Condition for the higher rank numerical range to be non-empty

Chi-Kwong Li,[1] Yiu-Tung Poon[2] and Nung-Sing Sze [3]

Abstract

It is shown that the rank-k numerical range of every n-by-n complex matrix is non-empty if $k < n/3+1$*. The proof is based on a recent characterization of the rank-k numerical range by Li and Sze, the Helly's theorem on compact convex sets, and some eigenvalue inequalities. In particular, the result implies that* $\Lambda_2(A)$ *is non-empty if* $n \geq 4$*. This confirms a conjecture of Choi et al. If* $k \geq n/3+1$*, an n-by-n complex matrix is given for which the rank-k numerical range is empty. Extension of the result to bounded linear operators acting on an infinite dimensional Hilbert space is also discussed.*

AMS Subject Classification 15A21, 15A24, 15A60, 15A90, 81P68.

Keywords Higher rank numerical range, eigenvalue inequalities, Helly's theorem.

1 Introduction

Let M_n be the algebra of $n \times n$ complex matrices. In [3], the authors introduced the notion of the *rank-k numerical range* of $A \in M_n$ defined and denoted by

$$\Lambda_k(A) = \{\lambda \in \mathbf{C} : X^*AX = \lambda I_k,\ X \text{ is } n \times k \text{ such that } X^*X = I_k\}$$

in connection to the study of quantum error correction; see [4]. Evidently, $\lambda \in \Lambda_k(A)$ if and only if there is a unitary matrix $U \in M_n$ such that U^*AU has λI_k as the leading principal submatrix. When $k = 1$, this concept reduces to the classical numerical range, which is well known to be convex by the Toeplitz-Hausdorff theorem; for example, see [7] for a simple proof. In [1] the authors conjectured that $\Lambda_k(A)$ is convex, and reduced the convexity problem to the problem of showing that $0 \in \Lambda_k(A)$ for

$$A = \begin{pmatrix} I_k & X \\ Y & -I_k \end{pmatrix}$$

for arbitrary $X, Y \in M_k$. They further reduced this problem to the existence of a Hermitian matrix H satisfying the matrix equation

$$I_k + MH + HM^* - HPH = H \tag{1}$$

for arbitrary $M \in M_k$ and positive definite $P \in M_k$. In [10], the author observed that equation (1) can be rewritten as the continuous Riccati equation

$$HPH - H(M^* - I_k/2) - (M - I_k/2)H - I_k = 0_k, \tag{2}$$

and existing results on Riccati equation will ensure its solvability; for example, see [5, Theorem 4]. This establishes the convexity of $\Lambda_k(A)$.

[1]Department of Mathematics, The College of William and Mary, Williamsburg, VA 23185 (ckli@math.wm.edu). Research of Li was partially supported by a USA NSF grant and a HK RGC grant. He is an honorary professor of the University of Hong Kong.

[2]Department of Mathematics, Iowa State University, Ames, IA 50051 (ytpoon@iastate.edu).

[3]Department of Mathematics, University of Connecticut, Storrs, CT 06269 (sze@math.uconn.edu).

For a Hermitian $X \in M_n$, let $\lambda_1(X) \geq \cdots \geq \lambda_n(X)$ be the eigenvalues of X. In [8], it was shown that

$$\Lambda_k(A) = \{\mu \in \mathbf{C} : e^{it}\mu + e^{-it}\overline{\mu} \leq \lambda_k(e^{it}A + e^{-it}A^*) \text{ for all } t \in [0, 2\pi)\} . \tag{3}$$

In particular, $\Lambda_k(A)$ is the intersection of closed half planes on $\mathbf{C}$, and therefore is always convex. Moreover, if $A \in M_n$ is normal with eigenvalues $\lambda_1, \ldots, \lambda_n$, then

$$\Lambda_k(A) = \bigcap_{1 \leq j_1 < \cdots < j_{n-k+1} \leq n} \operatorname{conv}\{\lambda_{j_1}, \ldots, \lambda_{j_{n-k+1}}\}.$$

This confirms a conjecture in [2].

While many interesting results have been obtained for $\Lambda_k(A)$, see [1, 2, 3, 4] for example, there are some basic questions whose answers are unknown. The purpose of this paper is to answer the following.

Problem Determine n and k such that $\Lambda_k(A)$ is non-empty for every $A \in M_n$.

It is well-known that the classical numerical range $\Lambda_1(A)$ is non-empty. For $k > n/2$, $\Lambda_k(A)$ has at most one element and one can easily construct $A \in M_n$ such that $\Lambda_k(A) = \emptyset$; see Proposition 2.2 and Corollary 2.3 in [3]. The situation for $\Lambda_k(A)$ with $n/2 \geq k > 1$ is not so clear. In [2], the authors conjectured that $\Lambda_2(A) \neq \emptyset$ for $n \geq 4$.

In the next section, we show that $\Lambda_k(A)$ is non-empty for every $A \in M_n$ if and only if $k < n/3+1$. In particular, it confirms the conjecture in [2] that $\Lambda_2(A) \neq \emptyset$ if $n \geq 4$. We also consider extension of the result to infinite dimensional bounded linear operators.

2 Results and proofs

Theorem 1 *Let $A \in M_n$, and let k be a positive integer such that $k < n/3 + 1$. Then $\Lambda_k(A)$ is non-empty.*

Proof. Evidently, $\Lambda_k(A) \subseteq \Lambda_1(A)$. Given $A \in M_n$ and $t \in [0, 2\pi)$, let $A(t) = e^{it}A + e^{-it}A^*$. Consider the compact convex sets

$$S(t) = \{\mu \in \Lambda_1(A) : e^{it}\mu + e^{-it}\overline{\mu} \leq \lambda_k(A(t))\}, \qquad t \in [0, 2\pi).$$

By (3),

$$\Lambda_k(A) = \bigcap_{t \in [0,2\pi)} S(t).$$

By Helly's Theorem [6, Theorem 24.9], it suffices to show that $S(t_1) \cap S(t_2) \cap S(t_3) \neq \emptyset$ for all choices of t_1, t_2, t_3 with $0 \leq t_1 < t_2 < t_3 < 2\pi$.

For $1 \leq j \leq 3$, let V_j be the subspace spanned by the eigenvectors of $A(t_j)$ corresponding to the eigenvalues $\lambda_k(A(t_j)), \ldots, \lambda_n(A(t_j))$. Then $\dim V_j \geq n - k + 1$. Hence, we have

$$\begin{aligned}
& \dim(V_1 \cap V_2 \cap V_3) \\
= \; & \dim(V_1 \cap V_2) + \dim V_3 - \dim((V_1 \cap V_2) + V_3) \\
= \; & \dim V_1 + \dim V_2 - \dim(V_1 + V_2) + \dim V_3 - \dim((V_1 \cap V_2) + V_3) \\
\geq \; & 3(n - k + 1) - 2n \\
= \; & n - 3k + 3 \\
\geq \; & 1.
\end{aligned}$$

Let v be a unit (column) vector in $V_1 \cap V_2 \cap V_3$. Then $\mu = v^*Av \in \Lambda_1(A)$ and for $1 \leq j \leq 3$, we have

$$e^{it}\mu + e^{-it}\overline{\mu} = v^*(A(t_j))v \leq \lambda_k(A(t_j)).$$

Hence, $\mu \in S(t_1) \cap S(t_2) \cap S(t_3)$. □

The following answers a question in [2].

Corollary 2 *Let $A \in M_n$ with $n \geq 4$. Then $\Lambda_2(A) \neq \emptyset$.*

Without additional information on $A \in M_n$, the bound on n in Theorem 1 is best possible as shown by the following result.

Theorem 3 *Suppose k is a positive integer such that $k \geq n/3 + 1$. There exists $A \in M_n$ such that $\Lambda_k(A) = \emptyset$.*

Proof. We first consider the case when $3k = n + 3$. Let $w = e^{i2\pi/3}$, and

$$A = I_{k-1} \oplus wI_{k-1} \oplus w^2 I_{k-1}.$$

Write $A = H + iG$ with $H = H^*$ and $G = G^*$. Then $H = I_{k-1} \oplus (-1/2)I_{2k-2}$. Thus, $\Lambda_k(H) = \{-1/2\}$; see also [3, Theorem 2.4]. So,

$$\Lambda_k(A) \subseteq \mathcal{L} = \{z : \operatorname{Re} z = -1/2\}.$$

By rotation of $2\pi/3$ and $4\pi/3$, one can show that $\Lambda_k(A) \subseteq w\mathcal{L}$ and $\Lambda_k(A) \subseteq w^2\mathcal{L}$. So,

$$\Lambda_k(A) \subseteq \mathcal{L} \cap w\mathcal{L} \cap w^2\mathcal{L} = \emptyset.$$

Now, suppose $3k > n + 3$. Then we can consider a principal submatrix $B \in M_n$ of the matrix $A \in M_{3k-3}$ constructed in the preceding paragraph. Then $\Lambda_k(B) \subseteq \Lambda_k(A) = \emptyset$. □

Note that we can perturb the example in the above proof to get a non-normal matrix $A \in M_n$ such that $\Lambda_k(A) = \emptyset$ if $k \geq n/3 + 1$. Also, Theorem 3 can be obtained from parts (1), (2), (3) of [2, Theorem 4.7] and the fact that $\Lambda_k(A)$ is a subset of

$$\bigcap_{1 \leq j_1 < \cdots < j_{n-k+1} \leq n} \operatorname{conv}\{\lambda_{j_1}, \ldots, \lambda_{j_{n-k+1}}\}$$

if $A \in M_n$ is normal with eigenvalues $\lambda_1, \ldots, \lambda_n$.

Let $\mathcal{B}(\mathcal{H})$ be the algebra of bounded linear operator acting on an infinite dimensional Hilbert space $\mathcal{H}$. One can extend the definition of $\Lambda_k(A)$ for a bounded linear operator $A \in \mathcal{B}(\mathcal{H})$ by

$$\Lambda_k(A) = \{\gamma \in \mathbf{C} : X^*AX = \gamma I_k,\ X : \mathbf{C}^k \to \mathcal{H},\ X^*X = I_k\}.$$

By Theorem 1, we have the following.

Corollary 4 *Suppose k is a positive integer and $A \in \mathcal{B}(\mathcal{H})$ for an infinite dimensional Hilbert space $\mathcal{H}$. Then*

$$\Lambda_k(A) \neq \emptyset.$$

Acknowledgment

We would like to thank the authors of [1] and [10] for sending us their preprints. We also thank Professor John Holbrook for some helpful comments, and the referee for a careful reading of the paper.

References

[1] M.D. Choi, M. Giesinger, J. A. Holbrook, and D.W. Kribs, Geometry of higher-rank numerical ranges, Linear and Multilinear Algebra, to appear.

[2] M.D. Choi, J.A. Holbrook, D. W. Kribs, and K. Życzkowski, Higher-rank numerical ranges of unitary and normal matrices, Operators and Matrices, to appear. http://arxiv.org/quant-ph/0608244.

[3] M.D. Choi, D. W. Kribs, and K. Życzkowski, Higher-rank numerical ranges and compression problems, Linear Algebra Appl., 418 (2006), 828–839.

[4] M.D. Choi, D. W. Kribs, and K. Życzkowski, Quantum error correcting codes from the compression formalism, Rep. Math. Phys., 58 (2006), 77–91.

[5] P. Lancaster and L. Rodman, Algebraic Riccati equations, Oxford Science Publications, The Clarendon Press Oxford University Press, New York, 1995.

[6] S.R. Lay, Convex Sets and their Applications, Pure and Applied Mathematics, John Wiley & Sons, Inc., New York, 1982.

[7] C.K. Li, A simple proof of the elliptical range theorem, Proc. of Amer. Math. Soc. 124 (1996), 1985–1986.

[8] C.K. Li and N.S. Sze, Canonical forms, higher rank numerical range, convexity, totally isotropic subspace, matrix equations, submitted.

[9] C.K. Li and N.K. Tsing, On the kth matrix numerical range, Linear and Multilinear Algebra 28 (1991), 229-239.

[10] H. Woerdeman, The higher rank numerical range is convex, Linear and Multilinear Algebra, to appear.