Inclusion regions for numerical ranges and Linear Preservers

Chi-Kwong Li[1] and Nung-Sing Sze[2]

Abstract

Let R be a proper subset of the complex plane, and let $\mathcal{S}_R$ be the set of $n \times n$ complex matrices A such that the numerical range $W(A)$ satisfies $W(A) \subseteq R$. Linear maps ϕ on matrices satisfying $\phi(\mathcal{S}_R) = \mathcal{S}_R$ are characterized. Denote by $\tilde{\mathcal{S}}_R$ the set of $n \times n$ complex matrices A such that the numerical radius $r(A)$ satisfies $r(A) \subseteq R$ for a proper subset R of nonnegative real numbers. Linear maps ϕ on matrices satisfying $\phi(\tilde{\mathcal{S}}_R) = \tilde{\mathcal{S}}_R$ are also characterized. Analogous results on Hermitian matrices are obtained.

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1 Introduction

Let $\mathbf{M}_n$ be the algebra of $n \times n$ complex matrices. Define the numerical range of $A \in \mathbf{M}_n$ by

$$W(A) = \{x^*Ax : x \in \mathbb{C}^n,\ x^*x = 1\},$$

and the numerical radius of A by

$$r(A) = \{|\mu| : \mu \in W(A)\}.$$

The numerical range and numerical radius are useful concepts in studying matrices; see [4, Chapter 1].

Let R be a proper subset of the complex plane, and let $\mathcal{S}_R$ be the subset of $\mathbf{M}_n$ consisting of matrices A such that $W(A) \subseteq R$, i.e.

$$\mathcal{S}_R = \{A \in M_n : W(A) \subseteq R\}.$$

There has been considerable interest in studying inclusion regions for numerical ranges. It is in fact very useful in knowing inclusion regions for $W(A)$. For example, it is well known (see [4, Chapter 1]) that $W(A) \subseteq \mathbb{R}$ if and only if $A = A^*$; $W(A) \subseteq [0, \infty)$ if and only if A is positive semidefinite; and $W(A) \subseteq (0, \infty)$ if and only if A is positive definite. Moreover, Ando [1] (see also [2]) showed that $W(A)$ is contained in the unit disk if and only if $A = X^*CX$ with a $2m \times n$ matrix X such that $X^*X = I_n$ and $C = \begin{pmatrix} 0_m & 2I_m \\ 0_m & 0_m \end{pmatrix}$ for some integer m; Mirman [6] showed that $W(A)$ is contained in a triangle with vertices a, b, c if and only if $A = X^*CX$ with $X^*X = I_n$ and $C \in M_m$ a normal matrix with eigenvalues a, b, c for some integer m; see [3] for further results along this direction.

[1]Department of Mathematics, College of William and Mary, Williamsburg, VA 23187-8795, USA. (ckli@math.wm.edu) Research partially supported by NSF.

[2]Department of Mathematics, University of Hong Kong, Hong Kong. (math2000@graduate.hku.hk)
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Let $\mathbf{V}_n$ be $\mathbf{M}_n$ or the real linear space $\mathbf{H}_n$ of $n \times n$ Hermitian matrices, and let $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$ according to $\mathbf{V}_n = \mathbf{M}_n$ or $\mathbf{H}_n$. In this paper, we study linear preservers of $\mathcal{S}_R$, i.e., $\mathbb{F}$-linear operators $\phi : \mathbf{V}_n \to \mathbf{V}_n$ satisfying $\phi(\mathcal{S}_R) = \mathcal{S}_R$.

Denote by $\mathbf{P}_n, \mathbf{P}_n^+, \mathbf{U}_n, \mathbf{GL}_n$, the sets of positive semidefinite matrices, positive definite matrices, unitary matrices, and invertible matrices in $\mathbf{M}_n$, respectively. Then $\mathcal{S}_{[0,\infty)} = \mathbf{P}_n$ is the set of positive semidefinite matrices; $\mathcal{S}_{(0,\infty)} = \mathbf{P}_n^+$ is the set of positive definite matrices; for $R = \{z \in \mathbb{C} : |z| \le 1\}$ the set $\mathcal{S}_R$ consists of matrices A satisfying $r(A) \le 1$. We have the following results on linear preservers of inclusion regions for numerical ranges.

Theorem 1.1 [8] *Let $\mathbf{V}_n = \mathbf{M}_n$ or $\mathbf{H}_n$, and let $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$ accordingly. Suppose $\phi : \mathbf{V}_n \to \mathbf{V}_n$ is an $\mathbb{F}$-linear operator. Then the following are equivalent.*

(a) $\phi(\mathbf{P}_n) = \mathbf{P}_n$.

(b) $\phi(\mathbf{P}_n^+) = \mathbf{P}_n^+$.

(c) *ϕ has the form $A \mapsto T^*AT$ or $A \mapsto T^*A^tT$ for some $T \in \mathbf{GL}_n$.*

Theorem 1.2 [5] *Let $\mathbf{V}_n = \mathbf{M}_n$ or $\mathbf{H}_n$, and let $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$ accordingly. Suppose $\phi : \mathbf{V}_n \to \mathbf{V}_n$ is an $\mathbb{F}$-linear operator. Then the following are equivalent.*

(a) *$r(\phi(A)) = r(A)$ for all $A \in \mathbf{V}_n$.*

(b) *$\phi(\mathcal{S}_R) = \mathcal{S}_R$ for $R = \{\mu \in \mathbb{F} : |\mu| \le 1\}$.*

(c) *there exists $\mu \in \mathbb{F}$ with $|\mu| = 1$ such that ϕ has the form $A \mapsto \mu U^*AU$ or $A \mapsto \mu U^*A^tU$ for some $U \in \mathbf{U}_n$.*

Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ according to $\mathbf{V}_n = \mathbf{H}_n$ or $\mathbf{M}_n$. In Sections 3 and 4, we shall solve the slightly more general problem, namely, characterization of linear operators $\phi : \mathbf{V}_n \to \mathbf{V}_n$ such that $\phi(\mathcal{S}_{R_1}) = \mathcal{S}_{R_2}$ for two given subsets $R_1, R_2 \subseteq \mathbb{F}$, after proving some preliminary results in Section 2. Denote by $\tilde{\mathcal{S}}_R$ the set of matrices A such that $r(A) \in R$ for a given proper subset R of $[0,\infty)$. In section 5, we characterize linear operators $\phi : \mathbf{V}_n \to \mathbf{V}_n$ such that $\phi(\tilde{\mathcal{S}}_{R_1}) = \phi(\tilde{\mathcal{S}}_{R_2})$ for two subsets $R_1, R_2 \subseteq [0,\infty)$.

Related to our investigation, one may also consider ϕ such that $\phi(\mathcal{S}_R) \subseteq \mathcal{S}_R$. But it is difficult. For example, if $R = [0,\infty)$, then $\phi(\mathcal{S}_R) \subseteq \mathcal{S}_R$ if and only if ϕ is a positive linear map. The structure of such maps are known to be very complicated, see [7, Chapter 3]. In connection to this, we have the following result.

Theorem 1.3 *Let $\mathbf{V}_n = \mathbf{H}_n$ or $\mathbf{M}_n$, and let $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$ accordingly. An $\mathbb{F}$-linear map $\phi : \mathbf{V}_n \to \mathbf{V}_n$ satisfies $W(\phi(A)) \subseteq W(A)$ for all $A \in \mathbf{V}_n$ if and only if ϕ is a unital positive linear map. Consequently, if $\phi : \mathbf{V}_n \to \mathbf{V}_n$ is a unital positive linear map, then $\phi(\mathcal{S}_R) \subseteq \mathcal{S}_R$ for any subset R of $\mathbb{C}$.*

Proof. (⇒) If A is positive semidefinite, then $W(\phi(A)) \subseteq W(A) \subseteq [0, \infty)$. Thus, $\phi(A)$ is positive semidefinite. Also, $W(\phi(I_n)) \subseteq W(I_n) = \{1\}$. Hence, $\phi(I_n) = I_n$.

(⇐) Suppose ϕ is a unital positive linear map. Then ϕ maps Hermitian matrices to Hermitian matrices in case $\mathbf{V}_n = \mathbf{M}_n$. Furthermore, if $\lambda I - (\mu A + (\mu A)^*) \in \mathbf{P}_n$, then $\lambda I - (\mu\phi(A) + (\mu\phi(A))^*) \in \mathbf{P}_n$ for any $\lambda \in \mathbb{R}$ and $\mu \in \mathbb{F}$. Since $\lambda I - (\mu B + (\mu B)^*) \in \mathbf{P}_n$ if and only if

$$W(B) \subseteq \{z \in \mathbb{F} : \lambda \geq (\mu z) + (\mu z)^*\},$$

we see that each half space of $\mathbb{F}$ containing $W(A)$ will also contain $W(\phi(A))$. It follows that $W(\phi(A)) \subseteq W(A)$. □

2 Preliminary Results

Lemma 2.1 *Let $\mathbf{V}_n = \mathbf{H}_n$ or $\mathbf{M}_n$, and let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ according to $\mathbf{V}_n = \mathbf{H}_n$ or $\mathbf{M}_n$. If $R \subseteq \mathbb{F}$ contains a nondegenerate line segment, then $\mathcal{S}_R \subseteq \mathbf{V}_n$ is a spanning set of $\mathbf{V}_n$. Consequently, if $\phi : \mathbf{V}_n \to \mathbf{V}_n$ is a linear operator such that $\phi(\mathcal{S}_{R_1}) = \mathcal{S}_R$ for some $R_1 \subseteq \mathbb{F}$, then $\mathcal{S}_{R_1}$ is a spanning set of $\mathbf{V}_n$ and ϕ is invertible.*

Proof. We prove the result for $\mathbf{M}_n$. The proof for $\mathbf{H}_n$ is similar.

Recall that a matrix $A \in \mathbf{M}_n$ has numerical range lying on a line segment L if and only if A is normal with eigenvalues contained in L. Thus, if R contains a line segment L, then $\mathcal{S}_R$ contains all normal matrices with eigenvalues in L. There exists some $A \in \mathcal{S}_R$ with eigenvalues in L and nonzero trace. By the main result in [9], $\{U^*AU : U \in \mathbf{U}_n\} \subseteq \mathcal{S}_R$ is a spanning set of $\mathbf{M}_n$.

Now, suppose $\phi : \mathbf{M}_n \to \mathbf{M}_n$ is a linear operator such that $\phi(\mathcal{S}_{R_1}) = \mathcal{S}_R$ for some $R_1 \subseteq \mathbb{C}$. Since $\phi(\mathcal{S}_{R_1})$ contains a spanning set of $\mathbf{M}_n$ the last assertion follows. □

The following lemma can be verified readily.

Lemma 2.2 *Let $\mathbf{V}_n = \mathbf{H}_n$ or $\mathbf{M}_n$, and let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ according to $\mathbf{V}_n = \mathbf{H}_n$ or $\mathbf{M}_n$. Suppose $\phi : \mathbf{V}_n \to \mathbf{V}_n$ is a linear operator satisfying $\phi(\mathcal{S}_{R_1}) = \mathcal{S}_{R_2}$. Then for any nonzero $\mu \in \mathbb{F}$*

$$\phi(\mathcal{S}_{\mu R_1}) = \mathcal{S}_{\mu R_2}.$$

For $R \subseteq \mathbb{F}$ and $\mu \in \mathbb{F}$, let

$$R + \mu = \{z + \mu \in \mathbb{F} : z \in R\}.$$

We have the following observation.

Lemma 2.3 *Let $\mathbf{V}_n = \mathbf{H}_n$ or $\mathbf{M}_n$, and let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ according to $\mathbf{V}_n = \mathbf{H}_n$ or $\mathbf{M}_n$. Suppose $\phi : \mathbf{V}_n \to \mathbf{V}_n$ is a linear operator satisfying $\phi(I_n) = \mu I_n$ for some $\mu \in \mathbb{F}$ and $\phi(\mathcal{S}_{R_1}) = \mathcal{S}_{R_2}$. Then $\mu R_1 = R_2$, and for any nonzero $\nu \in \mathbb{F}$*

$$\phi(\mathcal{S}_{R_1+\nu}) = \mathcal{S}_{R_2+\mu\nu}.$$

Proof. Note that $z \in R_1$ if and only if $zI \in \mathcal{S}_{R_1}$. Hence, $\mu z I \in \mathcal{S}_{R_2}$, or equivalently, $\mu z \in R_2$. Thus, $\mu R_1 = R_2$.

Let $A \in \mathbf{V}_n$ and $\mu \in \mathbb{F}$. Then $W(A) \subseteq R$ if and only if $W(A + \nu I_n) \subseteq R + \nu$. Hence, $\mathcal{S}_{R+\nu} = \{A + \nu I_n : A \in \mathcal{S}_R\}$. Since ϕ is linear and $\phi(I_n) = \mu I_n$, the result follows. □

Lemma 2.4 *Let $\mathbf{V}_n = \mathbf{H}_n$ or $\mathbf{M}_n$, and let $R_1, R_2 \subseteq \mathbb{F}$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ according to $\mathbf{V}_n = \mathbf{H}_n$ or $\mathbf{M}_n$. Suppose $\phi : \mathbf{V}_n \to \mathbf{V}_n$ is linear and satisfies $\phi(\mathcal{S}_{R_1}) = \mathcal{S}_{R_2}$. If C_1 is a connected component of R_1, then there is a connected component C_2 of R_2 such that $\phi(\mathcal{S}_{C_1}) \subseteq \mathcal{S}_{C_2}$. The set inclusion becomes a set equality if ϕ is invertible.*

Proof. Let C_1 be a connected component of R_1 and let $A \in \mathcal{S}_{C_1}$. For any $B \in \mathcal{S}_{C_1}$ we show that there is a continuous path $\gamma : [0,1] \to \mathcal{S}_{C_1}$ such that $\gamma(0) = A$ and $\gamma(1) = B$ as follows. First, by [4, Theorem 1.3.4], there is $U \in \mathbf{U}_n$ such that $A = U^*(aI_n + A_0)U$, where $a = (\operatorname{tr} A)/n$ and A_0 has zero diagonal entries. The path $\gamma_1(t) = U^*(aI_n + (1-t)A_0)U$, $t \in [0,1]$, connects A and aI_n. Moreover, since $a \in W(A)$, we see that

$$W(\gamma_1(t)) = W((1-t)A + taI_n) \subseteq (1-t)W(A) + tW(aI_n) \subseteq W(A) \subseteq C_1.$$

So, γ_1 is a path in $\mathcal{S}_{C_1}$. Similarly, there is a path γ_2 joining B and bI_n, where $b = (\operatorname{tr} B)/n$ in $\mathcal{S}_{C_1}$. Finally, if $a \in W(A) \subseteq C_1$ and $b \in W(B) \subseteq C_1$, there is a continuous path γ_3 in C_1 joining a and b. Then $\tilde{\gamma}_3$ defined by $\tilde{\gamma}_3(t) = \gamma_3(t)I_n$ is a continuous path in $\mathcal{S}_{C_1}$ connecting aI_n and bI_n. Combining $\gamma_1, \tilde{\gamma}_3$ and γ_2, we get a continuous path $\gamma(t)$ in $\mathcal{S}_{C_1}$ connecting A and B.

Now, $W(\gamma(t)) \subseteq \mathcal{S}_{R_2}$. We see that the set $\bigcup_{t\in[0,1]} W(\gamma(t))$ is a connected subset of R_2 containing both $W(\phi(A))$ and $W(\phi(B))$. Hence, they must lie in the same connected component C_2 of R_2. Thus for every $B \in \mathcal{S}_{C_1}$, we have $\phi(B) \in \mathcal{S}_{C_2}$. Thus $\phi(\mathcal{S}_{C_1}) \subseteq \mathcal{S}_{C_2}$.

Suppose ϕ is invertible. Then $\phi^{-1}(\mathcal{S}_{R_2}) = \mathcal{S}_{R_1}$. It follows that $\phi^{-1}(\mathcal{S}_{C_2}) \subseteq \mathcal{S}_{C_1}$. Hence the last assertion follows. □

The next two lemmas characterize linear operators ϕ satisfying $\phi(\mathcal{S}_{R_1}) = \mathcal{S}_{R_2}$ for some special R_1.

Lemma 2.5 *Let $\mathbf{V}_n = \mathbf{H}_n$ or $\mathbf{M}_n$, and $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ according to $\mathbf{V}_n = \mathbf{H}_n$ or $\mathbf{M}_n$. Suppose $R_1, R_2 \subseteq \mathbb{F}$ are non-empty such that R_1 does not contain any line segment, and $R_i \neq \{0\}$ for $i = 1, 2$. A linear operator $\phi : \mathbf{V}_n \to \mathbf{V}_n$ satisfies $\phi(\mathcal{S}_{R_1}) = \mathcal{S}_{R_2}$ if and only if $\phi(I) = \mu I$ for some $\mu \in \mathbb{F}$ satisfying $\mu R_1 = R_2$.*

Proof. Since R_1 does not contain any line segment, then $W(A)$ is a singleton for every $A \in \mathcal{S}_{R_1}$. Hence, $\mathcal{S}_{R_1} = \{\nu I_n : \nu \in R_1\}$ and the linear span of $\mathcal{S}_{R_1} = \mathbb{F} \cdot I$, is the 1-dimensional space of scalar matrices in $\mathbf{V}_n$. The ($\Leftarrow$) of the assertion is clear. To prove the implication ($\Rightarrow$), suppose $\nu_0 \in R_1$ and $B = \phi(\nu_0 I_n)$. Then for any $\nu \in R_1$, $\phi(\nu I_n) = (\nu/\nu_0)B$. If B is not a scalar matrix, then $W(B) \subseteq R_2$ contains some line segment L. By Lemma 2.1, the set $T = \{X \in \mathbf{V}_n : W(X) \subseteq L\}$ is a spanning set of $\mathbf{V}_n$. It follows that $\phi(\mathbb{F} \cdot I) = \phi(\operatorname{span} \mathcal{S}_{R_1}) = \operatorname{span} \mathcal{S}_{R_2} = \mathbf{V}_n$, which is a contradiction. □

Lemma 2.6 *Let $\mathbf{V}_n = \mathbf{H}_n$ or $\mathbf{M}_n$, and $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ according to $\mathbf{V}_n = \mathbf{H}_n$ or $\mathbf{M}_n$. Suppose $R_1 = \mathbb{F}$ and $R_2 \subseteq \mathbb{F}$ is non-empty and not equal to $\{0\}$. A linear operator $\phi : \mathbf{V}_n \to \mathbf{V}_n$ satisfies $\phi(\mathcal{S}_{R_1}) = \mathcal{S}_{R_2}$ if and only if ϕ is invertible and $R_2 = \mathbb{F}$.*

Proof. To prove the implication ($\Rightarrow$), take any nonzero element $\nu \in \mathbb{F}$ and $A \in \mathbf{V}_n = \mathcal{S}_{R_1}$ such that $\phi(A) = \nu I_n \in \mathcal{S}_{R_2}$. Then for any $\mu \in \mathbb{F}$, we have $\mu A \in \mathcal{S}_{R_1}$ and $\phi(\mu A) = \mu\nu I_n \in \mathcal{S}_{R_2}$. Thus, $\mu\nu \in R_2$. It follows that $R_2 = \mathbb{F}$. By Lemma 2.1, ϕ is invertible. The converse is clear. $\square$

3 Results on Hermitian matrices

In this section, we characterize linear maps ϕ on $\mathbf{H}_n$ satisfying $\phi(\mathcal{S}_{R_1}) = \mathcal{S}_{R_2}$ for two given subsets $R_1, R_2 \subseteq \mathbb{R}$. To avoid trivial consideration, we assume that R_1 and R_2 are non-empty. Furthermore, if $R_2 = \{0\}$ then ϕ can be any linear map such that $\phi(A) = 0$ for all $A \in \mathcal{S}_{R_1}$; one cannot say much about the structure of ϕ. If $R_1 = \{0\}$, then we must have $R_2 = \{0\}$ and ϕ can be any linear map. So, we also exclude these cases in our consideration.

Theorem 3.1 *Let R_1, R_2 be non-empty subsets of $\mathbb{R}$ such that $R_j \neq \{0\}$ for $j = 1, 2$. There is a linear operator $\phi : \mathbf{H}_n \to \mathbf{H}_n$ satisfying $\phi(\mathcal{S}_{R_1}) = \mathcal{S}_{R_2}$ if and only if there is a nonzero $\mu \in \mathbb{R}$ such that $\mu R_1 = R_2$ and one of the following conditions holds.*

1. *The set R_1 does not contain any line segment and $\phi(I_n) = \mu I_n$.*
2. *The set $R_1 = \mathbb{R}$ and ϕ is invertible.*
3. *The set R_1 equals $(0, \infty), [0, \infty), (-\infty, 0], (-\infty, 0]$ or $\mathbb{R} \setminus \{0\}$, and ϕ has the form $A \mapsto \mu T^* A T$ or $A \mapsto \mu T^* A^t T$ for some $T \in \mathbf{GL}_n$.*
4. *The set R_1 is not of any of the above forms, and ϕ has the form $A \mapsto \mu U^* A U$ or $A \mapsto \mu U^* A^t U$ for some $U \in \mathbf{U}_n$.*

Proof. The implication ($\Leftarrow$) can be readily verified. We consider the converse. The first two cases follow from Lemmas 2.5 and 2.6. In the other cases, R_1 contains a connected component L_1 which is neither $\mathbb{R}$ nor a singleton set. By Lemma 2.4, we have $\phi(\mathcal{S}_{L_1}) \subseteq \mathcal{S}_{L_2}$ for a connected component L_2 of R_2. Note that L_2 is not a singleton. Otherwise, $\mathcal{S}_{L_2} = \{\mu I_n\}$ for some $\mu \in \mathbb{R}$. Since $\mathcal{S}_{L_1}$ is a spanning set of $\mathbf{H}_n$, $\phi(\mathbf{H}_n) = \{\mu I_n\}$. It follows that $\mu = 0$, which is a contradiction. So, L_2 is a nontrivial interval, ϕ is invertible by Lemma 2.1, and $\phi(\mathcal{S}_{L_1}) = \mathcal{S}_{L_2}$.

Here we consider the following different types of proper intervals L in $\mathbb{R}$.

(a) $L = [0, \infty)$ or $(-\infty, 0]$;

(b) $L = (0, \infty)$ or $(-\infty, 0)$;

(c) There exists $(-a, a) \subseteq L$ for some $a > 0$ but $L \neq \mathbb{R}$;

(d) $L = [0,a), (0,a), [0,a], (0,a], (-a,0], (-a,0), [-a,0]$ or $[-a,0)$ for some $a > 0$;

(e) $L = (a,\infty), [a,\infty), (-\infty,-a)$ or $(-\infty,-a]$ for some $a > 0$;

(f) $L = (a,b), (a,b], [a,b)$ or $[a,b]$ for some $a,b \in \mathbb{R}$ with either $0 < a < b$ or $a < b < 0$.

In order that $\phi(\mathcal{S}_{L_1}) = \mathcal{S}_{L_2}$, L_1 and L_2 must be of the same type by the following character of intervals, which are invariant under an invertible linear map.

(a) $\mathcal{S}_L = \pm\mathbf{P}_n$ and for every $A \in \mathcal{S}_L$, $kA \in \mathcal{S}_L$ for all $k \geq 0$;

(b) $\mathcal{S}_L = \pm\mathbf{P}_n^+$ and for every $A \in \mathcal{S}_L$, $kA \in \mathcal{S}_L$ for all $k > 0$;

(c) $\mathcal{S}_L \neq \mathbf{H}_n$ and there exists $A \in \mathcal{S}_L$ such that $-A \in \mathcal{S}_L$;

(d) For every nonzero $A \in \mathcal{S}_L$, $-A \notin \mathcal{S}_L$. Moreover, there exist k_1 and k_2 with $0 < k_1 < k_2$ such that $kA \in \mathcal{S}_L$ for all $k \leq k_1$ while $kA \notin \mathcal{S}_L$ for all $k \geq k_2$;

(e) For every $A \in \mathcal{S}_L$, $-A \notin \mathcal{S}_L$. Also there exist k_1 and k_2 with $0 < k_1 < k_2$ such that $kA \notin \mathcal{S}_L$ for all $k \leq k_1$ while $kA \in \mathcal{S}_L$ for all $k \geq k_2$;

(f) $\mathcal{S}_L$ does not satisfy any of above properties.

Now, we are ready to characterize ϕ according to the different types of L_1. We have the following two cases.

(i) If L_1 is of the type (a) or (b), then ϕ has the form $A \mapsto \mu T^*AT$ or $A \mapsto \mu T^*A^tT$ for some $T \in \mathbf{GL}_n$ and $\mu \in \{1,-1\}$ such that $\mu R_1 = R_2$.

(ii) In the other cases, ϕ has the form $A \mapsto \mu U^*AU$ or $A \mapsto \mu U^*A^tU$ for some $U \in \mathbf{U}_n$ and $\mu \in \{1,-1\}$ such that $\mu R_1 = R_2$.

For type (a), note that $\mathcal{S}_{L_1}$ and $\mathcal{S}_{L_2}$ are either $\mathbf{P}_n$ or $-\mathbf{P}_n$. Hence, $\phi(\mathbf{P}_n) = \mathbf{P}_n$ or $\phi(\mathbf{P}_n) = -\mathbf{P}_n$. Replacing ϕ by $-\phi$ if necessary and using Theorem 1.1, we get the result.

For type (b), note that $\mathcal{S}_{L_1}$ and $\mathcal{S}_{L_2}$ are either $\mathbf{P}_n^+$ or $-\mathbf{P}_n^+$. The result again follows from Theorem 1.1.

For type (c), let $k_i = \sup\{k > 0 : (-k,k) \subseteq L_i\}$ for $i = 1,2$. Then both k_1 and k_2 are positive. Replacing (ϕ, L_1, L_2) by $(\frac{k_1}{k_2}\phi, \frac{1}{k_1}L_1, \frac{1}{k_2}L_2)$, we may assume $k_1 = k_2 = 1$. By the definition of k_1, we must have $[-k,k] \subseteq L_1$ for all $k < 1$; otherwise there is a $k < k' < 1$ such that $(-k',k') \nsubseteq L_1$.

For any $A \in \mathbf{H}_n$ and $k \in (-1,1)$, $W(\frac{k}{r(A)}A) \subseteq [-k,k] \subseteq L_1$. Then $W(\phi(\frac{k}{r(A)}A)) \subseteq L_2$. We claim that $W(\phi(\frac{1}{r(A)}A)) \subseteq [-1,1]$. Otherwise, there is $z \in W(\phi(\frac{1}{r(A)}A))$ such that $|z| > 1$. Since $kz \in W(\phi(\frac{k}{r(A)}A)) \subseteq L_2$ and k can be any value in $(-1,1)$, it follows that $(-|z|,|z|) \subseteq L_2$. It is impossible since $|z| > 1 = k_2$. Hence, we have $W(\phi(\frac{1}{r(A)}A)) \subseteq [-1,1]$, it follows that $r(\phi(A)) \leq r(A)$. By considering ϕ^{-1}, we have $r(\phi^{-1}(A)) \leq r(A)$. Hence, ϕ

is a numerical radius preserver on $\mathbf{H}_n$. By Theorem 1.2, ϕ has the asserted form, and the result follows.

For type (d), we may assume that $L_1 = L_2 = L$ is one of the following intervals:

$$[0,1], \quad [0,1), \quad (0,1], \quad (0,1).$$

Otherwise, replace (ϕ, L_1, L_2) by $(\frac{b}{a}\phi, aL_1, bL_2)$ for some suitable nonzero $a, b \in \mathbb{R}$. Then $A \in \mathbf{H}_n$ satisfies $r(A) < 1$ $(r(A) \leq 1)$ if and only if $A = A_1 - A_2$ with $A_1, A_2 \in \mathcal{S}_L$. Since $\phi(\mathcal{S}_L) = \mathcal{S}_L$, it follows that $r(\phi(A)) < 1$ $(r(\phi(A)) \leq 1)$ whenever $r(A) < 1$ $(r(A) \leq 1)$. Applying the argument to ϕ^{-1}, we see that $r(A) < 1$ $(r(A) \leq 1)$ whenever $r(\phi(A)) < 1$ $(r(\phi(A)) \leq 1)$. Consequently, ϕ preserves the numerical radius. The result follows from Theorem 1.2.

For type (e), we may assume that $L_1 = L_2 = L$ is the interval $[1, \infty)$ or $(1, \infty)$. Otherwise, replace (ϕ, L_1, L_2) by $(\frac{b}{a}\phi, aL_1, bL_2)$ for some suitable nonzero $a, b \in \mathbb{R}$. Then

$$\{kA : W(A) \subseteq L_i \text{ and } k > 0\} = \mathbf{P}_n^+, \quad i = 1, 2.$$

Since ϕ is linear, we see that $\phi(\mathbf{P}_n^+) = \mathbf{P}_n^+$. By Theorem 1.1, ϕ has the form $A \mapsto T^*A^+T$ for some $T \in \mathbf{GL}_n$, where A^+ denotes A or A^t.

Suppose T^*T has an eigenvalue $\gamma < 1$. Then $A = 2^{-1}(1 + 1/\gamma)I_n \in \mathcal{S}_{L_1}$, but $\phi(A) = 2^{-1}(1 + 1/\gamma)T^*T$ has an eigenvalue $2^{-1}(\gamma + 1) < 1$. Thus, $W(\phi(A)) \not\subseteq L_2$, which is a contradiction. Thus, all eigenvalues of T^*T are larger than or equal to 1, i.e., all singular values of T are larger than or equal to 1. Applying the argument to $\phi^{-1}(A) = (T^*)^{-1}A^+T^{-1}$, we see that the singular values of T^{-1} are larger than or equal to 1. As a result, all singular values of T equal 1, i.e., T is unitary.

For type (f), we may replace ϕ by $-\phi$ if necessary, and assume that $L_1, L_2 \subseteq (0, \infty)$. Let r_1, r_2, s_1 and s_2 denote $\inf L_1, \inf L_2, \sup L_1$ and $\sup L_2$, respectively. Then all of them are positive. Suppose $W(\phi(I_n)) = [a_1, b_1]$. Then as $z \in L_1$ if and only if $[za_1, zb_1] \subseteq L_2$, we have

$$0 < r_2 \leq a_1 r_1 \leq b_1 s_1 \leq s_2.$$

Similarly, if $W(\phi^{-1}(I_n)) = [a_2, b_2]$, then

$$0 < r_1 \leq a_2 r_2 \leq b_2 s_2 \leq s_1.$$

We can conclude that $1 \leq a_1 a_2 \leq b_1 b_2 \leq 1$, and that $a_1 a_2 = b_1 b_2$. As $0 < a_1 \leq b_1$ and $0 < a_2 \leq b_2$, we have $a_1 = b_1$ and $a_2 = b_2$. Thus, $\phi(I_n) = \mu I_n$ for some $\mu \in \mathbb{R}$. By lemma 2.3 with some suitable $\nu \in \mathbb{R}$, $\phi(\mathcal{S}_{L_1 - \nu}) = \mathcal{S}_{L_2 - \mu\nu}$, where $L_1 - \nu$ is of type (c).

It is easy to check that there is a nonzero $\mu \in \mathbb{R}$ such that $\mu R_1 = R_2$ in each case. □

4 Results on Complex Matrices

In this section, we characterize linear maps ϕ on $\mathbf{M}_n$ satisfying $\phi(\mathcal{S}_{R_1}) = \mathcal{S}_{R_2}$ for two given subsets $R_1, R_2 \subseteq \mathbb{C}$. Similar to section 3, we assume that R_1 and R_2 are non-empty. Also we exclude the cases that R_1 or R_2 equal to the set $\{0\}$ in our consideration.

Identify $\mathbf{U}_1$ with the unit circle in $\mathbb{C}$, we have the following result.

Theorem 4.1 *Let R_1, R_2 be non-empty subsets of $\mathbb{C}$ such that $R_j \neq \{0\}$ for $j = 1, 2$. There is a linear map $\phi : \mathbf{M}_n \to \mathbf{M}_n$ satisfying $\phi(\mathcal{S}_{R_1}) = \mathcal{S}_{R_2}$ if and only if there is a nonzero $\mu \in \mathbb{C}$ such that $\mu R_1 = R_2$ and one of the following conditions holds.*

1. *The set R_1 does not contain any nondegenerate line segment and $\phi(I_n) = \mu I_n$.*

2. *The set R_1 has no interior point and is the union of a collection of straight lines such that each of them passes through the origin; $\phi(\mathbf{H}_n) = \mu \mathbf{H}_n$.*

3. *The set R_1 has no interior point and equals $R_2 \cup R_3$, where R_2 is a non-empty collection of straight lines and R_3 does not contain any line segment so that either R_2 contains a line not passing the origin or $R_3 \setminus \{0\}$ is non-empty; $\phi(\mathbf{H}_n) = \mu \mathbf{H}_n$ and $\phi(I_n) = \mu I_n$.*

4. *The set $R_1 = \mathbb{C}$ and ϕ is invertible.*

5. *The set $R_1 \neq \mathbb{C}$ has interior points and is a union of sets of the forms: $w(0, \infty)$ or $w[0, \infty)$ with $w \in \mathbf{U}_1$; ϕ has the form $A \mapsto \mu T^* A T$ or $A \mapsto \mu T^* A^t T$ for some $T \in \mathbf{GL}_n$.*

6. *The set R_1 does not satisfy any of the conditions in (1)–(5), and ϕ has the form $A \mapsto \mu U^* A U$ or $A \mapsto \mu U^* A^t U$ for some $U \in \mathbf{U}_n$.*

Proof. The implication ($\Leftarrow$) can be readily verified except for Case (3). Note that in such case, if $A \in \mathcal{S}_{R_1}$, then $W(A)$ has no interior point and is a subset of $a + b\mathbb{R}$ for some $a, b \in \mathbb{C}$. Thus, $A = aI + bH$ for some $H \in \mathbf{H}_n$. So, $\phi(A) = \mu(aI + bK)$ for some $K \in \mathbf{H}_n$, and thus $W(\phi(A)) \subseteq \mu(a + b\mathbb{R}) \subseteq \mu R_1 = R_2$.

For the converse, Case (1) and Case (4) follow from Lemmas 2.5 and 2.6. We focus on the other cases.

Note that R_2 must contain some nondegenerate line segment. Otherwise, by lemma 2.4, there is a connected component C_1 in R_1 containing a nondegenerate line segment, and a singleton component C_2 in R_2 such that $\phi(\mathcal{S}_{C_1}) = \mathcal{S}_{C_2}$. Clearly, $\mathcal{S}_{C_2} = \{\mu I_n\}$ for some $\mu \in R_2$. Since $\mathcal{S}_{C_1}$ is a spanning set of $\mathbf{M}_n$, $\phi(\mathbf{M}_n) = \{\mu I_n\}$. It follows that $\mu = 0$, which is a contradiction. So, R_2 must contain some nondegenerate line segment, and ϕ is invertible by Lemma 2.1.

In the following, we establish a series of assertions leading to the conclusion that $\phi(\mathbf{H}_n) = w\mathbf{H}_n$ for some $w \in \mathbf{U}_1$ (Assertion 5).

For $i = 1, 2$, let J_i be the subset of R_i containing all elements z such that $rz \in R_i$ for all $r \in (0, 1]$. Also, let $\tilde{J}_i$ be the subset of R_i containing all elements z which $rz \in S$ for all $r \in [1, \infty)$. Also, for any $\alpha, \beta \in \mathbb{C}$, let $[\alpha, \beta] = \{\lambda\alpha + (1 - \lambda)\beta : \lambda \in [0, 1]\}$. We have the following assertions.

Assertion 1 *If J_1 is nonempty, then $\phi(\mathcal{S}_{J_1}) = \mathcal{S}_{J_2}$. Similarly, $\phi(\mathcal{S}_{\tilde{J}_1}) = \mathcal{S}_{\tilde{J}_2}$ if $\tilde{J}_1$ is nonempty.*

Proof. We shall prove the first implication, that of the second is similar. Let $A \in \mathcal{S}_{J_1}$. Then $W(A) \subseteq J_1 \subseteq R_1$, and $W(\phi(A)) \subseteq R_2$. By the definition of J_1, $W(rA) \subseteq J_1$ for all $r \in (0,1]$. Hence, for every $z \in W(\phi(A)) \subseteq R_2$, $rz \in R_2$ for all $r \in (0,1]$. We have $z \in J_2$ and $\phi(A) \in \mathcal{S}_{J_2}$. Therefore, $\phi(\mathcal{S}_{J_1}) \subseteq \mathcal{S}_{J_2}$. By considering ϕ^{-1}, we can deduce with a similar argument that $\phi^{-1}(\mathcal{S}_{J_2}) \subseteq \mathcal{S}_{J_1}$. The result follows.

Assertion 2 *If J_1 has nonzero elements, then there exists $w \in \mathbf{U}_1$ such that $\phi(\mathbf{H}_n) = w\mathbf{H}_n$.*

Proof. If J_1 has some nonzero elements, then so does J_2. Otherwise, $\phi(\mathcal{S}_{J_1}) = \{0\}$. Also, $R_2 \neq \mathbb{C}$ as that is R_1. Otherwise, $\phi(\mathcal{S}_{J_1}) = \mathbf{M}_n$ which implies $\mathcal{S}_{J_1} = \mathbf{M}_n$.

For $J = J_1$ or J_2, one of the following holds.

(a) $0 \in J$ and it is an interior point;

(b) $0 \in J$ and it is not an interior point;

(c) $0 \notin J$ and there is $r > 0$ such that $z \in J$ for all $0 < |z| < r$;

(d) $0 \notin J$ and no such $r > 0$ mentioned in (c) exists.

In order to have $\phi(\mathcal{S}_{J_1}) = \mathcal{S}_{J_2}$, J_1 and J_2 must be of the same type by the following character of regions, which are invariant under an invertible linear map.

(a) The zero matrix is in $\mathcal{S}_J$ and there exists some nonzero $A \in \mathcal{S}_J$ such that $wA \in \mathcal{S}_J$ for all $w \in \mathbf{U}_1$.

(b) The zero matrix is in $\mathcal{S}_J$ and there does not exist any nonzero $A \in \mathcal{S}_J$ such that $wA \in \mathcal{S}_J$ for all $w \in \mathbf{U}_1$.

(c) The zero matrix is not in $\mathcal{S}_J$ and there exists some nonzero $A \in \mathcal{S}_J$ such that $wA \in \mathcal{S}_J$ for all $w \in \mathbf{U}_1$.

(d) The zero matrix is not in $\mathcal{S}_J$ and there does not exist any nonzero $A \in \mathcal{S}_J$ such that $wA \in \mathcal{S}_J$ for all $w \in \mathbf{U}_1$.

Next, we prove that there is $w \in \mathbf{U}_1$ such that $\phi(\mathbf{H}_n) = w\mathbf{H}_n$ according to the different types of J_1.

For type (a), let $k_i = \sup\{k > 0 : B(0;k) \subseteq J_i\}$ for $i = 1,2$ where $B(a;k)$ is the open ball with center at a and radius k. Since the origin is an interior point and J_i is a proper subset of $\mathbb{C}$, k_i is a positive number for each $i = 1,2$. Replacing (ϕ, J_1, J_2) by $(\frac{k_1}{k_2}\phi, \frac{1}{k_1}J_1, \frac{1}{k_2}J_2)$, we may assume $\phi(\mathcal{S}_{J_1}) = \mathcal{S}_{J_2}$ and $k_1 = k_2 = 1$.

By the definition of J_1, we must have the closed ball $\overline{B}(0;k) \subseteq J_1$ for all $k < 1$; otherwise there is a $k < k' < 1$ such that $B(0;k') \nsubseteq J_1$.

We shall prove that ϕ is a numerical radius preserver on $\mathbf{M}_n$. For any $A \in \mathbf{M}_n$ and $k \in B(0;1)$, we have $W(\frac{k}{r(A)}A) \subseteq \overline{B}(0;k) \subseteq J_1$. Thus $W(\phi(\frac{k}{r(A)}A)) \subseteq J_2$. We claim that

$W(\phi(\frac{1}{r(A)}A)) \subseteq \overline{B}(0;1)$. Otherwise, there is $z \in W(\phi(\frac{1}{r(A)}A))$ such that $|z| > 1$. Since $kz \in W(\phi(\frac{k}{r(A)}A)) \subseteq J_2$ and k can be any value in $B(0;1)$, it follows that $B(0;|z|) \subseteq J_2$. But this is impossible since $|z| > 1 = k_2$. Hence, we have $W(\phi(\frac{1}{r(A)}A)) \subseteq \overline{B}(0;1)$. It follows that $r(\phi(A)) \leq r(A)$. By considering ϕ^{-1}, we have $r(\phi^{-1}(A)) \leq r(A)$. Hence, ϕ is a numerical radius preserver on $\mathbf{M}_n$. By Theorem 1.2, ϕ has the form $A \mapsto \mu UAU^*$ or $A \mapsto \mu UA^tU^*$ for some $U \in \mathbf{U}_n$. The result follows.

For any subset $C \subseteq \mathbb{C}$ and $k > 0$, let $\mathbf{U}_1(C) = \{w \in \mathbf{U}_1 : wr \in C \text{ for some } r > 0\}$ and $\mathbf{U}_1(C,k) = \{w \in \mathbf{U}_1 : wk \in C\}$. Clearly, $\mathbf{U}_1(C,k) \subseteq \mathbf{U}_1(C) \subseteq \mathbf{U}_1$. For any $w_1, w_2 \in \mathbf{U}_1$, let $[w_1 : w_2]$ be the arc joining w_1 and w_2 in the unit circle in the anticlockwise direction. Also, let $d(w_1 : w_2)$ be the length of the arc, i.e.

$$d(w_1 : w_2) = \begin{cases} \arg(w_1) - \arg(w_2) & \text{if } \arg(w_1) \geq \arg(w_2), \\ 2\pi + \arg(w_1) - \arg(w_2) & \text{if } \arg(w_1) < \arg(w_2). \end{cases}$$

For type (b), let $P \in \mathbf{P}_n$. Suppose $\phi(P) \notin w\mathbf{H}_n$ for any $w \in \mathbf{U}_1$. Then $\mathbf{U}_1(W(\phi(P)))$ must contain some nondegenerate arc, say $[w_1 : w_2]$. Suppose $w_1r_1, w_2r_2 \in W(\phi(P))$ for some $r_1, r_2 > 0$. Note that there exists $w' \in \mathbf{U}_1$ and $\epsilon > 0$ such that $W(w'\epsilon P) \subseteq J_1$. Thus, $W(w'\epsilon\phi(P)) \subseteq J_2$. By the definition of J_2, we have $[w'w_1 : w'w_2] \subseteq \mathbf{U}_1(J_2, k)$, where $k = \epsilon \min\{r_1, r_2\}$. Let $w_0 \in \mathbf{U}_1(\phi^{-1}(I_n))$. Then $w_0r_0 \in W(\phi^{-1}(I_n))$ for some $r_0 > 0$. Since $W(wk'I_n) \subseteq J_2$ for all $w \in [w'w_1 : w'w_2]$, we have $W(wk'\phi^{-1}(I_n)) \subseteq J_1$. Hence, $[w'w_1w_0 : w'w_2w_0] \subseteq \mathbf{U}_1(J_1, k_1)$, where $k_1 = kr_0$. So, $\mathbf{U}_1(J_1, k_1)$ contains a nondegenerate arc. We now show that it is impossible.

For simplicity, let

$$[w'w_1w_0 : w'w_2w_0] = [\mu_1 : \nu_1], \text{ and } d(\mu_1 : \nu_1) = d_1.$$

Since $W\left(\frac{wk_1}{r(P)}P\right) \subseteq J_1$ for $w \in [\mu_1 : \nu_1]$, we have $W\left(\frac{wk_1}{r(P)}\phi(P)\right) \subseteq J_2$ by Assertion 1. This implies that $[ww_1, ww_2] \subseteq \mathbf{U}_1(J_2, k_1')$, where $k_1' = \frac{k_1}{r(P)}(\min\{r_1, r_2\})$. As w varies in $[\mu_1 : \nu_1]$, we see that $[\mu_1w_1, \nu_1w_2] \subseteq \mathbf{U}_1(J_2, k_1')$. Since $W(wk_1'I_n) \subseteq J_2$ for $w \in [\mu_1w_1, \nu_1w_2]$, we have $W(wk_1'\phi^{-1}(I_n)) \subseteq J_1$, and hence $wk_1'w_0r_0 \in J_1$. It follows that $[\mu_1w_1w_0, \nu_1w_2w_0] \subseteq \mathbf{U}_1(J_1, k_2)$, where $k_2 = k_1'r_0$. If we call $\mu_2 = \mu_1w_1w_0$ and $\nu_2 = \nu_1w_2w_0$, then $d(\mu_2 : \nu_2) = d_1 + d$, where $d = d(w_1, w_2) > 0$. Inductively, we have $[\mu_n : \nu_n] \subseteq \mathbf{U}_1(J_1, k_n)$, and $d(\mu_n : \nu_n) = d_1 + (n-1)d$ for all $n \in \mathbb{N}$ if $d_1 + (n-1)d \leq 2\pi$. Take the largest n such that $d_1 + (n-1)d \leq 2\pi$. By the same argument above, we see that $\mathbf{U}_1(J_1, k_{n+1}) = \mathbf{U}_1(J_2, k_n') = \mathbf{U}_1$. That is, $wk_{n+1} \in J_1$ for all $w \in \mathbf{U}_1$. By the definition of J_1, the open ball $B(0; k_{n+1}) \subseteq J_1$. Hence the origin is an interior point, which is impossible. This contradiction shows that our assumption that $\phi(P) \notin w\mathbf{H}_n$ for any $w \in \mathbf{U}_1$ cannot hold. So, $\phi(P) \in w\mathbf{H}_n$ for some $w \in \mathbf{U}_1$.

Next, we show that $\phi(\mathbf{P}_n) \subseteq w\mathbf{H}_n$ for some $w \in \mathbf{U}_1$. Suppose there is a $P \in \mathbf{P}_n$ such that $\phi(P) \in w_1\mathbf{H}_n$ while $\phi(I_n) \in w_2\mathbf{H}_n$ for $w_1 \neq w_2$. Clearly, $\lambda P + (1-\lambda)I_n \in \mathbf{P}_n$ for all

$\lambda \in [0,1]$. We claim that there exists $x \in \mathbb{C}^n$ with $\|x\| = 1$ such that both $\alpha = x^*\phi(P)x$ and $\beta = x^*\phi(I_n)x$ are nonzero. Otherwise, we can find $x_1, x_2 \in \mathbb{C}^n$ with $\|x_1\| = \|x_2\| = 1$ such that $x_1^*\phi(P)x_1$ and $x_2^*\phi(I_n)x_2$ are nonzero while $x_1^*\phi(I_n)x_1 = x_2^*\phi(P)x_2 = 0$. Then both $x_1^*\phi(P)x_1$ and $x_2^*\phi(I_n)x_2$ lie in $W(\phi(P+I_n))$. But $x_1^*\phi(P)x_1 \in w_1\mathbb{R}$ while $x_2^*\phi(I_n)x_2 \in w_2\mathbb{R}$, which contradicts $\phi(P+I_n) \in w\mathbf{H}_n$ for some $w \in \mathbf{U}_1$.

Let $\mathcal{W} = \bigcup_{\lambda\in[0,1]} W(\lambda P + (1-\lambda)I_n)$ and $\mathcal{W}_\phi = \bigcup_{\lambda\in[0,1]} W(\lambda\phi(P) + (1-\lambda)\phi(I_n))$. Since $\lambda\alpha + (1-\lambda)\beta \in W(\lambda\phi(P) + (1-\lambda)\phi(I_n))$ for all $\lambda \in [0,1]$, we conclude that $[\alpha,\beta] \subseteq \mathcal{W}_\phi$. As $\frac{\alpha}{|\alpha|} = w_1 \neq w_2 = \frac{\beta}{|\beta|}$, $\mathbf{U}_1(\mathcal{W}_\phi, l)$ contains a nondegenerate arc for $l = \min\{|\alpha|, |\beta|\}$.

Clearly, $\mathcal{W} \subseteq [0,\infty)$. It is easy to see that for any $\mu \in \mathbb{C}$, if $\mu\mathcal{W} \subseteq J_1$, then $\mu\mathcal{W}_\phi \subseteq J_2$. By considering the set $\mathcal{W}$ instead of $W(P)$, we can show that $\mathbf{U}_1(J_1,k)$ does not contain any nondegenerate arc for all $k > 0$. However, by the definition of J_1, there exists $\mu \in \mathbb{C}$ such that $\mu\mathcal{W} \in J_1$. Hence, $\mu\mathcal{W}_\phi \subseteq J_2$. It follows that $\mathbf{U}_1(J_2,k')$ contains some nondegenerate arc for some $k' > 0$, and thus $\mathbf{U}_1(J_1,k)$ contains some nondegenerate arc for some $k > 0$. This is impossible, hence w_1 equals w_2.

Since P is arbitrary in $\mathbf{P}_n$, it follows that $\phi(\mathbf{P}_n) \subseteq w\mathbf{H}_n$ for some $w \in \mathbf{U}_1$. It can be further deduced that $\phi(\mathbf{H}_n) \subseteq w\mathbf{H}_n$. By considering ϕ^{-1}, we conclude that $\phi(\mathbf{H}_n) = w\mathbf{H}_n$.

For type (c), we can easily deduce that

$$\{kA : A \in S_{J_i} \text{ and } k > 0\} = \mathcal{S}_{\mathbb{C}\setminus\{0\}} \qquad i = 1,2.$$

As ϕ is linear and $\phi(\mathcal{S}_{J_1}) = \mathcal{S}_{J_2}$, $\phi(\mathcal{S}_{\mathbb{C}\setminus\{0\}}) = \mathcal{S}_{\mathbb{C}\setminus\{0\}}$. It suffices to assume $J_1 = J_2 = \mathbb{C}\setminus\{0\}$. Then ϕ satisfies $0 \in W(A)$ if and only if $0 \in W(\phi(A))$. Note that $0 \notin W(\phi(I_n))$. Let $H \in \mathbf{H}_n$. Then $0 \in W(H - \lambda I_n)$ if and only if $\lambda \in W(H)$. For any $x \in \mathbb{C}^n$ with $\|x\| = 1$, we have $0 \in W\left(\phi(H) - \frac{x^*\phi(H)x}{x^*\phi(I_n)x}\phi(I_n)\right)$, and thus $0 \in W\left(H - \frac{x^*\phi(H)x}{x^*\phi(I_n)x}I_n\right)$. Hence, we have

$$\frac{x^*\phi(H)x}{x^*\phi(I_n)x} \in W(H) \subseteq \mathbb{R} \quad \text{for every } \|x\| = 1. \tag{1}$$

Since $W(\phi(I_n))$ is convex and $0 \notin W(\phi(I_n))$, we may replace ϕ with some suitable $\mu\phi$ and assume that $W(\phi(I_n))$ is on the upper half plane and $\mathbf{U}_1(W(\phi(I_n))) = [1 : \nu]$ for some $\nu \in \mathbf{U}_1$ with $0 \le \arg(\nu) < \pi$. As a result, if $x^*\phi(H)x \neq 0$, then either

$$\frac{x^*\phi(H)x}{|x^*\phi(H)x|} = \frac{x^*\phi(I_n)x}{|x^*\phi(I_n)x|} \in \mathbf{U}_1(\phi(I_n)) \quad \text{or} \quad -\frac{x^*\phi(H)x}{|x^*\phi(H)x|} = \frac{x^*\phi(I_n)x}{|x^*\phi(I_n)x|} \in \mathbf{U}_1(\phi(I_n)).$$

Hence, $\mathbf{U}_1(\phi(H)) \subseteq [1:\nu] \cup [-1:-\nu]$. We see that $W(\phi(H))$ must lie in $\bigcup_{w\in[1:\nu]\cup[-1:\nu]} w\mathbb{R}$.

Now suppose $H \in \mathbf{H}_n$ is such that $W(H) = [\alpha,\beta]$ for $\alpha < 0 < \beta$, we shall show that $W(\phi(H)) \subseteq w\mathbb{R}$ for some $w \in \mathbf{U}_1$. As $0 \in W(H - \lambda I_n)$ for $\lambda = \alpha,\beta$, there exist x_1, $x_2 \in \mathbb{C}^n$ with $\|x_1\| = \|x_2\| = 1$ such that

$$x_1^*\phi(H)x_1 = \alpha x_1^*\phi(I_n)x_1 \quad \text{and} \quad x_2^*\phi(H)x_2 = \beta x_2^*\phi(I_n)x_2.$$

Then, we have $\frac{x_1^*\phi(H)x_1}{|x_1^*\phi(H)x_1|} \in [-1:-\nu]$ and $\frac{x_2^*\phi(H)x_2}{|x_2^*\phi(H)x_2|} \in [1:\nu]$. By the convexity of the numerical range, $W(\phi(H))$ can only be a line segment passing through the origin, say, $W(\phi(H)) \subseteq w\mathbb{R}$ for some $w \in \mathbf{U}_1$.

Next, we claim that $W(\phi(I_n)) \subseteq (0,\infty)$. Suppose $\nu \neq 1$. Then there exist x_1, x_2 such that

$$\frac{x_1^*\phi(I_n)x_1}{|x_1^*\phi(I_n)x_1|} = 1 \quad \text{and} \quad \frac{x_2^*\phi(I_n)x_2}{|x_2^*\phi(I_n)x_2|} = \nu.$$

We may assume that $x_1^*\phi(H)x_1, x_2^*\phi(H)x_2 \in W(\phi(H))$ are nonzero. Otherwise, replacing H by $H + \epsilon I_n$ for some small ϵ, and using (1), we have both $\frac{x_1^*\phi(H)x_1}{x_1^*\phi(I_n)x_1}$ and $\frac{x_2^*\phi(H)x_2}{x_2^*\phi(I_n)x_2}$ lie in $\mathbb{R}$. Hence, $x_1^*\phi(I_n)x_1, x_2^*\phi(I_n)x_2 \in w\mathbb{R}$ for some $w \in \mathbf{U}_1$ as $W(\phi(H)) \subseteq w\mathbb{R}$. But this contradicts $\nu \neq 1$. Therefore, $\mathbf{U}_1(W(\phi(I_n))) = \{1\}$, and $W(\phi(I_n)) \subseteq (0,\infty)$.

Take an arbitrary $P \in \mathbf{P}_n^+$. From (1), we have

$$\frac{x^*\phi(P)x}{x^*\phi(I_n)x} \in W(P) \subseteq (0,\infty) \quad \text{for every } \|x\| = 1.$$

Then $W(\phi(P)) \subseteq (0,\infty)$ since $W(\phi(I_n))$ does. This means $\phi(\mathbf{P}_n^+) \subseteq \mathbf{P}_n^+$. Since ϕ is invertible, and $\phi^{-1}(\mathcal{S}_{\mathbb{C}\setminus\{0\}}) = \mathcal{S}_{\mathbb{C}\setminus\{0\}}$, we have $\phi^{-1}(\mathbf{P}_n^+) \subseteq \mathbf{P}_n^+$. Hence, $\phi(\mathbf{P}_n^+) = \mathbf{P}_n^+$. By Theorem 1.1, the result follows.

The proof of type (d) is similar to that of case (b); one just have to replace $\mathbf{P}_n$ by $\mathbf{P}_n^+$ in the proof.

Assertion 3 *If $\tilde{J}_1$ contains some nonzero elements while J_1 does not, then there exists $w \in \mathbf{U}_1$ such that $\phi(\mathbf{H}_n) = w\mathbf{H}_n$.*

Proof. We may assume that $0 \notin \tilde{J}_1$. Otherwise, because of Lemma 2.4, either $\{0\}$ is a connected singleton component which we may ignored, or there exists $w(0,\infty) \subseteq \tilde{J}_1$ for some $w \in \mathbf{U}_1$ which means J_1 contains nonzero elements. It follows that $0 \notin \tilde{J}_2$ as ϕ is invertible, and has kernel $\{0\}$.

To prove that there is $w \in \mathbf{U}_1$ such that $\phi(\mathbf{H}_n) = w\mathbf{H}_n$, we consider the following two types of sets $\tilde{J}$ in $\mathbb{C}$.

(a) There is $r > 0$ such that $z \in \tilde{J}$ for all $|z| > r$.

(b) There is no positive real number r satisfying condition (a).

Note that $\tilde{J}$ satisfies (a) if and only if there exists some $A \in \mathcal{S}_{\tilde{J}}$ such that $wA \in \mathcal{S}_{\tilde{J}}$ for all $w \in \mathbf{U}_1$. Thus, $\tilde{J}_1$ satisfies (a) if and only if $\tilde{J}_2 = \phi(\tilde{J}_1)$ does. So, $\tilde{J}_1$ and $\tilde{J}_2$ must be of the same type.

If (a) holds, then

$$\{kA : W(A) \subseteq \tilde{J} \text{ and } k > 0\} = \{A : W(A) \subseteq \mathbb{C}\backslash\{0\}\} = \mathcal{S}_{\mathbb{C}\backslash\{0\}}.$$

Since ϕ is linear, $\phi(\mathcal{S}_{\mathbb{C}\backslash\{0\}}) = \mathcal{S}_{\mathbb{C}\backslash\{0\}}$. The proof is already done in type (c) of Assertion 2.

For situation (b), the proof is similar to type (b) of Assertion 2.

Assertion 4 *If both J_1 and $\tilde{J}_1$ do not contain any nonzero elements, then $\phi(I_n) = \mu I_n$ for some $\mu \in \mathbb{C}$ such that $\mu R_1 = R_2$.*

Proof. Suppose $\phi(I_n)$ is not a scalar matrix. Then $\phi^{-1}(I_n)$ is neither a scalar matrix. There exist nondegenerate line segments $[\alpha_1, \beta_1] \subseteq W(\phi(I_n))$ and $[\alpha_2, \beta_2] \subseteq W(\phi^{-1}(I_n))$.

By lemma 2.2, we may assume that $W(I_n) \subseteq R_1$. Then $[\alpha_1, \beta_1] \subseteq W(\phi(I_n)) \subseteq R_2$.

For every $\gamma \in [\alpha_1, \beta_1]$, $W(\gamma I_n) \subseteq R_2$ and hence $[\gamma\alpha_2, \gamma\beta_2] \subseteq W(\gamma\phi^{-1}(I_n)) \subseteq R_1$. As γ varies in $[\alpha_2, \beta_2]$, the set

$$\{\gamma_1\gamma_2 : \gamma_1 \in [\alpha_1, \beta_1] \text{ and } \gamma_2 \in [\alpha_2, \beta_2]\} = \operatorname{conv}\{\alpha_1\alpha_2, \alpha_1\beta_2, \beta_1\alpha_2, \beta_1\beta_2\}$$

lies in R_1. It follows that $[\alpha_1\alpha_2, \beta_1\beta_2] \subseteq R_1$.

Similarly, as $W(\gamma I_n) \subseteq R_1$ for all $\gamma \in [\alpha_1\beta_2, \beta_1\beta_2]$, $[\alpha_1^2\alpha_2, \beta_1^2\beta_2] \subseteq R_2$. Inductively, we can show that

$$[(\alpha_1\alpha_2)^n, (\beta_1\beta_2)^n] \subseteq R_1 \quad \text{and} \quad [\alpha_1^{n+1}\alpha_2^n, \beta_1^{n+1}\beta_2^n] \subseteq R_2 \quad \text{for all } n \in \mathbb{N}.$$

We may choose α_i and β_i such that $\arg(\alpha_1\alpha_2)$ and $\arg(\beta_1\beta_2)$ are rational multiples of π. Therefore, there exists $m \in \mathbb{N}$ such that both $m \arg(\alpha_1\alpha_2)$ and $m \arg(\beta_1\beta_2)$ are multiples of 2π. Then, $\alpha = (\alpha_1\alpha_2)^m$ and $\beta = (\beta_1\beta_2)^m$ are real numbers. Hence, $[\alpha^k, \beta^k]$ lies in $R_1 \cap \mathbb{R}$ for any $k \in \mathbb{N}$.

If $0 \leq \alpha < 1$, then there exists $K \in \mathbb{N}$ such that

$$\beta > \left(\frac{\alpha}{\beta}\right)^K > \left(\frac{\alpha}{\beta}\right)^k \quad \text{for all } k \in \mathbb{N} \text{ with } k \geq K. \tag{2}$$

For any $c \in (0, \beta^K]$, there exists $k \geq K$ such that $\alpha^{k+1} \leq c \leq \alpha^k$. With (2), $\alpha^{k+1} \leq c \leq \alpha^k < \beta^{k+1}$. Then $c \in [\alpha^{k+1}, \beta^{k+1}] \subseteq R_1$. Therefore, $(0, \beta^K] \subseteq R_1$. This means that J_1 has some nonzero elements, which is a contradiction.

Similarly, we can prove that $[\alpha^K, \infty) \subseteq R_1$ for some K if $1 \leq \alpha < \beta$, i.e., $\tilde{J}_1$ has some nonzero elements. This contradicts the assumption. Therefore, $\phi(I_n) = \mu I_n$ for some $\mu \in \mathbb{C}$. By Lemma 2.3, we have $\mu R_1 = R_2$.

Assertion 5 *There exists $w \in \mathbf{U}_1$ such that $\phi(\mathbf{H}_n) = w\mathbf{H}_n$.*

Proof. The result is clear if R_1 satisfies Assertion 2 or 3. Otherwise, $\phi(I_n) = \mu I_n$ for some $\mu \in \mathbb{C}$ by Assertion 4. Take any ν in a nondegenerate line segment of R_1. By Lemma 2.3, $\phi(\mathcal{S}_{R_1-\nu}) = \mathcal{S}_{R_2-\mu\nu}$. Then we can replace R_1 and R_2 by $R_1 - \nu$ and $R_2 - \mu\nu$ so that Assertion 2 holds after the replacement, and the result follows.

We are now ready to prove Conditions (2), (3), (5), (6). First, we consider the case when R_1 has no interior point.

Suppose R_1 satisfies condition (2). Then for any $\nu \in \mathbf{U}_1(R_1)$, we have $\pm\nu\mathbf{H}_n \subseteq \mathcal{S}_{R_1}$. Thus, $\pm\nu\phi(\mathbf{H}_n) = \pm\nu w\mathbf{H}_n \subseteq \mathcal{S}_{R_2}$. Hence, $wR_1 \subseteq R_2$. Applying the argument to ϕ^{-1}, we see that $w^{-1}R_2 \subseteq R_1$. Hence, $wR_1 = R_2$ and $\phi(\mathbf{H}_n) = w\mathbf{H}_n$, i.e., condition (2) holds with $\mu = w$.

Suppose R_1 does not satisfy (1) and (2). Then there exists $\nu \in \mathbf{U}_1$ such that $K_1 = \mathbb{R} \cap \nu R_1$ contains no line segment and $K_1 \setminus \{0\}$ is non-empty. Then for $K_2 = \mathbb{R} \cap w^{-1}\nu R_2$, we have $\phi(\mathcal{S}_{K_1}) = \mathcal{S}_{wK_2}$. So, the mapping Φ defined by $A \mapsto w^{-1}\phi(A)$ satisfies $\Phi(\mathbf{H}_n) = \mathbf{H}_n$ and $\Phi(\mathcal{S}_{K_1}) = \mathcal{S}_{K_2}$. By Theorem 3.1 (1), we see that $\Phi(I_n) = aI_n$ for some $a \in \mathbb{R}$. It follows that $\phi(I_n) = awI_n$. Let $\mu = aw$. Then $\mu R_1 = R_2$ by Lemma 2.3, and $\phi(\mathbf{H}_n) = w\mathbf{H}_n = \mu\mathbf{H}_n$.

If a nondegenerate line segment lying in R_1 always implies that the entire line containing such line segment also lies in R_1, then condition (3) holds. Otherwise, R_1 will contain a nondegenerate line segment such that the line containing such line segment is not a subset of R_1. We take any point ν from such a nondegenerate line segment. By Lemma 2.3, $\phi(\mathcal{S}_{R_1-\nu}) = \mathcal{S}_{R_2-\mu\nu}$, where $R_1 - \nu$ satisfies Assertion 2. Therefore, we can replace R_1 and R_2 by $R_1 - \nu$ and $R_2 - \mu\nu$. Furthermore, after the replacement, there is $\eta \in \mathbf{U}_1$ such that $L_1 = \mathbb{R} \cap \eta R_1$ does not satisfy Conditions (1) – (3) in Theorem 3.1. Since $\Phi(\mathcal{S}_{L_1}) = w^{-1}\phi(\mathcal{S}_{L_1}) = \mathcal{S}_{L_2}$, where $L_2 = \mathbb{R} \cap w^{-1}\eta R_2$, we see that Φ satisfies Theorem 3.1 (4), and hence ϕ satisfies condition (6).

Now, assume that R_1 contains some interior points. Suppose R_1 satisfies Condition (5). Note that if $\eta \in \mathbf{U}_1(R_1)$, then $\eta(0,\infty) \subseteq R_1$. Since $\phi(\mathbf{H}_n) = w\mathbf{H}_n$, it follows that $w\eta(0,\infty) \subseteq R_2$. Thus $wR_1 \subseteq R_2$. Applying the argument to ϕ^{-1}, we see that $w^{-1}R_2 \subseteq R_1$. Thus $wR_1 = R_2$.

If there exists $\nu \in \mathbf{U}_1$ such that $\mathbb{R} \cap \nu R_1 = (0,\infty)$, $[0,\infty)$ or $\mathbb{R} \setminus \{0\}$, then for $K_2 = \mathbb{R} \cap w^{-1}\nu R_2$, we have $\phi(\mathcal{S}_{K_1}) = \mathcal{S}_{wK_2}$. So, the mapping Φ defined by $A \mapsto w^{-1}\phi(A)$ satisfies $\Phi(\mathcal{S}_{K_1}) = \mathcal{S}_{K_1}$. By Theorem 3.1 (3), the result follows.

For the remaining cases in Condition (5), suppose $R_1 \neq \mathbb{C}$ is a union of $w\mathbb{R}$ with $w \in \mathbf{U}_1$ and has interior points. Since $\phi(\mathbf{H}_n) = w\mathbf{H}_n$, we see that $\nu \in \mathbf{U}_1(R_1)$ if and only if $w\nu \in \mathbf{U}_1(R_2)$. Thus, $wR_1 = R_2$. Let Φ be the map $A \mapsto w^{-1}\phi(A)$. Then $\Phi(\mathbf{H}_n) = \mathbf{H}_n$ and $\Phi(R_1) = R_1$. Since R_1 has interior points, there exists a nondegenerate arc in $\mathbf{U}_1(R_1)$, say $[w_1 : w_2] \subseteq \mathbf{U}_1(R_1)$.

For any $P \in \mathbf{P}_n$, there exists a sufficiently large $k > 0$ such that $W(w_1(iP + kI_n)) \subseteq R_1$. Then, $W(w_1\Phi(iP + kI_n)) \subseteq R_1$. Suppose the Hermitian matrix $\Phi(P)$ is indefinite, i.e., $W(\Phi(P)) = [a,b]$, for $a < 0 < b$. We have $\mathbf{U}_1(W(\Phi(iP + kI_n))) = [\nu_1 : \nu_2]$, where ν_1 and ν_2 lie on the lower and upper half plane respectively. In fact, $d(\nu_1 : \nu_2) > d(\nu_1 : 1)$. One can deduce that $[w_1\nu_1 : w_1\nu_2] \subseteq \mathbf{U}_1(R_1)$. We can further deduce that $[w_1\nu_1 : w_2] \subseteq [w_1\nu_1 : w_1\nu_2] \cup [w_1 : w_2] \subseteq \mathbf{U}_1(R_1)$. Hence,

$$d(w_1\nu_1 : w_2) = d(w_1\nu_1 : w_1) + d(w_1 : w_2) = d(w_1 : w_2) + d(\nu_1 : 1)$$

if $d(w_1 : w_2) + d(\nu_1 : 1) \leq 2\pi$. Inductively, we can show that $[w_1\nu_1^n : w_2] \subseteq R_1$, and that $d(w_1\nu_1^n : w_2) = d(w_1 : w_2) + nd(\nu_1 : 1)$ for all $n \in \mathbb{N}$ if $d(w_1 : w_2) + nd(\nu_1 : 1) \leq 2\pi$. Take the largest n satisfying this inequality, and apply the argument one more time. We deduce that $\mathbf{U}_1 \subseteq \mathbf{U}_1(R_1)$. But as R_1 is the union of $w\mathbb{R}$, $R_1 = \mathbb{C}$, which is impossible. Hence, either $a \leq b \leq 0$ or $0 \leq a \leq b$. This means that $\Phi(P)$ lies either in $\mathbf{P}_n$ or in $-\mathbf{P}_n$. Equivalently, $\Phi(\mathbf{P}_n) \subseteq \mathbf{P}_n \cup -\mathbf{P}_n$. It is easy to show that either $\Phi(\mathbf{P}_n) \subseteq \mathbf{P}_n$ or $\Phi(\mathbf{P}_n) \subseteq -\mathbf{P}_n$. By considering Φ^{-1} and replacing Φ by $-\Phi$ if necessary, we have $\Phi(\mathbf{P}_n) = \mathbf{P}_n$. The result follows from Theorem 1.1.

Finally, suppose R_1 has interior points, but (4)–(5) do not hold. Then there is $\eta \in \mathbf{U}_1$ such that $L_1 = \mathbb{R} \cap \eta R_1$ does not satisfy Conditions (1) – (3) in Theorem 3.1. Since $\Phi(\mathcal{S}_{L_1}) = w^{-1}\phi(\mathcal{S}_{L_1}) = \mathcal{S}_{L_2}$, where $L_2 = \mathbb{R} \cap w^{-1}\eta R_2$, we see that Φ satisfies Theorem 3.1 (4), and thus ϕ satisfies condition (6). $\square$

5 Results on Numerical Radius

Let $\mathbf{V}_n = \mathbf{H}_n$ or $\mathbf{M}_n$, and let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ according to $\mathbf{V}_n = \mathbf{H}_n$ or $\mathbf{M}_n$. For any subset R of $[0, \infty)$, let $\tilde{\mathcal{S}}$ be the set of $n \times n$ matrices on $\mathbf{V}_n$ such that $r(A) \in R$. In this section, we characterize linear maps ϕ on $\mathbf{V}_n$ satisfying $\phi(\tilde{\mathcal{S}}_{R_1}) = \tilde{\mathcal{S}}_{R_2}$ for two given subsets $R_1, R_2 \subseteq [0, \infty)$. Again, to avoid trivial consideration, we assume that R_1 and R_2 are non-empty. Furthermore, we exclude the cases that R_1 or R_2 equal to the set $\{0\}$ in our consideration.

Theorem 5.1 *Let R_1, R_2 be non-empty subsets of $[0, \infty)$ such that $R_j \neq \{0\}$ for $j = 1, 2$. Let $\mathbf{V}_n = \mathbf{H}_n$ or $\mathbf{M}_n$, and let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ according to $\mathbf{V}_n = \mathbf{H}_n$ or $\mathbf{M}_n$. Suppose $\phi : \mathbf{V}_n \to \mathbf{V}_n$ is an $\mathbb{F}$-linear operator satisfying $\phi(\tilde{\mathcal{S}}_{R_1}) = \tilde{\mathcal{S}}_{R_2}$. Then one of the following conditions holds.*

1. *$R_1 = R_2 = (0, \infty)$ or $R_1 = R_2 = [0, \infty)$, and ϕ is invertible.*
2. *The set R_1 is neither $(0, \infty)$ nor $[0, \infty)$, and ϕ has the form $A \mapsto \mu U^*AU$ or $A \mapsto \mu U^*A^tU$ for some $U \in \mathbf{U}_n$ and $\mu \in \mathbb{F}$ such that $|\mu|R_1 = R_2$.*

Proof. The ($\Leftarrow$) part of the result can be verified readily. We establish two assertions to prove the converse.

Assertion 1 *The set $\tilde{\mathcal{S}}_{R_2}$ is a spanning set of $\mathbf{V}_n$, and ϕ is invertible.*

Proof. Take a nonzero $k \in R_2$, then $\{kU^*E_{11}U : U \in \mathbf{U}_n\} \subseteq \tilde{\mathcal{S}}_{R_2}$ is a spanning set of $\mathbf{V}_n$ by the main result in [9]. Since $\phi(\tilde{\mathcal{S}}_{R_1}) = \tilde{\mathcal{S}}_{R_2}$ contains a spanning set, we conclude that ϕ is invertible.

Assertion 2 *If C_1 is a connected component of R_1, then $\phi(\tilde{\mathcal{S}}_{C_1}) = \tilde{\mathcal{S}}_{C_2}$, for a connected component C_2 of R_2.*

Proof. Suppose $a \in W(A)$ and $b \in W(B)$ such that $|a| = r(A)$ and $|b| = r(B)$ belong to C_1. If $r(\phi(A)) = c$ and $r(\phi(B)) = d$, we shall show that $[c, d] \subseteq R_2$.

First, we may assume that $a = |a|$; otherwise, replace A by μA for a suitable $\mu \in \mathbb{F}$ with $|\mu| = 1$. Similarly, we may assume that $b = |b|$. There is a unitary U such that $A = U^*(D + A_0)U$, for $D = \text{diag}(a, 0, \ldots, 0)$ and the $(1, 1)$ entry of A_0 is zero. Then $\gamma_1(t) = U^*(D + (1 - t)A_0)U$, $t \in [0, 1]$, is a path in $\tilde{\mathcal{S}}_{C_1}$ connecting A and U^*DU.

Let $U = e^{iH}$ where H is Hermitian. Then the path $\gamma_2(t) = e^{-itH}De^{itH}$, $t \in [0, 1]$, is a path in $\tilde{\mathcal{S}}_{C_1}$ connecting U^*DU and D. Similarly, one can construct a path in $\tilde{\mathcal{S}}_{C_1}$ connecting B and $\text{diag}(b, 0, \ldots, 0)$. Finally, one can construct a path in $\tilde{\mathcal{S}}_{C_1}$ connecting $\text{diag}(a, 0, \ldots, 0)$ and $\text{diag}(b, 0, \ldots, 0)$. So, we have a path in $\tilde{\mathcal{S}}_{C_1}$ connecting A and B. It follows that there is a path in $\tilde{\mathcal{S}}_{R_2}$ connecting $\phi(A)$ and $\phi(B)$. So, $\phi(A)$ and $\phi(B)$ belong to the $\tilde{\mathcal{S}}_{C_2}$ for a connected component C_2 of R_2. Since ϕ is invertible by Assertion 1, we have $\phi^{-1}(\tilde{\mathcal{S}}_{C_2}) \subseteq \tilde{\mathcal{S}}_{C_1}$, and hence $\phi^{-1}(\tilde{\mathcal{S}}_{C_2}) = \tilde{\mathcal{S}}_{C_1}$.

Now, we are ready to present the proof of Conditions (1) and (2). By Assertion 1, ϕ is invertible. If R_1 equals $(0, \infty)$ or $[0, \infty)$, then nothing else can be said about ϕ. Suppose R_1 does not satisfy Condition (1). By Assertion 2, we may assume that R_1 and R_2 are connected intervals. For any nonzero $A \in \mathbf{V}_n$, $r(A)$ and $r(\phi(A))$ are nonzero as ϕ is invertible by Assertion 1. Let $k_A = \frac{r(\phi(A))}{r(A)}$. Then

$$a \in R_1 \Leftrightarrow r\left(\frac{a}{r(A)}A\right) \in R_1 \Leftrightarrow r\left(\phi\left(\frac{a}{r(A)}A\right)\right) \in R_2 \Leftrightarrow k_A a \in R_2.$$

Hence, $k_A R_1 = R_2$. Since R_1 is neither $(0, \infty)$ nor $[0, \infty)$, we have $\sup R_1$ exists or $\inf R_1$ is nonzero. In both cases, we can deduce that k_A is a constant, say k, for all nonzero $A \in \mathbf{V}_n$.

Let Φ be the map $A \mapsto k^{-1}\phi(A)$ on $\mathbf{V}_n$. Then

$$\frac{r(\Phi(A))}{r(A)} = \frac{k^{-1}r(\phi(A))}{r(A)} = 1 \quad \text{for all } A \in \mathbf{V}_n \setminus \{0\}$$

Hence, Φ is a numerical radius preserver on $\mathbf{V}_n$. By Theorem 1.2, the result follows. □

References

[1] T. Ando, Structure of operators with numerical radius one, *Acta Sci. Math.* (*Szeged*) **34** (1973), 11-15.

[2] W.B. Arveson, Subalgebras of C^*-algebras II, *Acta Math.* **128** (1972), 271-308.

[3] M.D. Choi and C.K. Li, Numerical ranges and dilations, *Linear and Multilinear Algebra* **47** (2000), 35-48.

[4] R.A. Horn and C.R. Johnson, *Topics in Matrix Analysis*, Cambridge University Press, 1991.

[5] C.K. Li, Linear operators preserving the numerical radius of matrices, *Proc. of Amer. Math. Soc.* **99** (1987), 601-608.

[6] B.A. Mirman, Numerical range and norm of a linear operator, *Voronež. Gos. Univ. Trudy Sem. Funkcional Anal.*, **10** (1968), 51-55.

[7] Pierce et. al., A survey of linear preserver problems, *Linear and Multilinear Algebra* **33** (1992), 1-130.

[8] H. Schneider, Positive Operators and an Inertia Theorem, *Numerische Mathematik* **7** (1965), 11-17.

[9] B.S. Tam, A Simple Proof of the Goldberg-Straus Theorem on Numerical Radii, *Glasgow Math. J.* **28** (1986), 139-141.