HIGHER RANK NUMERICAL RANGES AND LOW RANK PERTURBATIONS OF QUANTUM CHANNELS

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Abstract. For a positive integer k, the rank-k numerical range $\Lambda_k(A)$ of an operator A acting on a Hilbert space $\mathcal{H}$ of dimension at least k is the set of scalars λ such that $PAP = \lambda P$ for some rank k orthogonal projection P. In this paper, a close connection between low rank perturbation of an operator A and $\Lambda_k(A)$ is established. In particular, for $1 \leq r < k$ it is shown that $\Lambda_k(A) \subseteq \Lambda_{k-r}(A+F)$ for any operator F with $\operatorname{rank}(F) \leq r$. In quantum computing, this result implies that a quantum channel with a k-dimensional error correcting code under a perturbation of rank $\leq r$ will still have a $(k-r)$-dimensional error correcting code. Moreover, it is shown that if A is normal or if the dimension of A is finite, then $\Lambda_k(A)$ can be obtained as the intersection of $\Lambda_{k-r}(A+F)$ for a collection of rank r operators F. Examples are given to show that the result fails if A is a general operator. The closure and the interior of the convex set $\Lambda_k(A)$ are completely determined. Analogous results are obtained for $\Lambda_\infty(A)$ defined as the set of scalars λ such that $PAP = \lambda P$ for an infinite rank orthogonal projection P. It is shown that $\Lambda_\infty(A)$ is the intersection of all $\Lambda_k(A)$ for $k = 1, 2, \ldots$. If $A - \mu I$ is not compact for any $\mu \in \mathbb{C}$, then the closure and the interior of $\Lambda_\infty(A)$ coincide with those of the essential numerical range of A. The situation for the special case when $A - \mu I$ is compact for some $\mu \in \mathbb{C}$ is also studied.

1. Introduction

Let $\mathcal{B}(\mathcal{H})$ be the algebra of bounded linear operators acting on a Hilbert space $\mathcal{H}$. We identify $\mathcal{B}(\mathcal{H})$ with M_n if $\mathcal{H}$ has dimension n. For $k \leq \dim \mathcal{H}$, define the *rank-k numerical range* of $A \in \mathcal{B}(\mathcal{H})$ by

$$\Lambda_k(A) = \{\lambda \in \mathbb{C} : PAP = \lambda P \text{ for some rank-}k \text{ orthogonal projection } P \in \mathcal{B}(\mathcal{H})\}.$$

Note that we allow $k = \infty$ if $\dim \mathcal{H} = \infty$. Evidently, $\lambda \in \Lambda_k(A)$ if and only if there is an orthogonal basis of $\mathcal{H}$ such that λI_k is the leading principal submatrix of the operator matrix of A with respect to the basis; equivalently, there is an isometry $X : \mathbb{C}^k \to \mathcal{H}$ such that $X^*AX = \lambda I_k$. (For $k = \infty$, we take $X : \ell^2 \to \mathcal{H}$.) When $k = 1$, this concept reduces to the *classical numerical range* of A defined by

$$W(A) = \{\langle Ax, x\rangle : x \in \mathcal{H},\ \langle x, x\rangle = 1\},$$

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which is useful in studying operators and matrices; for example see [11].

The higher rank numerical range was introduced in connection to the construction of quantum error correction code in the study of quantum information theory; see [7]. In quantum computing, information is stored in qubits (quantum bits). Mathematically, the state of a qubit is represented by a 2×2 rank one Hermitian matrix Q satisfying $Q^2 = Q$. A state of N-qubits $Q_1, \ldots, Q_N$ is represented by their tensor products in M_n with $n = 2^N$. A quantum channel for states of N-qubits corresponds to trace preserving completely positive linear map $\Phi : M_n \to M_n$. By the structure theory of completely positive linear map [3], there are $T_1, \ldots, T_m \in M_n$ with $\sum_{j=1}^m T_j^* T_j = I_n$ such that

$$\Phi(X) = \sum_{j=1}^{m} T_j X T_j^*. \tag{1.1}$$

Let $\mathbf{V}$ be a subspace of $\mathbb{C}^n$ and $P_{\mathbf{V}}$ the orthogonal projection of $\mathbb{C}^n$ onto $\mathbf{V}$. Then $\mathbf{V}$ is a quantum error correction code for Φ if there exists a trace preserving completely positive linear map $\Psi : M_n \to M_n$ such that $\Psi \circ \Phi(A) = A$ for all $A \in P_{\mathbf{V}} M_n P_{\mathbf{V}}$. This happens if and only if there are scalars γ_{ij} with $1 \le i, j \le m$ such that

$$P_{\mathbf{V}} T_i^* T_j P_{\mathbf{V}} = \gamma_{ij} P_{\mathbf{V}}, \quad 1 \le i, j \le m;$$

see [7, 12]. It turns out that even for a single matrix A, determining $\Lambda_k(A)$ is highly non-trivial, and the results are useful in quantum computing, say, in constructing binary unitary channels; see [5]. In a sequence of papers [4, 5, 6, 7, 9, 13, 14, 18], researchers studied the set $\Lambda_k(A)$ for $A \in \mathcal{B}(\mathcal{H})$. Many interesting results (see P1-P8 below) were obtained.

In the study of operator theory and applications, it is often useful to study the properties of an operator which are stable under different kinds of perturbation. For example, the *essential numerical range* of an infinite dimensional operator $A \in \mathcal{B}(\mathcal{H})$ can be defined as

$$W_e(A) = \cap\{\mathbf{Cl}(W(A + F)) : F \in \mathcal{B}(\mathcal{H}) \text{ has finite rank}\}, \tag{1.2}$$

which captures many important properties of A (see [1, 8, 17, 19]). Here $\mathbf{Cl}(S)$ denotes the closure of the set S. In fact, one can include all compact operators F in $\mathcal{B}(\mathcal{H})$ on the right hand side of (1.2). If $\mathcal{K}$ is the algebra of compact operators in $\mathcal{B}(\mathcal{H})$ and if $\psi : \mathcal{B}(\mathcal{H}) \mapsto \mathcal{B}(\mathcal{H})/\mathcal{K}$ is the canonical homomorphism of $\mathcal{B}(\mathcal{H})$ onto the Calkin algebra $\mathcal{B}(\mathcal{H})/\mathcal{K}$, then $W_e(A)$ is the closure of the numerical range of $\psi(A)$. In [1, Theorem 4], it was also proven that

$$\Lambda_\infty(A) = \cap\{W(A + F) : F \in \mathcal{B}(\mathcal{H}) \text{ has finite rank}\}. \tag{1.3}$$

In this paper, we study the change of the higher rank numerical range of an operator under low rank perturbation. For instance, we show in Theorem 3.1 that

for $1 \leq r < k < \infty$, if $A, F \in \mathcal{B}(\mathcal{H})$ with $\operatorname{rank}(F) \leq r$, then

$$\Lambda_k(A) \subseteq \Lambda_{k-r}(A+F). \tag{1.4}$$

In Theorem 5.1, we refine the set equalities (1.2) and (1.3) by using smaller sets of operators F for the intersection on the right hand sides of the equalities.

It is worth noting that the inclusion (1.4) has the following implication in the theory of quantum computing. Suppose $A \in M_n$ corresponds to a quantum channel with a k-dimensional error correcting code (realized as a subspace of $\mathbb{C}^n$), then for any perturbation of the channel A by an operator F of rank bounded by r, the resulting channel $A+F$ will have a $(k-r)$-dimensional error correcting code. More generally, if the matrices $T_1, \ldots, T_m$ correspond to quantum channel (1.1) with a k-dimensional error correcting code, and if T_j is changed to $T_j + F_j$ such that the sum of the range spaces of

$$(T_i + F_i)^*(T_j + F_j) - T_i^* T_j = T_i^* F_j + F_i T_j^* + F_i F_j^*, \qquad 1 \leq i, j \leq n,$$

has dimension bounded by r, then the resulting quantum channel will still have a $(k-r)$-dimensional error correcting code.

Our paper is organized as follows. First, we study $\Lambda_k(A)$ for $A \in \mathcal{B}(\mathcal{H})$ when k is finite in Sections 2 – 4. In Section 2, we give a complete description of the closure and interior of $\Lambda_k(A)$. In Section 3, we establish inclusion (1.4) for any operators $A, F \in \mathcal{B}(\mathcal{H})$ with $\operatorname{rank}(F) \leq r$, where $1 \leq r < k < \infty$. It follows that

$$\Lambda_k(A) \subseteq \cap\{\Lambda_{k-r}(A+F) : F \in \mathcal{B}(\mathcal{H}) \text{ has rank } \leq r\}. \tag{1.5}$$

In particular, taking $r = k-1$, we have

$$\Lambda_k(A) \subseteq \cap\{W(A+F) : F \in \mathcal{B}(\mathcal{H}) \text{ has rank } < k\}. \tag{1.6}$$

We show that the inclusions in (1.5) and (1.6) become inequalities if $\dim \mathcal{H}$ is finite. Examples are given to show that these not true for infinite dimensional operators. Nevertheless, we show that equalities also hold in (1.5) and (1.6) for infinite dimensional normal operators in Section 4. The set equalities in (1.5) and (1.6) can be viewed as refinements of (1.3). Similar set equality results are given in Corollary 3.3, which can be viewed as refinements of (1.2). In Section 5, we extend the results in Sections 2 – 4 to $\Lambda_\infty(A)$. In particular, we show in Theorem 5.1 and 5.2 that

$$\Lambda_\infty(A) = \bigcap_{k \geq 1} \Lambda_k(A) = \cap\{W(A+F) : F \in \mathcal{B}(\mathcal{H}) \text{ has finite rank}\}, \tag{1.7}$$

and $\Lambda_\infty(A) \neq \emptyset$ if and only if the closure of $\Lambda_\infty(A)$ is the essential numerical range of A. Then we determine the condition under which $\Lambda_\infty(A)$ is nonempty. The first equality in (1.7) gives an affirmative answer to a question of Martinez-Avendano [16].

We close this section by listing some basic properties for the higher rank numerical range; see [4, 5, 6, 7, 9, 13, 14, 18].

P1. For any $a, b \in \mathbb{C}$, $\Lambda_k(aA + bI) = a\Lambda_k(A) + b$.

P2. For any unitary $U \in \mathcal{B}(\mathcal{H})$, $\Lambda_k(U^*AU) = \Lambda_k(A)$.

P3. If A_0 is a compression of A on a subspace $\mathcal{H}_0$ of $\mathcal{H}$ such that $\dim \mathcal{H}_0 \geq k$, then $\Lambda_k(A_0) \subseteq \Lambda_k(A)$.

P4. Suppose $\dim \mathcal{H} < 2k$. The set $\Lambda_k(A)$ has at most one element.

P5. If $\dim \mathcal{H} \geq 3k - 2$ then $\Lambda_k(A)$ is non-empty. Otherwise, there is $B \in \mathcal{B}(\mathcal{H})$ such that $\Lambda_k(B) = \emptyset$.

P6. $\Lambda_k(A)$ is always convex.

P7. If $\dim \mathcal{H} < \infty$, then $\Lambda_k(A) = \Omega_k(A)$ with

$$\Omega_k(A) = \bigcap_{\xi \in [0,2\pi)} \left\{ \mu \in \mathbb{C} : e^{i\xi}\mu + e^{-i\xi}\bar{\mu} \leq \lambda_k(e^{i\xi}A + e^{-i\xi}A^*) \right\},$$

where $\lambda_k(H)$ denotes the k-th largest eigenvalue of the Hermitian matrix $H \in M_n$.

P8. If $A \in M_n$ is a normal matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$, then

$$\Lambda_k(A) = \bigcap_{1 \leq j_1 < \cdots < j_{n-k+1} \leq n} \operatorname{conv}\{\lambda_{j_1}, \ldots, \lambda_{j_{n-k+1}}\}.$$

2. The interior and closure of $\Lambda_k(A)$

First, we extend the definition of $\Omega_k(A)$ to infinite dimensional operators. For a self-adjoint operator H, let

$$\lambda_k(H) = \sup\{\lambda_k(X^*HX) : X \text{ is an isometry from } \mathbb{C}^k \text{ to } \mathcal{H} \text{ so that } X^*X = I_k\}.$$

For $A \in \mathcal{B}(\mathcal{H})$, let $\operatorname{Re}(A) = (A + A^*)/2$ be the real part of A and

$$\Omega_k(A) = \bigcap_{\xi \in [0,2\pi)} \left\{ \mu \in \mathbb{C} : \operatorname{Re}(e^{i\xi}\mu) \leq \lambda_k(\operatorname{Re}(e^{i\xi}A)) \right\}.$$

By definition, $\Omega_k(A)$ is a compact convex set. It may be empty if $\dim \mathcal{H} \leq 3k-3$; see [5, Theorem 4.7]. In the finite dimensional case, we have $\Lambda_k(A) = \Omega_k(A)$ as noted in property (P7). Let $A = I_k \oplus \operatorname{diag}(1, 1/2, \ldots)$. One easily checks that $\Omega_k(A) = [0, 1]$ and $\Lambda_k(A) = (0, 1]$. (See also Example 3.5.) Hence, property (P7) may not hold for infinite dimensional operator A.

We continue to use $\mathbf{Cl}(S)$ to denote the closure of a set S in $\mathbb{C}$. Let $\mathbf{Int}(S)$ denote the relative interior of S. We have the following.

Theorem 2.1. *Let $A \in \mathcal{B}(\mathcal{H})$ be an infinite dimensional operator, and let k be a positive integer. Then*

$$\mathbf{Int}(\Omega_k(A)) \subseteq \Lambda_k(A) \subseteq \Omega_k(A) = \mathbf{Cl}(\Lambda_k(A)).$$

Proof. First, we establish the inclusion $\Lambda_k(A) \subseteq \Omega_k(A)$. By [13, Corollary 4], $\Lambda_k(A)$ is always non-empty. Suppose $\mu \in \Lambda_k(A)$. Then there is an isometry $X : \mathbb{C}^k \to \mathcal{H}$ such that $X^*X = I_k$ and $X^*AX = \mu I_k$. As a result, for any $t \in [0, 2\pi)$ we have

$$\mathrm{Re}\,(e^{it}\mu) \leq \lambda_k(\mathrm{Re}\,(e^{it}A)).$$

Thus, $\mu \in \Omega_k(A)$.

Next, we turn to the equality $\Omega_k(A) = \mathbf{Cl}(\Lambda_k(A))$ and the inclusion $\mathbf{Int}(\Omega_k(A)) \subseteq \Lambda_k(A)$. By Corollary 4 in [13], $\Lambda_k(A)$ is non-empty. We consider three cases.

Case 1. Suppose $\Omega_k(A)$ is a singleton. Then $\Lambda_k(A) = \Omega_k(A)$ because $\Lambda_k(A)$ is non-empty, and $\mathbf{Int}(\Lambda_k(A)) = \mathbf{Int}(\Omega_k(A)) = \emptyset$.

Case 2. Suppose $\Omega_k(A)$ has non-empty interior in $\mathbb{C}$. Let μ be an interior point of $\Omega_k(A)$. We may replace A by $A - \mu I$ and assume that $\mu = 0$, i.e., $0 \in \mathbf{Int}(\Omega_k(A))$. Therefore, there exists $d > 0$ such that

$$\{\mu \in \mathbb{C} : |\mu| \leq d\} \subseteq \Omega_k(A).$$

Thus, for all $t \in [0, 2\pi)$, $\mu = de^{it} \in \Omega_k(A)$. Write $A = H + iG$ where H and G are self-adjoint. Then

$$e^{-it}A + e^{it}A^* = 2(\cos tH + \sin tG).$$

Hence,

$$\lambda_k(e^{-it}A + e^{it}A^*) \geq e^{-it}\mu + e^{it}\overline{\mu} \quad \Rightarrow \quad \lambda_k(\cos tH + \sin tG) \geq d\,.$$

Then, for each $t \in [0, 2\pi)$ there is $X_t : \mathbb{C}^k \to \mathcal{H}$ with $X_t^*X_t = I_k$ such that $\lambda_k(\cos tX_t^*HX_t + \sin tX_t^*GX_t) > d/2$. Furthermore, there is $\delta_t > 0$ such that for each $s \in (t - \delta_t, t + \delta_t)$,

$$\|(\cos tX_t^*HX_t + \sin tX_t^*GX_t) - (\cos sX_t^*HX_t + \sin sX_t^*GX_t)\| < d/4.$$

Note that $|\lambda_k(R) - \lambda_k(S)| \leq \|R - S\|$ for any two Hermitian matrices R and S by the Weyl's inequality; for example, see [2, III.2]. It follows that

$$|\lambda_k(\cos tX_t^*HX_t + \sin tX_t^*GX_t) - \lambda_k(\cos sX_t^*HX_t + \sin sX_t^*GX_t)| < d/4.$$

Consequently,

$$\lambda_k(\cos sX_t^*HX_t + \sin sX_t^*GX_t) > \lambda_k(\cos tX_t^*HX_t + \sin tX_t^*GX_t) - d/4 > d/4.$$

Since $[0, 2\pi]$ is compact, there exists a finite sequence $0 \leq t_1 < \cdots < t_m < 2\pi$ so that

$$[0, 2\pi] \subseteq \bigcup_{j=1}^{m} (t_j - \delta_{t_j}, t_j + \delta_{t_j}).$$

Let $A_0 = H_0 + iG_0$ be a compression of A onto a subspace spanned by the range spaces of $X_{t_1}, \ldots, X_{t_m}$. Then $\lambda_k(\cos tH_0 + \sin tG_0) > d/4$ for all $t \in [0, 2\pi)$ and

so $0 \in \Omega_k(A_0)$. Thus, $0 \in \Lambda_k(A_0) \subseteq \Lambda_k(A)$ by Theorem 2..2 in [14]. Hence, $\mathbf{Int}(\Omega_k(A)) \subseteq \Lambda_k(A)$ and thus $\mathbf{Cl}(\Lambda_k(A)) = \Omega_k(A)$.

Case 3. Suppose $\Omega_k(A)$ is not a singleton and has no interior in $\mathbb{C}$. Since $\Omega_k(A)$ is a compact convex set in $\mathbb{C}$, if it is not a singleton and has no interior in $\mathbb{C}$, then it is a non-degenerate line segment. We will show that $\Lambda_k(A)$ contains all the relative interior points of $\Omega_k(A)$. The result will then follow.

Assume γ is a (relative) interior point of the line segment. By property (P1), we may assume that $[-1, 1] \subseteq \Omega_k(A) \subseteq \mathbb{R}$ and $\gamma = 0$. Write $A = H + iG$ where H and G are self-adjoint. Since $-1, 1 \in \Omega_k(A)$, we have $\lambda_k(\cos tH + \sin tG) \geq |\cos t|$ for all $t \in [0, 2\pi]$. We claim that $\lambda_k(G) = 0$. If it is not true, then there is $\delta > 0$ such that $\lambda_k(\cos tH + \sin tG) \geq \varepsilon > 0$ for each $t \in [\pi/2 - \delta, \pi/2 + \delta]$. By decreasing ε, if necessary, we may assume that $|\cos(\pi/2 + \delta)| = |\cos(\pi/2 - \delta)| \geq \varepsilon$. Therefore, we have $\lambda_k(\cos tH + \sin tG) \geq \varepsilon$ for all $t \in [0, \pi]$. Let $\mu = i\varepsilon$. Then, we have

$$\mathrm{Re}(\mu e^{-it}) \leq \begin{cases} \varepsilon \leq \lambda_k(\cos tH + \sin tG) & \text{if } t \in [0, \pi], \\ 0 \leq \lambda_k(\cos tH + \sin tG) & \text{if } t \in [\pi, 2\pi]. \end{cases}$$

Therefore, $i\varepsilon \in \Omega_k(A)$. This contradicts that $\Omega_k(A)$ is a line segment in $\mathbb{R}$. Similarly, we can show that $\lambda_k(-G) = 0$. So, we may assume that G has operator matrix $D \oplus 0$ with

$$D = \mathrm{diag}\,(d_1, \ldots, d_{p+q})$$

such that $d_1, \ldots, d_p > 0$ and $d_{p+1}, \ldots, d_{p+q} < 0$, where $p < k$ and $q < k$. Let H_0 and A_0 be the compressions of H and A to the kernel of G, respectively.

Suppose $\lambda_k(H_0) > 0$ and $\lambda_k(-H_0) > 0$. Then H_0 has a compression $\tilde{H}_0 \in M_{2k}$ such that $\tilde{H}_0$ has k positive eigenvalues and k negative eigenvalues. Clearly, $0 \in \Lambda_k(\tilde{H}_0)$ and $\tilde{H}_0$ is also a compression of A. Then $0 \in \Lambda_k(H_0) \subseteq \Lambda_k(A)$. So, we assume that $\lambda_k(H_0) \leq 0$ without loss of generality.

Suppose the kernel of H_0 has dimension at least k. Then again we have $0 \in \Lambda_k(H_0) = \Lambda_k(A_0) \subseteq \Lambda_k(A)$. Thus, we may assume that the kernel of H_0 has dimension $< k$. Then H_0 has operator matrix of the form

$$H_{22} \oplus H_{33}$$

so that $H_{22} \in M_r$ with $r < 2k - 1$ is positive semi-definite and H_{33} is negative definite such that the kernel of H_{33} is the zero space. Clearly, there is a negative real number in $\Lambda_k(H_{33}) \subseteq \Lambda_k(A_0) \subseteq \Lambda_k(A)$. We will show that $\Lambda_k(A)$ also contains a positive real number. By the convexity of $\Lambda_k(A)$, it will then follow that $0 \in \Lambda_k(A)$.

Note that 0 is an interior point, and H_{22} is finite dimensional. We may find a small $\varepsilon > 0$ such that $\varepsilon \in \Omega_k(A)$ and $H_0 - \varepsilon I = \hat{H}_{22} \oplus \hat{H}_{33}$ so that $\hat{H}_{22}$ is positive semi-definite and $\hat{H}_{33}$ is negative definite bounded above by $-\varepsilon < 0$. Thus, $\hat{H}_{33}$ is

invertible, and there is an orthonormal basis of $\mathcal{B}(\mathcal{H})$ so that the operator matrices of G and $\hat{H} = H - \varepsilon I$ equal

$$D \oplus 0 \qquad \text{and} \qquad \begin{bmatrix} \hat{H}_{11} & \hat{H}_{12} & \hat{H}_{13} \\ \hat{H}_{21} & \hat{H}_{22} & 0 \\ \hat{H}_{31} & 0 & \hat{H}_{33} \end{bmatrix}$$

for $\hat{H}_{22} \in M_{r'}$ with $r' \leq r$. For notational simplicity, we rename r' as r. Suppose $S \in \mathcal{B}(\mathcal{H})$ has operator matrix

$$\begin{bmatrix} I_{p+q} & 0 & -\hat{H}_{13}\hat{H}_{33}^{-1} \\ 0 & I_r & 0 \\ 0 & 0 & I \end{bmatrix}.$$

Then SGS^* and $S\hat{H}S^*$ have operator matrices

$$D \oplus 0 \quad \text{and} \quad \begin{bmatrix} \hat{H}_{11} - \hat{H}_{13}\hat{H}_{33}^{-1}\hat{H}_{31} & \hat{H}_{12} \\ \hat{H}_{21} & \hat{H}_{22} \end{bmatrix} \oplus \hat{H}_{33}.$$

Since $0 \in \Omega_k(A - \varepsilon I)$, we see that for each $t \in [0, 2\pi)$, we have $\lambda_k(\cos t\hat{H} + \sin tG) \geq 0$ and hence $\lambda_k(\cos tS\hat{H}S^* + \sin tSGS^*) \geq 0$. Consequently, if we let $\tilde{H}_{33}$ be the leading $k \times k$ submatrix of $\hat{H}_{33}$ and let $\tilde{A} = \tilde{H} + i\tilde{G} \in M_{p+q+r+k}$ with

$$\tilde{H} = \begin{bmatrix} \hat{H}_{11} - \hat{H}_{13}\hat{H}_{33}^{-1}\hat{H}_{31} & \hat{H}_{12} \\ \hat{H}_{21} & \hat{H}_{22} \end{bmatrix} \oplus \tilde{H}_{33} \quad \text{and} \quad \tilde{G} = D \oplus 0_{r+k},$$

then $\lambda_k(\cos t\tilde{H} + \sin t\tilde{G}) \geq 0$ for all $t \in [0, 2\pi)$ and hence $0 \in \Omega_k(\tilde{A})$. By Theorem 2.2 in [14], there is a $(p + q + r + k) \times k$ matrix X such that

$$X^*X = I_k \quad \text{and} \quad X^*\tilde{A}X = 0_k.$$

Consequently, as $\tilde{A}$ is a finite compression of $S(A - \varepsilon I)S^*$, there is a partial isometry $Y : \mathbb{C}^k \to \mathcal{H}$ such that $Y^*S(A - \varepsilon I)S^*Y = 0_k$. Note that $S^*Y = ZT$ with $Z^*Z = I_k$ for some invertible $T \in M_k$. Thus, $Z^*(A - \varepsilon I)Z = 0_k$, i.e., $\varepsilon \in \Lambda_k(A)$. □

In the finite dimensional case, $\Lambda_k(A)$ is always closed. If $\dim \mathcal{H}$ is uncountable, then for any bounded convex set S in $\mathbb{C}$ one can construct a normal operator B using the points in S as diagonal elements so that $\Lambda_k(A) = S$ for $A = B \otimes I$. In the following, we give examples of A acting on a separable Hilbert space such that $\Lambda_k(A)$ has non-empty interior with no, some or all its boundary points. It is known that $\Lambda_k(A)$ is a singleton if A is a scalar operator, and that $\Lambda_k(A) \subseteq \mathbb{R}$ if $A = A^*$. We give examples different from these trivial cases.

Example 2.2. In the following examples, let $B = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$.

(a) Let $A = B \otimes I_{k-1} \oplus 0$. Then $\Omega_k(A) = \Lambda_k(A) = \{0\}$.

(b) Let $A = B \otimes I_{k-1} \oplus \operatorname{diag}(1, 1/2, 1/3, \dots)$. Then $\Lambda_k(A) = (0, 1]$. One can easily modify the example so that $\Lambda_k(A) = [0, 1]$ or $\Lambda_k(A) = (0, 1)$.

(c) Let $A = B \otimes I_{k-1} \oplus C$. If $C = B \oplus 0$ then $\Lambda_k(A)$ is the closed unit disk; if C is the unilateral shift, then $\Lambda_k(A)$ is the open unit disk; if $C = \operatorname{diag}(-1, i, -i, 1/2, 2/3, 3/4, 4/5, \dots)$ then $\Omega_k(A)$ is the convex hull of $\{-1, i, -i, 1\}$, and $\Lambda_k(A)$ is the union of the interior of $\Omega_k(A)$ and the convex hull of $\{-1, i, -i\}$.

3. Low rank perturbations of general operators

For a positive integer r, let $\mathcal{F}_r$ be the set of operators in $\mathcal{B}(\mathcal{H})$ with rank at most r, and let $\mathcal{P}_r$ be the set of rank r orthogonal projections in $\mathcal{B}(\mathcal{H})$.

Theorem 3.1. *Let $1 \le r < k < \infty$. Suppose $A \in \mathcal{B}(\mathcal{H})$ and $F \in \mathcal{F}_r$. Then $\Lambda_k(A) \subseteq \Lambda_{k-r}(A + F)$. Consequently,*

$$\Lambda_k(A) \subseteq \cap\{\Lambda_{k-r}(A + F) : F \in \mathcal{F}_r\}.$$

Proof. Suppose $\lambda \in \Lambda_k(A)$. Let $X : \mathbb{C}^k \to \mathcal{H}$ be an isometry such that $X^*AX = \lambda I_k$. Then X^*FX has rank at most r. There is a unitary $U \in M_k$ such that

$$U^*X^*FXU = \begin{bmatrix} 0_{k-r} & * \\ 0 & * \end{bmatrix}.$$

Let U_1 be obtained by taking the first $k - r$ column of U, and $V = XU_1$. Then $V^*(A + F)V = \lambda I_{k-r}$ so that $\lambda \in \Lambda_{k-r}(A + F)$. □

Note that one can easily adapt the above proof to show that for $A_1, \dots, A_m \in \mathcal{B}(\mathcal{H})$, if $X^*A_jX = \lambda_j I_k$ with $X^*X = I_k$ and if $F_1, \dots, F_m \in \mathcal{B}(\mathcal{H})$ are such that

$$U^*X^*F_jXU = \begin{bmatrix} 0_{k-r} & * \\ 0 & * \end{bmatrix} \quad j = 1, \dots, m,$$

then $V^*A_jV = \lambda_j I_{k-r}$ for all $j = 1, \dots, m$. So, the comment about a low rank perturbation of a quantum channel in Section 1 follows.

If $1 \le r < k \le \dim \mathcal{H}$ and $A \in \mathcal{B}(\mathcal{H})$, then $\Omega_k(A)$ can be written as the intersection of $\Omega_{k-r}(A + F)$ for a collection of rank r operators F as shown in the following.

Theorem 3.2. *Suppose $A \in \mathcal{B}(\mathcal{H})$ and $1 \le r < k < \infty$. Let $\mathcal{S}$ be a subset of $\mathcal{F}_r$ containing the set $\mathcal{S}_0 = \{2e^{i\xi}\|A\|P : P \in \mathcal{P}_r \text{ and } \xi \in [0, 2\pi)\}$. Then*

$$\Omega_k(A) = \cap\{\Omega_{k-r}(A + F) : F \in \mathcal{S}\}.$$

Proof. The inclusion ($\subseteq$) follows from Theorem 3.1 and the fact that $\Omega_k(A) = \mathbf{Cl}(\Lambda_k(A))$ by Theorem 2.1.

Suppose $\lambda \notin \Omega_k(A)$. Then there exists $t \in \mathbb{R}$ such that $\lambda_k(\mathrm{Re}\left(e^{it}A\right)) < \mathrm{Re}\left(e^{it}\lambda\right)$. Let $e^{it}A = H + iG$ be with $H = H^*$ and $G = G^*$. Then H has an operator matrix $\mathrm{diag}\,(\lambda_1, \ldots, \lambda_m) \oplus H_2$ with $m \leq k-1$ such that $\sup \sigma(H_2) < \mathrm{Re}\,(e^{it}\lambda)$. Let $F = -2e^{-it}\|A\|(I_m \oplus I_{k-m} \oplus 0) \in \mathcal{B}(\mathcal{H})$. Then $\lambda_{k-r}(\mathrm{Re}\,(e^{it}(A+F)) < \mathrm{Re}\left(e^{it}\lambda\right)$. Hence, $\lambda \notin \Omega_{k-r}(A+F)$. □

Note that for the set $\mathcal{S}$ in the above theorem, we can take the whole $\mathcal{F}_r$ or the much smaller subset $\mathcal{S}_0$. We have the following corollary.

Corollary 3.3. *Under the same setting as in Theorem 3.2. Each of the following sets is equal to $\Omega_k(A)$.*

(a) $\cap\{\Omega_{k-1}(A + 2e^{i\xi}\|A\|P) : \xi \in [0, 2\pi),\ P \in \mathcal{P}_1\}$.
(b) $\cap\{\Omega_1(A + 2e^{i\xi}\|A\|P) : \xi \in [0, 2\pi),\ P \in \mathcal{P}_{k-1}\}$.

With Theorem 2.1, the above result also holds if we replace $\Omega_m(B)$ by $\mathbf{Cl}\,(\Lambda_m(B))$. Using the fact that $\Lambda_k(A) = \Omega_k(A)$ when $A \in M_n$, we have the following result.

Theorem 3.4. *Suppose $A \in M_n$ and $1 \leq r < k \leq n$. Let $\mathcal{S}$ be a subset of $\mathcal{F}_r$ containing the set $\mathcal{S}_0 = \{2e^{i\xi}\|A\|P : P \in \mathcal{P}_r \text{ and } \xi \in [0, 2\pi)\}$. Then*

(a) $\Lambda_k(A) = \cap\{\Lambda_{k-r}(A + F) : F \in \mathcal{S}\}$.
(b) $\Lambda_k(A) = \cap\{\Lambda_{k-1}(A + 2e^{i\xi}\|A\|P) : \xi \in [0, 2\pi), P \in \mathcal{P}_1\}$.
(c) $\Lambda_k(A) = \cap\{W(A + 2e^{i\xi}\|A\|P) : \xi \in [0, 2\pi),\ P \in \mathcal{P}_{k-1}\}$.

The following example shows that Theorem 3.4 does not hold for infinite dimensional operators.

Example 3.5. Let $A = A_1 \oplus A_2$, where

$$A_1 = \begin{bmatrix} 0 & i \\ i & 2 \end{bmatrix} \quad \text{and} \quad A_2 = \mathrm{diag}(b_2, \bar{b}_2, b_3, \bar{b}_3, \cdots) \oplus \mathrm{diag}(b_2, \bar{b}_2, b_3, \bar{b}_3, \cdots)$$

with $b_m = -1 + e^{i\pi/m}$ for $m = 2, \ldots$. Then $0 \in \mathbf{Cl}(\Lambda_2(A))$ and $0 \notin \Lambda_2(A)$, but $0 \in \cap\{W(A+F) : F \text{ is rank one}\}$.

Verification. Note that every $\mu \in \Lambda_1(A_2)$ is an element of $\Lambda_2(A_2)$, and hence $\Lambda_1(A_2) = \Lambda_2(A_2)$. Clearly, $0 \in \mathbf{Cl}(\Lambda_1(A)) = \mathbf{Cl}(\Lambda_2(A))$.

Next, we show that $0 \notin \Lambda_2(A)$. Suppose $0 \in \Lambda_2(A)$. Then $0 \in \Lambda_2(H)$ for $H = (A + A^*)/2$. Let U be unitary such that $U^*AU = \begin{bmatrix} 0_2 & * \\ * & * \end{bmatrix}$. Then U^*HU has the same form. Since H has spectrum $\{2, 0\} \cup \{-1 + \cos\pi/m : m = 2, \ldots\}$, we may assume that U has the form $[1] \oplus U_1$ such that the $(1,1)$ entry of U_1 is nonzero. But then U^*GU will have non-zero $(1,2)$ entry for $G = (A - A^*)/(2i)$. This contradicts the fact that U^*AU has zero $(1,2)$ entry. So, we see that $0 \notin \Lambda_2(A)$.

Now, suppose $F = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}$ is a rank one operator with $F_{11} \in M_2$. Let $x \in \mathbb{C}^2$ be a nonzero vector such that $F_{11}x = 0$. If x is a multiple of e_1, then the $(1,1)$ entry of $A_1 + F_{11}$ equals 0 and we have $0 \in W(A_1 + F_{11}) \subseteq W(A + F)$. If x is not a multiple of e_1, then $\mu_0 = x^*(A_1 + F_{11})x = x^*A_1x \in W(A_1)$ has positive real part and $\mu_0 \in W(A_1 + F_{11}) \subseteq W(A + F)$. Since F_{22} has rank at most one, by Theorem 3.1 we have

$$W(A_2) = \Lambda_2(A_2) \subseteq \Lambda_1(A_2 + F_{22}) = W(A_2 + F_{22}) \subseteq W(A + F).$$

So there exist $\mu_1, \mu_2 \in W(A_2) \subseteq W(A+F)$ on the different sides of the line passing through μ_0 and the origin. It follows that $0 \in \operatorname{conv}\{\mu_0, \mu_1, \mu_2\} \subseteq W(A + F)$ by the convexity of $W(A + F)$. Consequently, we have

$$0 \in \cap\{W(A + F) : F \text{ has rank one}\}. \qquad \square$$

4. Low rank perturbations of infinite dimensional normal operators

In the following, we prove that Theorem 3.4 is valid for (infinite dimensional) normal operators. We first establish some auxiliary results showing that one can refine the spectral decomposition of a normal operator using the geometrical information of its numerical range.

Let $\mathcal{P} = \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$ be the open upper half plane of $\mathbb{C}$. For $A \in \mathcal{B}(\mathcal{H})$ and $k \le \dim \mathcal{H}$, let

$$\mu_k(A, t) = \lambda_k((e^{-it}A - e^{it}A^*)/(2i)).$$

Notice also that

$$\Omega_k(A) = \bigcap_{t \in [0, 2\pi)} \left\{\mu \in \mathbb{C} : \operatorname{Im}(e^{-it}\mu) \le \mu_k(A, t)\right\}.$$

Lemma 4.1. *Suppose $A \in \mathcal{B}(\mathcal{H})$ is normal. If $\mu_m(A, t) \le 0$ for some $m \ge 1$ and $t \in \mathbb{R}$. Then A has a decomposition $A_1 \oplus A_2 \oplus \hat{A}$ such that $\dim A_1 < m$,*

$$W(A_1) \subseteq e^{it}\mathcal{P}, \quad W(A_2) \subseteq -e^{it}\mathcal{P} \quad \text{and} \quad W(\hat{A}) \subseteq e^{it}\mathbb{R}.$$

Furthermore, if $\lambda_\ell(e^{-it}\hat{A} + e^{it}\hat{A}^*)/2 \le 0$ for some $\ell \ge 1$, then $\hat{A}$ has a decomposition $A_3 \oplus A_4 \oplus 0$ such that dim $A_3 < \ell$,

$$W(A_3) \subseteq e^{it}(0, \infty), \quad \text{and} \quad W(A_4) \subseteq e^{it}(-\infty, 0).$$

Note that each of the summands $A_1, A_2, \hat{A}, A_3, A_4$ may be vacuous.

Proof. Without loss of generality, we may assume that $t = 0$. Let $A = H + iG$, where H, G are self-adjoint. Then $G = G_1 \oplus G_2 \oplus 0$ such that G_1 is positive definite

with dimension $p < m$ and G_2 is negative definite. Let

$$H = \begin{bmatrix} H_{11} & H_{12} & H_{13} \\ H_{12}^* & H_{22} & H_{23} \\ H_{13}^* & H_{23}^* & H_{33} \end{bmatrix}$$

such that $H_{12} = [D \,|\, 0]$, where $D = \operatorname{diag}(d_1, \ldots, d_p)$ with $d_1 \geq \cdots \geq d_p \geq 0$. Since $GH = HG$, it follows that $G_1[D \,|\, 0] = [D \,|\, 0]G_2$. Since G_1 is positive definite and G_2 is negative definite, the $(1,1)$ entry on the left side is nonnegative, and the $(1,1)$ entry on the right side is nonpositive. Thus, $d_1 = 0$ and hence $H_{12} = 0$. Since $G_1 H_{13} = 0$ and $G_2 H_{23} = 0$, we have $H_{13} = 0$ and $H_{23} = 0$. So, $H = H_{11} \oplus H_{22} \oplus H_{33}$ and A has asserted properties, with $A_1 = H_{11} + iG_1$, $A_2 = H_{22} + iG_2$, and $\hat{A} = H_{33}$.

If $\lambda_\ell(e^{-it}\hat{A} + e^{it}\hat{A}^*)/2 \leq 0$ for some ℓ, then we can apply the above result to $\hat{A}$ and get the desired decomposition for $\hat{A}$. □

The following result [1, Lemma 2 and Corollary] will be needed in later discussion.

Lemma 4.2. *Suppose $\dim \mathcal{H}$ is infinite. Let $T \in \mathcal{B}(\mathcal{H})$. Then the following are equivalent.*

(a) $\lambda \in W_e(T)$.

(b) *There is an orthonormal set $\{e_n\}$ such that $\langle Te_n, e_n \rangle \to \lambda$.*

(c) *There is a decomposition of $\mathcal{H}$ as $\mathcal{H}_1 \oplus \mathcal{H}_2$ and a sequence $\{\lambda_i\}$ in $\mathbb{C}$, such that $\lambda_i \to \lambda$ and*

$$T = \left[\begin{array}{ccc|c} \lambda_1 & & 0 & \\ & \lambda_2 & & * \\ & & \ddots & \\ \hline 0 & * & & * \end{array}\right]$$

Furthermore, if α, $\beta \in W_e(T)$, then there exist two sequences $\{\alpha_i\}$ and $\{\beta_i\}$ in $\mathbb{C}$, such that $\alpha_i \to \alpha$, $\beta_i \to \beta$ and a decomposition of $\mathcal{H}$ as $\mathcal{H}_1 \oplus \mathcal{H}_2$ such that

$$T = \left[\begin{array}{cccc|c} \alpha_1 & & & & \\ & \beta_1 & & 0 & \\ & & \alpha_2 & & * \\ & 0 & & \beta_2 & \\ & & & & \ddots \\ \hline & & * & & * \end{array}\right]$$

In both cases, we may take $\mathcal{H}_2$ to be infinite dimensional.

Lemma 4.3. *Suppose $T \in \mathcal{B}(\mathcal{H})$ is a normal operator such that for some $\pi \leq s_1 < s_2 \leq 2\pi$,*

$$\sigma(T) \subseteq \{\rho e^{it} \in \mathbb{C} : \rho > 0,\ t \in [s_1, s_2]\}.$$

Let k be a positive integer and $s_3 \in (s_1, s_2)$,

$$\mathcal{L} = \{\rho e^{it} \in \mathbb{C} : \rho > 0,\ t \in (s_1, s_3)\} \quad \textit{and} \quad \mathcal{R} = \{\rho e^{it} \in \mathbb{C} : \rho > 0,\ t \in (s_3, s_2)\}.$$

We have

(a) *If $\mathcal{L} \cap \sigma(T)$ is infinite or contains an eigenvalue of T with infinite multiplicity, then T has a compression $T_1 \in M_k$ such that $W(T_1) \subseteq \mathcal{L}$.*

(b) *If $\mathcal{R} \cap \sigma(T)$ is infinite or contains an eigenvalue of T with infinite multiplicity, then T has a compression $T_2 \in M_k$ such that $W(T_2) \subseteq \mathcal{R}$.*

If both hypotheses in (a) and (b) hold, then T has a compression of the form $T_1 \oplus T_2$ such that $\dim T_1 = \dim T_2 = k$ and

$$W(T_1) \subseteq \mathcal{L} \quad \text{and} \quad W(T_2) \subseteq \mathcal{R}.$$

Proof. We will prove the last assertion. The proof of (a) and (b) are similar. Suppose both $\mathcal{R} \cap \sigma(T)$ and $\mathcal{L} \cap \sigma(T)$ contain only isolated points of $\sigma(T)$. Then we can construct T_1 (respectively, T_2) from any k (counting multiplicity) eigenvalues of T in $\mathcal{L}$ (respectively, in $\mathcal{R}$) and the corresponding eigenvectors.

Suppose one of the sets $\mathcal{L} \cap \sigma(T)$ or $\mathcal{R} \cap \sigma(T)$, say, $\mathcal{L} \cap \sigma(T)$, contains only isolated points of $\sigma(T)$, and the other set contains an accumulation point of $\sigma(T)$. Then we can construct T_1 from any k eigenvalues of T in $\mathcal{L}$ and the corresponding eigenvectors. Let $\mathcal{H}_1$ be the k-dimensional subspace spanned by the k-eigenvectors. Then with respect to the decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_1^{\perp}$, $T = T_1 \oplus S$ for some normal S. Since $\mathcal{R}$ contains an accumulation point of $\sigma(S)$ and S is normal, by Lemma 4.2, S has a k-dimensional compression T_2 with $W(T_2) \subseteq \mathcal{R}$.

Finally, suppose both $\mathcal{L}$ and $\mathcal{R}$ contain an accumulation point of $\sigma(T)$. Then the result follows from the last statement in Lemma 4.2. □

Theorem 4.4. *Suppose $A \in B(H)$ is normal. Then $\lambda \notin \Lambda_k(A)$ if and only if A can be decomposed into $\tilde{A}_1 \oplus \tilde{A}_2$ such that $\tilde{A}_1$ has dimension at most $k-1$, $W(\tilde{A}_1) \subseteq \lambda + S$ and $W(\tilde{A}_2) \subseteq \mathbb{C} \setminus (\lambda + S)$, where $S = e^{it}\left(\mathcal{P} \cup \tilde{L}\right)$ with $\mathcal{P} = \{z \in \mathbb{C} : \mathrm{Im}\,(z) > 0\}$ and $\tilde{L} = (-\infty, 0]$ or $[0, \infty)$ for some $t \in \mathbb{R}$.*

Proof. Suppose A has the decomposition as stated with $\dim \tilde{A}_1 = m \leq k - 1$. Take $F \in M_m$ such that $W(\tilde{A}_1 + F) \subseteq \mathbb{C} \setminus (\lambda + S)$. By Theorem 3.1,

$$\Lambda_k(A) \subseteq W(A + (F \oplus 0)) = W((\tilde{A}_1 + F) \oplus \tilde{A}_2) \subseteq \mathbb{C} \setminus (\lambda + S).$$

Hence, $\lambda \notin \Lambda_k(A)$.

Conversely, suppose $\lambda \notin \Lambda_k(A)$. Without loss of generality, we may assume that $\lambda = 0$.

Case 1. Suppose $\lambda = 0 \notin \Omega_k(A)$, then $\mu_k(A, t) < 0$ for some $t \in [0, 2\pi)$. By Lemma 4.1, $A = A_1 \oplus A_2 \oplus \hat{A}$ with $\dim A_1 < k$, $W(A_1) \subseteq e^{it}\mathcal{P}$, $W(A_2) \subseteq -e^{it}\mathcal{P}$ and $W(\hat{A}) \subseteq e^{it}\mathbb{R}$. Furthermore, as $\mu_k(A, t) < 0$, we must have $\dim A_1 + \dim \hat{A} < k$. Then $\hat{A} = A_3 \oplus A_4$ so that $W(A_3) \subseteq e^{it}[0, \infty)$ and $W(A_4) \subseteq -e^{it}(0, \infty)$. Then the result follows with $\tilde{A}_1 = A_1 \oplus A_3$ and $\tilde{A}_2 = A_2 \oplus A_4$.

Case 2. Suppose $\lambda = 0 \in \Omega_k(A)$ and such decomposition mentioned in the theorem does not exist. Suppose $\ker A$, the kernel of A, has dimension $p < k$. We may assume that $p = 0$. Otherwise, replace A by the compression of A on $(\ker A)^\perp$ and replace k by $k - p$. We are going to derive a contradiction by showing that A has a finite dimensional compression B such that $0 \in \Omega_k(B) = \Lambda_k(B) \subseteq \Lambda_k(A)$.

To construct the matrix B, we first show that there exist $s_1 \leq 0 \leq \pi \leq s_2$ with $s_2 - s_1 \leq 2\pi$ such that $A = A_1 \oplus A_2 \oplus A_3 \oplus A_4$, where

$$\begin{aligned} \dim A_1 < \infty, \quad W(A_1) &\subseteq \{\rho e^{it} : \rho > 0,\ t \in (s_1, s_2)\}, \\ W(A_2) &\subseteq \{\rho e^{it} : \rho > 0,\ t \in (s_2, s_1 + 2\pi)\}, \\ W(A_3) &\subseteq e^{is_1}(0, \infty), \quad \text{and} \quad W(A_4) \subseteq e^{is_2}(0, \infty). \end{aligned} \tag{4.1}$$

Then we show that $A_2 \oplus A_3 \oplus A_4$ has a finite dimensional compression $B_2 \oplus B_3 \oplus B_4$ such that $B = A_1 \oplus B_2 \oplus B_3 \oplus B_4$ has $0 \in \Omega_k(B)$.

Since $0 \in \Omega_k(A)$, we have $\mu_k(A, t) \geq 0$ for all $t \in [0, 2\pi)$. If $\mu_k(A, t) > 0$ for all $t \in [0, 2\pi)$, then 0 lies in the interior of $\Omega_k(A)$. Hence, $0 \in \Lambda_k(A)$. So, we may assume that there is $t_0 \in [0, 2\pi)$ such that $\mu_k(A, t_0) = 0$. We may further assume that $t_0 = 0$.

As $\mu_k(A, t_0) = 0$, A has at most $k - 1$ eigenvalues in the open upper half plane. Suppose these eigenvalues have arguments $0 < t_1 \leq t_2 \leq \cdots \leq t_p < \pi$, $p < k$. Take $t_{p+1} = \pi$. Let $g \in \{1, \ldots, p+1\}$ be the smallest integer such that $\mu_m(A, t_g - \pi) = 0$ for some m and $h \in \{0, 1, \ldots, p\}$ be the largest integer satisfying $\mu_m(A, t_h) = 0$ for some m. Let $s_1 = t_g - \pi$ and $s_2 = t_h + \pi$. We are going to find A_j for $j = 1, 2, 3, 4$ satisfying (4.1).

By Lemma 4.1 with $t = t_h$, we have $A = \hat{A}_1 \oplus \hat{A}_2 \oplus \hat{A}$ such that

$$\dim \hat{A}_1 < \infty, \quad W(\hat{A}_1) \subseteq e^{it_h}\mathcal{P}, \quad W(\hat{A}_2) \subseteq -e^{it_h}\mathcal{P}, \quad \text{and} \quad W(\hat{A}) \subseteq e^{it_h}\mathbb{R}.$$

Let $\hat{H} = \left(e^{-it_h}\hat{A} + e^{-it_h}\hat{A}^*\right)/2$. If both $\lambda_k(\hat{H})$ and $\lambda_k(-\hat{H})$ are nonnegative, then we have $0 \in \Lambda_k(\hat{H})$, which implies that $0 \in \Lambda_k(A)$, a contradiction. So, we have either $\lambda_k(\hat{H})$ or $\lambda_k(-\hat{H})$ is negative. By Lemma 4.1 and the assumption that $\ker A = 0$, we have $\hat{A} = \hat{A}_3 \oplus \hat{A}_4$ with

$$W(\hat{A}_3) \subseteq e^{it_h}(0, \infty) \quad \text{and} \quad W(\hat{A}_4) \subseteq -e^{it_h}(0, \infty).$$

If $t_h = t_g - \pi$, we take $A_j = \hat{A}_j$ for $j = 1, 2, 3, 4$. Then A_1, A_2, A_3, A_4 satisfy (4.1) with $s_1 = t_g - \pi = 0$ and $s_2 = t_h + \pi = \pi$.

Suppose $t_h > t_g - \pi$. Then $\dim \hat{A}_3$ is finite. We further apply Lemma 4.1 to $\hat{A}_2$ with $t = t_g - \pi$, we have $\hat{A}_2 = A_1' \oplus A_2' \oplus A_3'$, with

$$\dim A_1' < \infty, \quad W(A_1') \subseteq \{\rho e^{it} : \rho > 0,\ t \in (t_g - \pi, t_h)\},$$
$$W(A_2') \subseteq \{\rho e^{it} : \rho > 0,\ t \in (t_h - \pi, t_g - \pi)\}, \text{ and } W(A_3') \subseteq e^{i(t_g - \pi)}(0, \infty).$$

Note that A_4' is vacuous because $t_h \le t_g \le t_h + \pi$. Then $A_1 = \hat{A}_1 \oplus \hat{A}_3 \oplus A_1'$, $A_2 = A_2'$, $A_3 = A_3'$, and $A_4 = \hat{A}_4$ will satisfy (4.1) with $s_1 = t_g - \pi$ and $s_2 = t_h + \pi$.

Now we choose a finite dimensional compression $B_2 \oplus B_3 \oplus B_4$ of $A_2 \oplus A_3 \oplus A_4$ and show that $\mu_k(A_1 \oplus B_2 \oplus B_3 \oplus B_4, t) \ge 0$ for all $t \in [0, 2\pi)$. Observe that

$$\mu_k(A_1, t) \ge 0 \quad \text{for all} \quad t_g - \pi < t < t_h. \tag{4.2}$$

Let B_3 be a k-dimensional compression of A_3, if $\dim A_3$ is infinite and $B_3 = A_3$, otherwise. We claim that

$$\mu_k(A_1 \oplus B_3, t) \ge 0 \quad \text{for all} \quad t_{g-1} - \pi \le t \le t_g - \pi. \tag{4.3}$$

The claim is clear if $\dim A_3$ is infinite. Suppose $\dim A_3$ is finite and $\mu_k(A_1 \oplus A_3, t) < 0$ for some $t \in [t_{g-1} - \pi, t_g - \pi]$. Since $\dim(A_1 \oplus A_3)$ is finite and $W(A_1 \oplus A_3) \subseteq \{\rho e^{it} : \rho > 0,\ t \in [t_g - \pi, t_h + \pi)\}$, $A_1 \oplus A_3$ has a decomposition $A_1'' \oplus A_3''$, with

$$\dim A_1'' < k, \quad W(A_1'') \subseteq S, \quad \text{and} \quad W(A_3'') \subseteq \mathbb{C} \setminus S,$$

where $S = e^{i(t_g - \pi)}(\mathcal{P} \cup [0, \infty)) = \{\rho e^{it} : \rho > 0,\ t \in [t_g - \pi, t_g)\}$. Notice also that $W(A_2 \oplus A_4) \subseteq \mathbb{C} \setminus S$. Then if we take $\tilde{A}_1 = A_1''$ and $\tilde{A}_2 = A_2 \oplus A_3'' \oplus A_4$, we have $W(\tilde{A}_1) \subseteq S$ and $W(\tilde{A}_2) \subseteq \mathbb{C} \setminus S$, which contradicts our assumption that such decomposition does not exist.

Next, let B_4 be a k-dimensional compression of A_4, if $\dim A_4$ is infinite and $B_4 = A_4$, otherwise. By a similar argument as in the previous paragraph, we can show that

$$\mu_k(A_1 \oplus B_4, t) \ge 0 \quad \text{for all} \quad t_h \le t \le t_{h+1}. \tag{4.4}$$

In the following, we will choose a finite dimension compression of B_2 of A_2 so that

$$\mu_k(B_2 \oplus B_3 \oplus B_4, t) \ge 0 \quad \text{for all} \quad t_{h+1} \le t \le t_{g-1} + \pi. \tag{4.5}$$

Suppose $\dim A_2$ is finite. Then by the definition of t_h and t_g, both $\dim A_3$ and $\dim A_4$ are infinite. Then $\mu_k(B_3 \oplus B_4, t) \ge 0$ for all $t \in [t_{h+1}, t_{g-1} + \pi]$ and so (4.5) holds with vacuous B_2.

Now suppose $\dim A_2$ is infinite. We consider the following three cases.

Case 1. $t_g = t_h$**.** In this case, the summand $A_2' \oplus A_3'$ is vacuous and so as $A_2 \oplus A_3$. Also $\dim A_4$ is infinite. Then (4.5) holds with vacuous B_2 and B_3.

Case 2. $t_g = t_{h+1}$**.** Let B_2 be a k-dimensional compression of A_2. Then $\mu_k(B_2, t) \geq 0$ for all $t \in [t_{h+1}, t_{g-1} - \pi]$ and so (4.5) holds.

Case 3. $t_g > t_{h+1}$**.** Because of the choice t_g and t_h, both $\mathcal{L} \cap \sigma(A_2)$ and $\mathcal{R} \cap \sigma(A_2)$ are infinite or contains an eigenvalue of A_2 with infinite multiplicity, where

$$\mathcal{L} = \{\rho e^{it} : \rho > 0,\ t \in (t_h + \pi, t_{h+1} + \pi)\} \text{ and } \mathcal{R} = \{\rho e^{it} : \rho > 0,\ t \in (t_{g-1} + \pi, t_g + \pi)\}.$$

By Lemma 4.3, we can get finite dimensional compressions T_1 and T_2 of A_2 such that $\dim(T_1) = \dim(T_2) = k$, $W(T_1) \subseteq \mathcal{L}$ and $W(T_2) \subseteq \mathcal{R}$. Then $\mu_k(T_1, t) \geq 0$ for all $t \in [t_{h+1}, t_h + \pi]$ and $\mu_k(T_2, t) \geq 0$ for all $t \in [t_g, t_{g-1} + \pi]$. Thus, $B_2 = T_1 \oplus T_2$ will satisfy (4.5).

Now let $B = A_1 \oplus B_2 \oplus B_3 \oplus B_4$. By (4.2), (4.3), (4.4), and (4.5), we conclude that $\mu_k(B, t) \geq 0$ for all $t \in [0, 2\pi)$ and hence $0 \in \Omega_k(B) = \Lambda_k(B)$. □

Theorem 4.5. *Suppose $A \in \mathcal{B}(\mathcal{H})$ is normal and $1 \leq r < k$. Let $\mathcal{S}$ be a subset of $\mathcal{F}_r$ containing the set $\mathcal{S}_0 = \{2e^{i\xi}\|A\|P : P \in \mathcal{P}_r \text{ and } \xi \in [0, 2\pi)\}$. Then*

$$\Lambda_k(A) = \cap\{\Lambda_{k-r}(A + F) : F \in \mathcal{S}\}.$$

Proof. The inclusion ($\subseteq$) follows from Theorem 3.1. Suppose $\lambda \notin \Lambda_k(A)$. By Theorem 4.4, A has a decomposition $A_1 \oplus A_2$ with $A_1 \in M_m$, $W(A_1) \subseteq \lambda + S$ and $W(A_2) \subseteq \mathbb{C} \setminus (\lambda + S)$, where $m \leq k - 1$ and S is defined as in Theorem 4.4. Let $F = -2ie^{it}\|A\|(I_r \oplus 0) \in \mathcal{S}_0$. Then $A + F$ has less than $k - r$ eigenvalues in $\lambda + \mathcal{S}$. Thus, $A + F$ has a decomposition $B_1 \oplus B_2$ with $\dim B_1 < k - r$ such that $W(B_1) \subseteq \lambda + S$ and $W(B_2) \subseteq \mathbb{C} \setminus (\lambda + S)$. By Theorem 4.4, $\lambda \notin \Lambda_{k-r}(A + F)$. □

If $A \in \mathcal{B}(\mathcal{H})$ is self-adjoint, Theorem 4.4 reduces to the following corollary.

Corollary 4.6. *Suppose $A \in \mathcal{B}(\mathcal{H})$ is self-adjoint and $1 \leq r \leq k$. Then $\lambda \in \Lambda_k(A)$ if and only if A can be decomposed into $\tilde{A}_1 \oplus \tilde{A}_2$ such that $\dim \tilde{A}_1 < k$, $W(\tilde{A}_1) \subseteq L$ and $W(\tilde{A}_2) \subseteq \mathbb{R} \setminus L$, where $L = [\lambda, \infty)$ or $(-\infty, \lambda]$.*

Using a similar argument as in the proof of Theorem 4.5, an analogue result can also be obtained for self-adjoint operators.

Theorem 4.7. *Suppose $A \in \mathcal{B}(\mathcal{H})$ is self-adjoint and $1 \leq r < k \leq \dim \mathcal{H}$. Let $\mathcal{S}$ be a subset of $\mathcal{F}_r$ containing the set $\{\pm 2\|A\|P : P \in \mathcal{P}_r\}$. Then*

$$\Lambda_k(A) = \cap\{\Lambda_{k-r}(A + F) : F \in \mathcal{S}\}.$$

In [16, Proposition 2.3], the author showed that $\Lambda_k(A) \subseteq \cap_{X \in \mathcal{V}_{k-1}} W(X^*AX)$, where $\mathcal{V}_m$ is the set of $X : \mathcal{H} \to \mathcal{H}$ with $X^*X = I_{\mathcal{H}}$ and $X(\mathcal{H}) = \mathcal{H}_1^{\perp}$ for some subspace $\mathcal{H}_1$ of $\mathcal{H}$ satisfying $\dim \mathcal{H}_1 \leq m$. In general, we have the following.

Proposition 4.8. *Suppose $A \in \mathcal{B}(\mathcal{H})$ and $1 \leq r < k < \infty$. Then*

$$\Lambda_k(A) \subseteq \cap\{\Lambda_{k-r}(X^*AX) : X \in \mathcal{V}_r\}.$$

Proof. Let $\lambda \in \Lambda_k(A)$. Then there exists a rank k orthogonal projection P such that $PAP = \lambda P$. Suppose $X \in \mathcal{V}_r$. Then there exists a subspace $\mathcal{H}_1$ of $\mathcal{H}$ with $\dim \mathcal{H}_1 \leq r$ satisfying $X^*X = I_{\mathcal{H}}$ and $X(\mathcal{H}) = \mathcal{H}_1^\perp$. Therefore, $\dim \left(P(\mathcal{H}) \cap \mathcal{H}_1^\perp\right) \geq k - r$. Choose a $k-r$ dimensional subspace $\mathcal{H}_2$ of $\mathcal{H}$ such that $X(\mathcal{H}_2) \subseteq P(\mathcal{H}) \cap \mathcal{H}_1^\perp$. Let $\{y_i\}_{i=1}^{k-r}$ be an orthogonal basis of $\mathcal{H}_2$. Then $\{X(y_i)\}_{i=1}^{k-r}$ is an orthonormal subset of $P(\mathcal{H})$. So, for $1 \leq i,\ j \leq k-r$, we have

$$\langle X^*AXy_i, y_j\rangle = \langle A(Xy_i), (Xy_j)\rangle = \delta_{ij}\lambda.$$

Hence, $\lambda \in \Lambda_{k-r}(X^*AX)$. □

Using Theorems 3.4 and 4.5, we have

Corollary 4.9. *Suppose $A \in \mathcal{B}(\mathcal{H})$ and $1 \leq r < k < \infty$. If $\dim \mathcal{H} < \infty$ or A is normal, then*

$$\Lambda_k(A) = \cap\{\Lambda_{k-r}(X^*AX) : X \in \mathcal{V}_r\}.$$

Proof. For each $F \in \mathcal{F}_r$, there is $X \in \mathcal{V}_r$ such that $X^*FX = 0$. Then

$$\Lambda_k(A) \subseteq \bigcap_{X\in\mathcal{V}_r} \Lambda_{k-r}(X^*AX) \subseteq \bigcap_{F\in\mathcal{F}_r} \Lambda_{k-r}(X^*(A+F)X) \subseteq \bigcap_{F\in\mathcal{F}_r} \Lambda_{k-r}(A+F).$$

By Theorems 3.4 and 4.5, the inclusions are indeed equalities. □

Similarly, using Theorems 2.1 and Corollary 3.3, we have the last corollary in this section.

Corollary 4.10. *Suppose $A \in \mathcal{B}(\mathcal{H})$ and $1 \leq r < k < \infty$. Then*

$$\Omega_k(A) = \cap\{\Omega_{k-r}(X^*AX) : X \in \mathcal{V}_r\}.$$

5. Results on $\Lambda_\infty(A)$

Suppose $\mathcal{H}$ is infinite dimensional and $A \in \mathcal{B}(\mathcal{H})$. It is clear that $\Lambda_\infty(A)$ can be viewed as the set of $\lambda \in \mathbb{C}$ for which there exists an infinite orthonormal set $\{x_i \in \mathcal{H} : i \geq 1\}$ such that $\langle Ax_i, x_j\rangle = \delta_{ij}\lambda$ for all $i, j \geq 1$. Extend the definition of $\Omega_k(A)$ to

$$\Omega_\infty(A) = \bigcap_{\xi\in[0,2\pi)} \left\{\mu \in \mathbb{C} : \mathrm{Re}\,(e^{i\xi}\mu) \leq \lambda_k(\mathrm{Re}\,(e^{i\xi}A)) \text{ for all } k \geq 1\right\}.$$

We have the following result.

Theorem 5.1. *Suppose $\dim \mathcal{H}$ is infinite and $A \in \mathcal{B}(\mathcal{H})$. Let $\mathcal{S}$ be a set of finite rank operators on $\mathcal{B}(\mathcal{H})$ containing the set*

$$\{2e^{i\xi}\|A\|P : \xi \in [0, 2\pi),\ P \text{ is a finite rank orthogonal projection}\}.$$

Then we have the following equalities.

(1) $\Omega_\infty(A) = \bigcap_{k\geq 1} \Omega_k(A) = \bigcap\{\mathbf{Cl}(W(A+F)) : F \in \mathcal{S}\} = W_e(A)$.
(2) $\Lambda_\infty(A) = \bigcap_{k\geq 1} \Lambda_k(A) = \bigcap\{W(A+F) : F \in \mathcal{S}\}$.

Proof. (1) By the definition of $\Omega_\infty(A)$, we have $\Omega_\infty(A) = \bigcap_{k\geq 1} \Omega_k(A)$. By (1.2) and Corollary 3.3, we have

$$\begin{aligned} W_e(A) &= \bigcap\{\mathbf{Cl}(W(A+F)) : F \in \mathcal{B}(\mathcal{H}) \text{ has finite rank } \} \\ &= \bigcap_{k\geq 1}\bigcap\{\mathbf{Cl}(W(A+F)) : F \in \mathcal{B}(\mathcal{H}) \text{ has rank } k-1\} \\ &= \bigcap_{k\geq 1} \Omega_k(A) \\ &= \bigcap_{k\geq 1}\bigcap\{\mathbf{Cl}(W(A+F)) : F \in \mathcal{S} \text{ has rank } k-1\} \\ &= \bigcap\{\mathbf{Cl}(W(A+F)) : F \in \mathcal{S}\}. \end{aligned}$$

So, the second and third equalities in (1) hold.

(2) By Theorem 4 in [1],we have

$$\Lambda_\infty(A) = \bigcap\{W(A+F) : F \in \mathcal{B}(\mathcal{H}) \text{ has finite rank } \}. \tag{5.1}$$

Clearly, we have the inclusion

$$\Lambda_\infty(A) \subseteq \bigcap_{k\geq 1} \Lambda_k(A).$$

To prove the reverse inclusion, suppose $\lambda \in \bigcap_{k\geq 1} \Lambda_k(A)$. Let $F \in \mathcal{B}(\mathcal{H})$ of rank m. Choose $k \geq m+1$. Then $\lambda \in \Lambda_k(A)$. By Theorem 3.1, we have

$$\lambda \in \Lambda_k(A) \subseteq \Lambda_1(A+F) = W(A+F).$$

Hence,

$$\bigcap_{k\geq 1} \Lambda_k(A) \subseteq \bigcap\{W(A+F) : F \in \mathcal{B}(\mathcal{H}) \text{ is of finite rank}\} = \Lambda_\infty(A).$$

Thus, we get the first equality in (2).

Next, we show that one only needs to use $F \in \mathcal{S}$ for the intersection on the right side of (5.1). To this end, note that

$$\bigcap\{W(A+F) : F \in \mathcal{B}(\mathcal{H}) \text{ is of finite rank}\} \subseteq \bigcap\{W(A+F) : F \in \mathcal{S}\}.$$

To prove the reverse inclusion, assume that

$$\lambda \notin \bigcap\{W(A+F) : F \in \mathcal{B}(\mathcal{H}) \text{ is of finite rank}\}.$$

If $\lambda \notin \Omega_\infty(A)$, then there is a finite rank $F \in \mathcal{B}(\mathcal{H})$ such that $\lambda \notin \mathbf{Cl}(W(A+F))$ and hence $\lambda \notin W(A+F)$. So, assume that

$$\lambda \in W_e(A) \subseteq W(A) \text{ and thus } |\lambda| \leq \sup\{|\mu| : \mu \in W(A)\} \leq \|A\|. \tag{5.2}$$

Then there is $\xi \in [0, 2\pi)$ and a finite rank operator $F \in \mathcal{B}(\mathcal{H})$ such that

$$e^{i\xi} W(A + F - \lambda I) \subseteq \{\mu \in \mathbb{C} : \operatorname{Im}(\mu) < 0\} \cup \mathcal{L} \tag{5.3}$$

with $\mathcal{L} = (0, \infty)$ or $\mathcal{L} = (-\infty, 0)$. We may replace A by $e^{i\xi}A$ and assume that $\xi = 0$. Without loss of generality, assume that $\mathcal{L} = (-\infty, 0)$.

Let $\lambda = a + ib$ with $a, b \in \mathbb{R}$ and $A = H + iG$ with $H = H^*$ and $G = G^*$. Since (5.3) holds with $\xi = 0$ for a finite rank operator $F \in \mathcal{B}(\mathcal{H})$, there is r not larger than the rank of $\operatorname{Im} F$ such that G has an operator matrix of the form

$$\operatorname{diag}(g_1, \ldots, g_r) \oplus bI_s \oplus G_2 \tag{5.4}$$

with $g_1 \geq \cdots \geq g_r > b$, $W(G_2) \subseteq (-\infty, b)$ and $0 \leq s \leq \infty$. By (5.2), we have

$$g_1 - b \leq |g_1| + |b| \leq 2\|G\| \leq 2\|A\|.$$

We consider two cases.

Case 1. Suppose $g_1 - b = 2\|A\|$. Then $g_1 = \|A\| = -b$. Since $\lambda \in W_e(A)$ and

$$\|A\| = |b| \leq |a + ib| = |\lambda| \leq \|A\|,$$

it follows that

$$a = 0 \quad \text{and} \quad \lambda = ib = -i\|A\|$$

is the only element in $\mathbf{Cl}(W(A)) \cap \{\mu \in \mathbb{C} : \operatorname{Im}(\mu) \leq -\|A\|\}$. Thus, G_2 in (5.4) is vacuous, i.e., G has operator matrix $\operatorname{diag}(g_1, \ldots, g_r) \oplus bI_s$. Using the same basis, we let H have the operator matrix

$$\begin{bmatrix} H_{11} & H_{12} \\ H_{12}^* & H_{22} \end{bmatrix}.$$

Since $\|H_{22} + ibI\| \leq \|A\| = |b|$, we see that $H_{22} = 0$. By the fact that

$$|b|^2 = \|A^*A\| = \|(H + iG)^*(H + iG)\|,$$

we see that H_{12} is zero as well. Thus, A has operator matrix

$$A_1 \oplus ibI_s \qquad \text{with} \quad A_1 \in M_m.$$

Since (5.3) holds for a finite operator F with $\xi = 0$ and $\mathcal{L} = (-\infty, 0)$, we see that $s \neq \infty$. But then $\dim \mathcal{H}$ is finite, which is a contradiction.

Case 2. Suppose $g_1 - b < 2\|A\|$. If s is finite in (5.4), then

$$\tilde{F} = i2\|A\|(I_{r+s} \oplus 0) \in \mathcal{S} \quad \text{and} \quad W(A - \tilde{F}) \subseteq \{\mu \in \mathbb{C} : \operatorname{Im}(\mu) < b\}.$$

Thus, $\lambda = a + ib \notin W(A - \tilde{F})$.

Next, assume that $s = \infty$. Suppose the compression of H on the null space of $G - bI$ equals H_0. Then there is a positive integer m such that H_0 has operator matrix $\operatorname{diag}(h_1, \ldots, h_m) \oplus H_1$ such that $h_1 \geq \cdots \geq h_m \geq a$ and $W(H_1) \subseteq (-\infty, a)$. Otherwise, (5.3) cannot hold for a finite operator F with $\xi = 0$ and $\mathcal{L} = (-\infty, 0)$. Let $\tilde{F} = i2\|A\|(I_{r+m} \oplus 0) \in \mathcal{S}$, and let $\hat{A} = A - \tilde{F} - \lambda I$. Then $\operatorname{Im}(\hat{A}) = (\hat{A} - \hat{A}^*)/2i$

has an operator matrix $\hat{G}_1 \oplus 0_{s-m} \oplus \hat{G}_2$ with $W(\hat{G}_1 \oplus \hat{G}_2) \subseteq (-\infty, 0)$. Moreover, the compression of $\mathrm{Re}\,(\hat{A}) = (\hat{A} + \hat{A}^*)/2$ on the null space of $\mathrm{Im}\,(\hat{A})$ equal $H_1 - aI$. As a result, if $\mu = \langle \hat{A}x, x\rangle \in W(\hat{A})$ has imaginary part 0, then x must lie in the null space of $\mathrm{Im}\,(\hat{A})$, and hence the real part of μ lies in $W(H_1 - aI) \subseteq (-\infty, 0)$. Thus, $0 \notin W(\hat{A})$, equivalently, $\lambda \notin W(A - \tilde{F})$. Consequently,

$$\bigcap\{W(A+F) : F \in \mathcal{S}\} \subseteq \bigcap\{W(A+F) : F \in \mathcal{B}(\mathcal{H}) \text{ is of finite rank}\}. \qquad \square$$

In [16], Martinex-Avendano asked whether $\Lambda_\infty(A) = \bigcap_{k\geq 1} \Lambda_k(A)$. Assertion (2) answers the question affirmatively.

Theorem 5.2. *Suppose $A \in \mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ is infinite dimensional. Then*

$$\mathbf{Int}(\Omega_\infty(A)) \subseteq \Lambda_\infty(A) \subseteq \Omega_\infty(A).$$

Moreover, $\mathbf{Cl}(\Lambda_\infty(A)) = \Omega_\infty(A)$ if and only if $\Lambda_\infty(A) \neq \emptyset$.

Proof. By the Corollary after Theorem 4 in [1], we see that $\mathbf{Int}(\Omega_\infty(A)) \subseteq \Lambda_\infty(A)$. The inclusion $\Lambda_\infty(A) \subseteq \Omega_\infty(A)$ is clear.

Note that $\Omega_\infty(A)$ is always a non-empty compact convex set. If $\Lambda_\infty(A) = \emptyset$ then $\mathbf{Cl}(\Lambda_\infty(A)) \neq \Omega_\infty(A)$. Conversely, suppose $\Lambda_\infty(A) \neq \emptyset$. If $\mathbf{Int}(\Lambda_\infty(A)) = \mathbf{Int}(\Omega_\infty(A))$ is non-empty, then $\mathbf{Cl}(\Lambda_\infty(A)) = \Omega_\infty(A)$. If $\mathbf{Int}(\Omega_\infty(A))$ is empty, then $\Omega_\infty(A) = \{\mu\}$ is a singleton and so is the non-empty set $\Lambda_\infty(A)$. Hence $\mathbf{Cl}(\Lambda_\infty(A)) = \Lambda_\infty(A) = \{\mu\}$. $\square$

The next example show that $\Lambda_\infty(A)$ may indeed be empty so that $\mathbf{Cl}(\Lambda_\infty(A)) \neq \Omega_\infty(A)$.

Example 5.3. Let $A = \bigoplus_{n\geq 2} \mathrm{diag}\,\left(e^{i\pi/n}/n, -1/n\right) \in \mathcal{B}(\mathcal{H})$. Then $\Omega_\infty(A) = \{0\}$ but $0 \notin \Lambda_1(A)$ so that $\Lambda_\infty(A) = \cap\{\Lambda_k(A) : k = 1, 2, \dots\} = \emptyset$. On the other hand, if $B = A \oplus 0_{\mathcal{H}}$, then $\Lambda_\infty(B) = \{0\}$.

From the proof of Theorem 5.2, we see that if $\Lambda_\infty(A)$ is a singleton, then $\Omega_\infty(A)$ is also a singleton, which can happen if and only if $A - \mu I$ is a compact operator for some $\mu \in \mathbb{C}$ by the corollary after Lemma 3 in [1]. In connection to this comment and Example 5.3, we have the following.

Proposition 5.4. *Let $A \in \mathcal{B}(\mathcal{H})$ and $\mu \in \mathbb{C}$ be such that $A - \mu I$ is compact. Then the following are equivalent.*

(a) *$\Lambda_\infty(A)$ is non-empty.*
(b) *$\Lambda_\infty(A) = \{\mu\}$.*
(c) *$\mu \in \Lambda_k(A)$ for each $k = 1, 2, \dots$.*

Proof. The implications "(a) $\iff$ (b)" is clear. We have "(c) $\iff$ (b)" because $\Lambda_\infty(A) = \bigcap_k \Lambda_k(A)$ by Theorem 5.1. $\square$

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