) $$\phi(X) = UX^tU^*.$$

Here, P is a projection and U is unitary.

Proof of Proposition 5.1. Start with Theorem 2.1. Since unitary similarity and transposition preserve the zeros of Jordan product we may already assume $\phi(X) = t_X X$ for every rank one X. It remains to show that t_X is constant.

Choose any orthogonal rank one projections P, Q and a scalar $a \neq 0$ to form a rank two, trace-zero matrix $X = a(P - Q)$. By additivity, $\phi(X) = \phi(aP) - \phi(aQ) = t_1 P - t_2 Q$. However, it follows by Lemma 2.7 and its proof that ϕ preserves the set of trace-zero matrices with rank two. So $\operatorname{Sp}\phi(X) = t\{-1, 0, 1\}$ which gives $t_1 = t_2 = t(a)$. If P, Q are not orthogonal rank one projections we can easily construct another rank one projection R, which is orthogonal to both P and Q and then repeat the above arguments on (aP, aR) and on (aR, aQ) to find that

$$\phi(aP) = t(a)P; \qquad \text{for every} \quad P^2 = P \in \mathcal{H}_n,\ \operatorname{rk} P = 1.$$

Since ϕ is additive, so is t.

Next, $D_{12} = E_{12} - E_{21} = P - Q$, where $P = \frac{1}{2}\left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right) \oplus 0$ and $Q = \frac{1}{2}\left(\begin{smallmatrix} 1 & -1 \\ -1 & 1 \end{smallmatrix}\right) \oplus 0$ are rank one projections. Hence, $\phi(aD_{12}) = \phi(aP) - \phi(aQ) =$

$t(a)(P-Q) = t(a)D_{12}$. Likewise $\phi(aD_{ij}) = t(a)D_{ij}$ for any $i \neq j$. Moreover, $\mathbf{i}(E_{12} - E_{21}) = P_2 - Q_2$ where $P_2 = \frac{1}{2}\left(\begin{smallmatrix} 1 & \mathbf{i} \\ -\mathbf{i} & 1 \end{smallmatrix}\right) \oplus 0$ and $Q_2 = \frac{1}{2}\left(\begin{smallmatrix} 1 & -\mathbf{i} \\ \mathbf{i} & 1 \end{smallmatrix}\right) \oplus 0$, and yet again, $\phi(aF_{ij}) = t(a)F_{ij}$.

Given $a \in \mathbb{R}$ we derive that a rank one Hermitian $R = a^2E_{11} + aD_{12} + E_{22}$ is mapped into a rank one $\phi(R) = t(a^2)E_{11} + t(a)D_{12} + t(1)E_{22}$. But this is Hermitian only when $t(a) \in \mathbb{R}$, and is of rank one only when $t(a^2)t(1) = t(a)^2$. Clearly, $t(1) = 0$ implies $\phi(E_{11}) = 0$, which contradicts injectivity. But then, $f : \mathbb{R} \to \mathbb{R}$, defined by $f : x \mapsto t(x)/t(1)$ is additive and multiplicative. It is well known that $\mathbb{R}$ admits only one such nonzero function, namely identity. Thus, $t(a) = t(1) \cdot a$ for $a \in \mathbb{R}$, and $A = \sum a_{ii}E_{ii} + \sum \beta_{ij}D_{ij} + \sum \delta_{ij}F_{ij}$ is mapped into $\phi(A) = \sum t(a_{ii})E_{ii} + \sum t(\beta_{ij})D_{ij} + \sum t(\delta_{ij})F_{ij} = t(1)\,A$, for any Hermitian A. □

We proceed with the proof of Proposition 5.2. We will rely on Lemma 5.3 below, which says that every $A \in \mathcal{H}_n$ can be written as a Jordan product of a finitely many rank one Hermitian matrices.

Denote by $\Delta_n \subseteq \mathcal{H}_n$ be the Jordan hull of rank one Hermitian matrices, that is, the smallest subset in $\mathcal{H}_n$ closed under the operation $(A, B) \mapsto A \circ B$ and containing all rank one Hermitian matrices.

Lemma 5.3. $\Delta_n = \mathcal{H}_n$.

Proof. Induction on the rank.

By its definition, if $\operatorname{rk} A = 1$ then $A \in \Delta_n$. Also, $0 = E_{11} \circ E_{22} \in \Delta_n$. So assume we have already shown that Δ_n contains all Hermitian matrices of rank at most k. If $k = n$ we are done. If $k < n$, pick any Hermitian A with $\operatorname{rk} A = k+1$. Using unitary similarity we may assume $A = \operatorname{diag}(a_1, \ldots, a_k, a_{k+1}) \oplus 0_{n-k-1} = \operatorname{diag}(\vec{a}, a_k, a_{k+1}, \mathbf{0})$, where a_i are nonzero, $\vec{a} = (a_1, \ldots, a_{k-1})$, and $\mathbf{0}$ is the zero vector of length $n-k-1$. We first show that

$$\operatorname{diag}(\vec{c}, c_k, xc_k, \mathbf{0}) \in \Delta_n \qquad \text{for any } x \in (-1, 0) \text{ and } (\vec{c}, c_k) \in \mathbb{R}^k. \tag{18}$$

There is nothing to do if $c_k = 0$ since then, $\operatorname{rk}\operatorname{diag}(\vec{c}, c_k, xc_k, \mathbf{0}) \leq k-1$. Otherwise, let $X_{k-1} = \operatorname{diag}(\vec{c}) \in \mathcal{H}_{k-1}$, let $X = \operatorname{diag}(\vec{c}, c_k, 0, \mathbf{0}) = X_{k-1} \oplus c_k \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \oplus 0_{n-k-1}$, and define for every $t \in (0, 1)$ the Hermitian matrix

$$Q_t = (\tfrac{1}{2}\operatorname{Id}_{k-1}) \oplus \tfrac{1}{t+\sqrt{t}} \begin{pmatrix} t & \sqrt{(1-t)t} \\ \sqrt{(1-t)t} & 1-t \end{pmatrix} \oplus 0_{n-k-1}.$$

It is easy to see that $\operatorname{rk} X \leq \operatorname{rk} Q_t = k$. Therefore, by the inductive hypothesis, $X, Q_t \in \Delta_n$. Since Δ_n is closed under Jordan product we get

$$X \circ Q_t = X_{k-1} \oplus \tfrac{c_k}{t+\sqrt{t}} \begin{pmatrix} 2t & \sqrt{(1-t)t} \\ \sqrt{(1-t)t} & 0 \end{pmatrix} \oplus 0_{n-k-1} \in \Delta_n.$$

The eigenvalues of the middle summand equal $\frac{c_k}{t+\sqrt{t}}(t \pm \sqrt{t}) = c_k \cdot (1, x)$ where $x = \frac{t-\sqrt{t}}{t+\sqrt{t}}$. As t ranges over $(0, 1)$, $x = \frac{t-\sqrt{t}}{t+\sqrt{t}}$ takes the values from $(-1, 0)$. So, given any $x \in (-1, 0)$ there exists a suitably chosen $t \in (0, 1)$, and a unitary matrix of the form $U = \mathrm{Id}_{k-1} \oplus U' \oplus \mathrm{Id}_{n-k-1}$ such that $\mathrm{diag}(\vec{c}, c_k, xc_k, \mathbf{0}) = (U^*XU) \circ (U^*Q_tU) \in \Delta_n$, as anticipated.

Next, $\left(\begin{smallmatrix} 1 & 0 \\ 0 & y \end{smallmatrix}\right)$ is unitarily equivalent to $\left(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Therefore, it follows from the above that also $\mathrm{diag}(\frac{1}{2}\vec{1}, y, 1, \mathbf{0}) = (\frac{1}{2}\mathrm{Id}_{k-1}) \oplus \mathrm{diag}(y, 1, \mathbf{0}) \in \Delta_n$ for any $y \in (-1, 0)$. But then,

$$\mathrm{diag}(\vec{c}, 2yc_k, 2xc_k, \mathbf{0}) = \mathrm{diag}(\vec{c}, c_k, xc_k, \mathbf{0}) \circ \mathrm{diag}(\tfrac{1}{2}\vec{1}, y, 1, \mathbf{0}) \in \Delta_n$$

for any $x, y \in (-1, 0)$. Given arbitrary positive number z, there exists $x, y \in (-1, 0)$ such that $x/y = z$. We may further let $\vec{c} = \vec{a}$ and choose $c_k = \frac{a_k}{2y}$ to get $\mathrm{diag}(\vec{a}, a_k, za_k, \mathbf{0}) \in \Delta_n$, for every $z > 0$.

From Eq. (18) we already know that $\mathrm{diag}(\frac{1}{2}\vec{1}, \frac{1}{2}, \frac{-1}{4}, \mathbf{0}) \in \Delta_n$; combined with the above result we find

$$\mathrm{diag}(\vec{a}, a_k, \tfrac{-z}{2}a_k, \mathbf{0}) = \mathrm{diag}(\vec{a}, a_k, za_k, \mathbf{0}) \circ \mathrm{diag}(\tfrac{1}{2}\vec{1}, \tfrac{1}{2}, \tfrac{-1}{4}, \mathbf{0}) \in \Delta_n; \quad (z > 0).$$

But then, $\mathrm{diag}(\vec{a}, a_k, ta_k, \mathbf{0}) \in \Delta_n$ for every nonzero $t \in \mathbb{R}$, and in particular also $A = \mathrm{diag}(\vec{a}, a_k, a_{k+1}, \mathbf{0}) \in \Delta_n$. □

Proof of Proposition 5.2. For brevity let us say that Hermitian matrices A, B are Jordan–orthogonal if $A \circ B = 0$. We proceed in three steps.

Assertion 1. Let $m \geq 1$ and let $\phi' : \mathcal{H}_n \to \mathcal{H}_m$ be a Jordan homomorphism. Then either $\phi'(X) = 0$ for every X or else $\phi'(A) = 0$ implies $A = 0$.

Namely assume that ϕ' annihilates nonzero A. Among all these matrices, take a matrix A with maximum rank value. That is, $\mathrm{rk}(A) \geq \mathrm{rk}(B)$ for all B with $\phi'(B) = 0$. Suppose $k = \mathrm{rk}(A) < n$. Without loss of generality, we may assume that $A = \mathrm{diag}(a_1, \dots, a_k) \oplus 0_{n-k}$ for some nonzero $a_j \in \mathbb{R}$. Consider $B = 2A + a_k(E_{k,k+1} + E_{k+1,k}) = A \circ (\mathrm{Id} + E_{k,k+1} + E_{k+1,k})$. Then $\phi'(B) = \phi'(A) \circ \phi'(\mathrm{Id} + E_{k,k+1} + E_{k+1,k}) = 0$. But $\mathrm{rk}(B) = k + 1 > \mathrm{rk}(A)$, which contradicts to the choice of A. Therefore, $\mathrm{rk}(A) = n$. It follows that $\phi'(\frac{1}{2}\mathrm{Id}) = \phi'(A \circ (\frac{1}{4}A^{-1})) = 0$ and hence $\phi'(X) = \phi'(X \circ \frac{1}{2}\mathrm{Id}) = 0$ for all $X \in \mathcal{H}_n$.

Assertion 2. Either ϕ is constant or $\phi(0) = 0$.
Namely, if $P = \phi(0) = \phi(0) \circ \phi(0) = 2\phi(0)^2$, so $2P = (2P)^2$ is a projection. Using unitary similarity we may assume $P = \frac{1}{2}\mathrm{Id}_k \oplus 0_{n-k}$. Then, given any X we have $P = \phi(X \circ 0) = \phi(X) \circ P$, which is possible only when $\phi(X) = P \oplus \phi'(X)$ for some Jordan homomorphism $\phi' : \mathcal{H}_n \to \mathcal{H}_{n-k}$, which satisfies $\phi'(0) = 0$. Combining with the previous step, ϕ' is either zero or else it preserves zeros of Jordan product in both directions. In the later case, ϕ' maps $\lfloor \frac{3n}{2} \rfloor$ pairwise Jordan–orthogonal matrices from $\mathcal{H}_n$ into the same number of pairwise Jordan–orthogonal matrices inside $\mathcal{H}_{n-k}$. By Lemma 2.5 this is possible only when $k = 0$. So either $\phi(0) = 0$ or $\phi(X) = P \oplus 0_{n-k}$ for all $X \in \mathcal{H}_n$.

Assertion 3. Assume ϕ is nonconstant. By Assertions 1-2 it preserves zeros of Jordan product in both directions. So, by Corollary 2.2, it takes the forms (i)–(ii) of Theorem 2.1 on rank one Hermitian matrices. Replacing ϕ by $U^*\phi(\cdot)U$ or by $(U^*\phi(\cdot)U)^t$ we obtain a Jordan homomorphism which fixes all rank one Hermitian matrices modulo scalars. We will denote the new map again by ϕ. The set $\Xi = \{X \in \mathcal{H}_n \,:\, \exists c_X \in \mathbb{R}\backslash\{0\} \text{ such that } \phi(X) = c_X X\}$ contains all rank one Hermitian matrices and is closed under Jordan product. Hence, Lemma 5.3 gives

$$\phi(A) = c_A A; \qquad A \in \mathcal{H}_n.$$

It remains to show that $c_A = 1$. Firstly, given any projection P we easily deduce that $(\frac{1}{2}P) \circ (\frac{1}{2}P) = (\frac{1}{2}P)$ implies $\phi(\frac{1}{2}P) = \frac{1}{2}P$. Next, arguing as in the proof of Lemma 5.3, we see that $(\frac{t-\sqrt{t}}{4}E_{11} + \frac{t+\sqrt{t}}{4}E_{22}) = (\frac{1}{2}VE_{11}V^*) \circ (\frac{1}{2}VQ_tV^*) \in \Delta_n$ for suitably chosen unitary V, where $Q_t = \begin{pmatrix} t & \sqrt{(1-t)t} \\ \sqrt{(1-t)t} & (1-t) \end{pmatrix} \oplus 0_{n-2}$ is a rank one projection for $t \in (0,1)$. Hence

$$\begin{aligned}\phi(\tfrac{t-\sqrt{t}}{4}E_{11} + \tfrac{t+\sqrt{t}}{4}E_{22}) &= \phi(\tfrac{1}{2}VE_{11}V^*) \circ \phi(\tfrac{1}{2}VQ_tV^*) \\ &= (\tfrac{1}{2}VE_{11}V^*) \circ (\tfrac{1}{2}VQ_tV^*) = (\tfrac{t-\sqrt{t}}{4}E_{11} + \tfrac{t+\sqrt{t}}{4}E_{22}); \quad t \in (0,1),\end{aligned}$$

and therefore also

$$\begin{aligned}\phi(\tfrac{x}{2}E_{11}) &= \phi(\tfrac{t-\sqrt{t}}{4}E_{11} + \tfrac{t+\sqrt{t}}{4}E_{22}) \circ \phi(\tfrac{1}{2}E_{11}) \\ &= (\tfrac{t-\sqrt{t}}{4}E_{11} + \tfrac{t+\sqrt{t}}{4}E_{22}) \circ (\tfrac{1}{2}E_{11}) = \tfrac{x}{2}E_{11}; \quad x = \tfrac{t-\sqrt{t}}{2} \in (-\tfrac{1}{8}, 0).\end{aligned}$$

Next, $\phi(c\frac{1}{2}\,\mathrm{Id}) = f(c)(\frac{1}{2}\,\mathrm{Id})$ for some scalar function $f : \mathbb{R} \to \mathbb{R}$. Due to $ab(\frac{1}{2}\,\mathrm{Id}) = a(\frac{1}{2}\,\mathrm{Id}) \circ b(\frac{1}{2}\,\mathrm{Id})$, the function f is multiplicative. However, $\frac{x}{2}E_{11} = (x\frac{1}{2}\,\mathrm{Id}) \circ (\frac{1}{2}E_{11})$ implies that $\frac{x}{2}E_{11} = \phi(\frac{x}{2}E_{11}) = f(x)(\frac{1}{2}\,\mathrm{Id}) \circ (\frac{1}{2}E_{11})$, so $f(x) = x$ for any $x \in (-\frac{1}{8}, 0)$. Being multiplicative this gives $f(z) = z$ for any real z. So, given any rank one projection P and any $z \in \mathbb{R}$ we have

$$\phi(zP) = \phi(2z\tfrac{1}{2}\,\mathrm{Id}) \circ \phi(\tfrac{1}{2}P) = f(2z)(\tfrac{1}{2}\,\mathrm{Id}) \circ (\tfrac{1}{2}P) = 2z(\tfrac{1}{2}P) = zP.$$

In particular, ϕ fixes every rank one Hermitian matrix. By Lemma 5.3 again, $\phi(A) = A$ for every $A \in \mathcal{H}_n$. □

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