Linear maps transforming the higher numerical ranges

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Abstract

Let $k \in \{1, \ldots, n\}$. The k-numerical range of $A \in M_n$ is the set

$$W_k(A) = \{(\operatorname{tr} X^*AX)/k : X \text{ is } n \times k,\ X^*X = I_k\},$$

and the k-numerical radius of A is the quantity

$$w_k(A) = \max\{|z| : z \in W_k(A)\}.$$

Suppose $k > 1$, $k' \in \{1, \ldots, n'\}$ and $n' < C(n,k) \min\{k', n' - k'\}$. It is shown that there is a linear map $\phi : M_n \to M_{n'}$ satisfying $W_{k'}(\phi(A)) = W_k(A)$ for all $A \in M_n$ if and only if $n'/n = k'/k$ or $n'/n = k'/(n-k)$ is a positive integer. Moreover, if such a linear map ϕ exists, then there is a unitary matrix $U \in M_{n'}$ and nonnegative integers p, q with $p + q = n'/n$ such that ϕ has the form

$$A \mapsto U^*[\underbrace{A \oplus \cdots \oplus A}_{p} \oplus \underbrace{A^t \oplus \cdots \oplus A^t}_{q}]U$$

or

$$A \mapsto U^*[\underbrace{\psi(A) \oplus \cdots \oplus \psi(A)}_{p} \oplus \underbrace{\psi(A)^t \oplus \cdots \oplus \psi(A)^t}_{q}]U,$$

where $\psi : M_n \to M_n$ has the form $A \mapsto [(\operatorname{tr} A)I_n - (n-k)A]/k$. Linear maps $\widetilde{\phi} : M_n \to M_{n'}$ satisfying $w_{k'}(\widetilde{\phi}(A)) = w_k(A)$ for all $A \in M_n$ are also studied. Furthermore, results are extended to triangular matrices.

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1 Introduction

There has been a great deal of interest in studying linear operator $\phi : \mathcal{M} \to \mathcal{M}$, where $\mathcal{M}$ is a matrix algebra or space, with a certain special property such as:

(a) $f(\phi(A)) = f(A)$ for all $A \in \mathcal{M}$, where f is a given function on $\mathcal{M}$;

(b) $\phi(\mathcal{S}) \subseteq \mathcal{S}$ or $\phi(\mathcal{S}) = \mathcal{S}$ for a certain subset $\mathcal{S} \subseteq \mathcal{M}$;

(c) $\phi(A) \sim \phi(B)$ in $\mathcal{M}$ whenever $A \sim B$ in $\mathcal{M}$ for a certain relation $\sim$ on $\mathcal{M}$.

Very often, ϕ has nice forms such as

$$A \mapsto MAN \qquad \text{or} \qquad A \mapsto MA^tN$$

for some suitable $M, N \in \mathcal{M}$. One may see [19] for a survey on the subject. Recently, there has been research on more general problems concerning linear transformations $\phi : \mathcal{M} \to \mathcal{M}'$ with some special properties such as

(a) $f'(\phi(A)) = f(A)$ for all $A \in \mathcal{M}$, where f and f' are appropriate functions on $\mathcal{M}$ and $\mathcal{M}'$;

(b) $\phi(\mathcal{S}) \subseteq \mathcal{S}'$ or $\phi(\mathcal{S}) = \mathcal{S}'$ for certain subsets $\mathcal{S} \subseteq \mathcal{M}$ and $\mathcal{S}' \subseteq \mathcal{M}'$;

(c) $\phi(A) \sim' \phi(B)$ in $\mathcal{M}'$ whenever $A \sim B$ in $\mathcal{M}$ for certain relations $\sim$ on $\mathcal{M}$ and $\sim'$ on $\mathcal{M}'$.

Such problems are more challenging and their study often lead to the discovery of unexpected results and hidden structures of the matrix algebras $\mathcal{M}$ and $\mathcal{M}'$; see [6, 10]. In this paper, we consider these types of problems. We solve a specific problem and develop some proof techniques that may be useful for future study in this area.

Let us first introduce some notations and definitions. Denote by M_n the algebra of $n \times n$ complex matrices. For $1 \le k \le n$, define (see Halmos [11]) the k-numerical range of $A \in M_n$ as

$$W_k(A) = \{(\operatorname{tr} X^*AX)/k : X \text{ is } n \times k,\ X^*X = I_k\}.$$

Since $W_n(A) = \{\operatorname{tr} A/n\}$, we always assume that $k < n$ to avoid trivial consideration. When $k = 1$, we have the classical numerical range $W_1(A)$, which is useful in studying matrices and operators; see [11]. Researchers have studied linear maps $\phi : M_n \to M_n$ such that

$$W_k(\phi(A)) = W_k(A) \quad \text{for all } A \in M_n. \tag{1.1}$$

By a result of Pellegrini [18], a linear map $\phi : M_n \to M_n$ satisfies (1.1) for $k = 1$ if and only if there is a unitary $U \in M_n$ such that ϕ has the form

(S1) $A \mapsto U^*AU$ or $A \mapsto U^*A^tU$.

Pierce and Watkins [20] extended the result of Pellegrini to other values of k as long as $k \neq n/2$, and raised the open problem for the case $k = n/2$. In [12] (see also [17]), it was shown that for $k = n/2$, a linear map $\phi : M_n \to M_n$ satisfies (1.1) if and only if there is a unitary $U \in M_n$ such that ϕ has the form (S1), or

(S2) $n = 2k$ and

$$A \mapsto (\operatorname{tr} A/k)I_n - U^*AU \quad \text{or} \quad A \mapsto (\operatorname{tr} A/k)I_n - U^*A^tU. \tag{1.2}$$

In fact, for any $k \in \{1, \dots, n-1\}$, a mapping ϕ of the form (1.2) satisfies

$$(n-k)W_{n-k}(A) = kW_k(\phi(A)) \quad \text{for all } A \in M_n.$$

In [6] the authors studied linear maps $\phi : M_n \to M_{n'}$ such that (1.1) holds with $k = 1$. It was shown that for $n' \le 2n - 2$, a linear map $\phi : M_n \to M_{n'}$ satisfies (1.1) if and only if $n' \ge n$, there exist a unitary $U \in M_{n'}$ and a unital positive linear map $f : M_n \to M_{n'-n}$ such that ϕ has the form

$$A \mapsto U^*[A \oplus f(A)]U \quad \text{or} \quad A \mapsto U^*[A^t \oplus f(A)]U.$$

However, for $n' > 2n-2$, there are other linear maps $\phi : M_n \to M_{n'}$ satisfying (1.1) with complicated structure. The complete characterization of $\phi : M_n \to M_{n'}$ satisfying (1.1) is unknown.

The purpose of this paper is to study those linear operators $\phi : M_n \to M_{n'}$ satisfying

$$W_{k'}(\phi(A)) = W_k(A) \quad \text{for all } A \in M_n .$$

By modifying the map in [6], we can easily get a map $\phi : M_n \to M_{n'}$ with complicated structure satisfying

$$W_{k'}(\phi(A)) = W_1(A) \quad \text{for all } A \in M_n .$$

Therefore, we will only study the case when $k > 1$. It turns out that we also need to impose some conditions on n and n' to avoid the pathetic situation described below.

Let $k \in \{1, \dots, n\}$, and (α, β) be a pair of length k increasing subsequences of $\{1, \dots, n\}$. Denote by $A[\alpha, \beta]$ the submatrix of $A \in M_n$ lying in rows and columns indexed by α and β, respectively. Then the kth compound matrix of A is the $C(n,k) \times C(n,k)$ matrix $C_k(A)$ whose entries equal $\det A[\alpha, \beta]$ arranged in lexicographic order of α and β. The kth additive compound is defined by

$$\Delta_k(A) = \frac{\mathrm{d}}{\mathrm{dt}} C_k(I + tA)|_{t=0}.$$

It is known (e.g., see [16]) that the mapping

$$A \mapsto \Delta_k(A)$$

from M_n to $M_{C(n,k)}$ is linear and satisfies

$$W_k(A) = W_1(\Delta_k(A)) \qquad \text{for any } A \in M_n.$$

So, if $n' \geq 2C(n,k) - 1$ then there is a linear map $\psi : M_{C(n,k)} \to M_{n'}$ satisfying $W_1(\psi(X)) = W_1(X)$ for all $X \in M_n$ without nice structure. Thus, the linear map $\phi : M_n \to M_{n'}$ defined by

$$A \mapsto \psi(\Delta_k(A))$$

satisfies $W_k(A) = W_1(\phi(A))$ for all $A \in M_n$ and does not have nice structure. For larger n' one can extend the above idea to construct ϕ of the form

$$A \mapsto \psi_1(\Delta_k(A)) \oplus \cdots \oplus \psi_{k'}(\Delta_k(A))$$

satisfying $W_k(A) = W_{k'}(\phi(A))$ for all $A \in M_n$ without nice structure.

By the above discussion, we see that it is reasonable to impose appropriate assumption on n, n', k, k' to obtain nice characterizations of linear map $\phi : M_n \to M_{n'}$ satisfying $W_k(A) = W_{k'}(\phi(A))$ for all $A \in M_n$. This is done in Section 2. In fact, we show that the same result is valid for real linear map $\phi : H_n \to H_{n'}$, where H_m denotes the real linear space of all $m \times m$ complex Hermitian matrices. In Section 3, we extend the result to triangular matrices. Define the k-numerical radius of $A \in M_n$ by

$$w_k(A) = \max\{|z| : z \in W_k(A)\}.$$

In Section 4, we study those linear maps $\widetilde{\phi}$ satisfying

$$w_{k'}(\widetilde{\phi}(A)) = w_k(A) \qquad \text{for all } A \in M_n.$$

Some open problems are mentioned in Section 5.

Note that recently, researchers have also considered mappings preserving the classical numerical range and radius on more general operator algebras; see [1, 2, 4, 5, 7, 8, 14].

2 Results on Hermitian and Complex Matrices

The main theorem of this section is the following.

Theorem 2.1 *Let $(\mathcal{M}, \mathcal{M}') = (H_n, H_{n'})$ or $(M_n, M_{n'})$. Suppose $k \in \{2, \dots, n-1\}$, $k' \in \{1, \dots, n'\}$ and $n' < C(n,k) \min\{k', n'-k'\}$. There exists a linear map $\phi : \mathcal{M} \to \mathcal{M}'$ such that*

$$W_{k'}(\phi(A)) = W_k(A) \quad \text{for all } A \in \mathcal{M} \tag{2.1}$$

if and only if there is a unitary $U \in M_{n'}$ and nonnegative integers p, q with $p + q = n'/n$ such that one of the following holds:

(W1) *$n'/n = k'/k$ and ϕ has the form*

$$A \mapsto U^*[\underbrace{A \oplus \cdots \oplus A}_{p} \oplus \underbrace{A^t \oplus \cdots \oplus A^t}_{q}]U.$$

(W2) *$n'/n = k'/(n-k)$ and ϕ has the form*

$$A \mapsto U^*[\underbrace{\psi(A) \oplus \cdots \oplus \psi(A)}_{p} \oplus \underbrace{\psi(A)^t \oplus \cdots \oplus \psi(A)^t}_{q}]U,$$

where $\psi : \mathcal{M} \to \mathcal{M}$ is the mapping $A \mapsto [(\operatorname{tr} A)I_n - (n-k)A]/k$.

Proof of the sufficiency part. Suppose $n'/n = k'/k$. Then any mapping described in (W1) satisfies (2.1). If $n'/n = k'/(n-k)$, then the mapping $\phi : \mathcal{M} \to \mathcal{M}'$ described in (W2) satisfies

$$W_{k'}(\phi(A)) = W_{n-k}(\psi(A)) = W_k(A) \quad \text{for all } A \in \mathcal{M}. \qquad \blacksquare$$

In the following, we consider the converse. Suppose there exists a linear map $\phi : \mathcal{M} \to \mathcal{M}'$ such that $W_{k'}(\phi(A)) = W_k(A)$ for all $A \in \mathcal{M}$. We will show that n'/n is an integer and one of conditions (W1) or (W2) holds by establishing a sequence of lemmas.

Let $X \in H_m$. Denote the eigenvalues of X by

$$\lambda_1(X) \geq \cdots \geq \lambda_m(X).$$

Suppose $r \in \{1, \ldots, m-1\}$. Let

$$S_r(X) = \sum_{j=1}^{r} \lambda_j(X) \quad \text{and} \quad s_r(X) = \sum_{j=1}^{r} \lambda_{m-j+1}(X).$$

We have the following result concerning the r-numerical range; see [11, 12, 20] and their references.

Lemma 2.2 *Suppose $r \in \{1, \ldots, m-1\}$.*

(a) *$W_r(A) = W_r(U^*AU)$ for any unitary $U \in M_m$.*

(b) *$W_r(\alpha A + \beta I_m) = \alpha W_r(A) + \beta$ for any $\alpha, \beta \in \mathbb{C}$.*

(c) *If $A \in H_m$, then*

$$W_r(A) = [s_r(A)/r, S_r(A)/r]\,.$$

(d) *A matrix $B \in M_m$ satisfies $W_r(B) \subseteq \mathbb{R}$ if and only if $B = B^*$.*

(e) *A matrix $C \in M_m$ satisfies $W_r(C) = \{\lambda\}$ if and only if $C = \lambda I_m$.*

Lemma 2.3 *The mapping ϕ satisfies $\phi(H_n) \subseteq H_{n'}$ and $\phi(I_n) = I_{n'}$.*

Proof. If $A \in H_n$, then $W_{k'}(\phi(A)) = W_k(A) \subseteq \mathbb{R}$. By Lemma 2.2 (d), $\phi(A) \in H_{n'}$. Furthermore, since $W_{k'}(\phi(I_n)) = W_k(I_n) = \{1\}$, we have $\phi(I_n) = I_{n'}$ by Lemma 2.2 (e). ∎

By the above lemma, we can focus on proving the result for the Hermitian case. Once it is done, the result on complex matrices will follow from the fact that $\phi(A) = \phi(H) + i\phi(G)$ for any complex matrix $A = H + iG \in M_n$ with $H, G \in H_n$.

A key step in our proof is to show that ϕ or $\phi \circ \psi^{-1}$ will map idempotents in H_n to idempotents in $H_{n'}$. Two idempotents $F, G \in H_m$ are said to be disjoint if $FG = GF = 0_m$.

Lemma 2.4 *Suppose $A, B \in H_m$ and $r \in \{1, \ldots, m-1\}$. The following conditions are equivalent.*

(a) *The sum of the first r diagonal entries of $A + B$ equals $S_r(A) + S_r(B)$.*

(b) *The sum of the last $m - r$ diagonal entries of $A + B$ equals $s_{m-r}(A) + s_{m-r}(B)$.*

(c) *$A = A_1 \oplus A_2$ and $B = B_1 \oplus B_2$ such that A_1 has the r largest eigenvalues of A and B_1 has the r largest eigenvalues of B.*

Proof. Clearly, (a) and (b) are equivalent, and (c) implies (a). To prove (a) implies (c), let d_A be the sum of the first r diagonal entries of A, and d_B be the sum of the first r diagonal entries of B. Then $d_A \leq S_r(A)$ and $d_B \leq S_r(B)$. Now, the sum of the first r diagonal entries of $A + B$ equals $d_A + d_B = S_r(A) + S_r(B)$. So, $d_A = S_r(A)$ and $d_B = S_r(B)$. By [13, Lemma 4.1], $A = A_1 \oplus A_2$ and $B = B_1 \oplus B_2$ with $A_1, B_1 \in M_r$ such that A_1 has the r largest eigenvalues of A and B_1 has the r largest eigenvalues of B. ∎

By Lemma 2.4, one readily deduces the following.

Lemma 2.5 *Suppose $A, B \in H_m$ and $1 < r < m$. The following conditions are equivalent.*

(a) $S_r(A+B) = S_r(A) + S_r(B)$.

(b) $s_{m-r}(A+B) = s_{m-r}(A) + s_{m-r}(B)$.

(c) *If $V \in M_m$ is unitary such that $V^*(A+B)V = \mathrm{diag}\,(c_1, \ldots, c_m)$ with $c_1 \geq \cdots \geq c_m$, then $V^*AV = A_1 \oplus A_2$ and $V^*BV = B_1 \oplus B_2$ with $A_1, B_1 \in M_r$ such that A_1 (respectively, B_1) has the r largest eigenvalues of A (respectively, B).*

Lemma 2.6 *Suppose $k < n/2$ and $k' \leq n'/2$. If $E \in H_n$ is a rank one idempotent, then $\phi(E)$ is positive semidefinite.*

Proof. Suppose $Q \in H_n$ is a rank k idempotent such that $QE = EQ = 0$. Then $Q + E$ is a Hermitian idempotent with trace $k+1$. Since $k < n/2$,

$$W_{k'}(\phi(Q+E)) = W_k(Q+E) = [0,1],$$

$$W_{k'}(\phi(Q)) = W_k(Q) = [0,1], \quad \text{and} \quad W_{k'}(\phi(E)) = W_k(E) = [0, 1/k].$$

Then $S_{k'}(\phi(Q)) = k'$, $s_{k'}(\phi(Q)) = 0$, $S_{k'}(\phi(E)) = k'/k$, and $s_{k'}(\phi(E)) = 0$. So,

$$s_{k'}(\phi(Q) + \phi(E)) = s_{k'}(\phi(Q)) + s_{k'}(\phi(E)).$$

Let $V \in M_{n'}$ be unitary such that $V^*(\phi(Q+E))V = \mathrm{diag}\,(c_1, \ldots, c_{n'})$ with $c_1 \geq \cdots \geq c_{n'}$. By Lemma 2.5, $V^*\phi(Q)V = Y_1 \oplus Y_2$ and $V^*\phi(E)V = Z_1 \oplus Z_2$ with $Y_2, Z_2 \in M_{k'}$ such that Z_2 has the k' smallest eigenvalues of $\phi(E)$. If Z_2 is the zero matrix, then Z_1 and hence, $\phi(E)$ is positive semidefinite as asserted. If Z_2 has a negative eigenvalue, then the largest eigenvalue of Z_2 is positive. Hence, Z_1 is positive definite. Now, suppose $V_1 \in M_{n'-k'}$ such that $V_1^* Y_1 V_1 = \mathrm{diag}\,(b_1, \ldots, b_{n'-k'})$ with $b_1 \geq \ldots \geq b_{n'-k'}$. Then $\sum_{j=1}^{k'} b_j = S_{k'}(\phi(Q)) = k'$. Since Z_1 is positive definite and $k' \leq n'/2$, the sum of the first k' diagonal entries of $V_1^* Z_1 V_1 = a > 0$. Let X be the matrix consisting of the first k' columns of the unitary matrix $V(V_1 \oplus I_{k'})$. Then

$$1 < (k'+a)/k' = \mathrm{tr}\,(X^*\phi(Q+E)X)/k' \in W_{k'}(\phi(Q+E)),$$

which contradicts the fact that $W_{k'}(\phi(Q+E)) = [0,1]$. ∎

Lemma 2.7 *Suppose $k \leq n/2$ and $k' \leq n'/2$. Let $E_1, \ldots, E_n \in H_n$ be rank one idempotents such that $E_1 + \cdots + E_n = I_n$.*

(a) *Suppose $s \in \{1, \ldots, k'-1\}$ such that*

$$\lambda_{n'-s}(\phi(E_1)) > \lambda_{n'-s+1}(\phi(E_1)).$$

Then there is an $n' \times s$ matrix S whose columns are orthonormal eigenvectors of the eigenvalues $\lambda_{n'-s+1}(\phi(E_1)), \ldots, \lambda_{n'}(\phi(E_1))$ such that $S^*\phi(E_j)S = \gamma_j I_s$ with

$$\gamma_1 = 1 - \sum_{j=2}^{n} \gamma_j = \lambda_{n'-s+1}(\phi(E_1)) = \cdots = \lambda_{n'}(\phi(E_1)),$$

and

$$\gamma_j = \lambda_{k'-s+1}(\phi(E_j)) = \cdots = \lambda_{n'-k'+s}(\phi(E_j)), \qquad j = 2, \ldots, n.$$

(b) *Suppose $r \in \{1, \ldots, k'-1\}$ such that*

$$\lambda_r(\phi(E_1)) > \lambda_{r+1}(\phi(E_1)).$$

Then there is an $n' \times r$ matrix R whose columns are orthonormal eigenvectors of the eigenvalues $\lambda_1(\phi(E_1)), \ldots, \lambda_r(\phi(E_1))$ such that $R^*\phi(E_j)R = \tilde{\gamma}_j I_r$ with

$$\tilde{\gamma}_1 = 1 - \sum_{j=2}^{n} \tilde{\gamma}_j = \lambda_1(\phi(E_1)) = \cdots = \lambda_r(\phi(E_1)),$$

and

$$\tilde{\gamma}_j = \lambda_{k'-r+1}(\phi(E_j)) = \cdots = \lambda_{n'-k'+r}(\phi(E_j)), \qquad j = 2, \ldots, n.$$

Proof. Assume that $\phi(E_1)$ has eigenvalues $a_1 \geq \cdots \geq a_{n'}$. Since

$$W_{k'}(\phi(E_1)) = W_k(E_1) = [0, 1/k],$$

$a_{n'} + \cdots + a_{n'-k'+1} = 0$. Let $B \in \{E_2, \ldots, E_n\}$. Then

$$W_{k'}(\phi(B)) = W_k(B) = [0, 1/k] \quad \text{and} \quad W_{k'}(\phi(E_1 + B)) = W_k(E_1 + B) = [0, 2/k].$$

Suppose $\phi(B)$ has eigenvalues $b_1 \geq \cdots \geq b_{n'}$.

(a) Note that $s_{k'}(\phi(E_1) + \phi(B)) = s_{k'}(\phi(E_1)) + s_{k'}(\phi(B))$. By Lemma 2.5, there is a unitary $V \in M_{n'}$ such that $V^*\phi(E_1)V = A_1 \oplus A_2$ and $V^*\phi(B)V = B_1 \oplus B_2$, where A_2 has eigenvalues $a_{n'}, \ldots, a_{n'-k'+1}$ and B_2 has eigenvalues $b_{n'}, \ldots, b_{n'-k'+1}$. Now,

$$W_{k'}(\phi(E_1) - \phi(B)) = W_k(E_1 - B) = [-1/k, 1/k].$$

Then $s_{k'}(\phi(E_1)) + s_{k'}(-\phi(B)) = s_{k'}(\phi(E_1 - B))$. By Lemma 2.5, there is a unitary $\tilde{V} \in M_{n'}$ such that $\tilde{V}^*\phi(E_1)\tilde{V} = \tilde{A}_1 \oplus \tilde{A}_2$ and $\tilde{V}^*\phi(B)\tilde{V} = \tilde{B}_1 \oplus \tilde{B}_2$, where $\tilde{A}_2$ has eigenvalues $a_{n'}, \ldots, a_{n'-k'+1}$ and $\tilde{B}_2$ has eigenvalues $b_1, \ldots, b_{k'}$. Since $a_{n'-s} > a_{n'-s+1}$, we may assume that the last s columns of V and $\tilde{V}$ are the eigenvectors of $\phi(E_1)$ corresponding to the eigenvalues $a_{n'-s+1}, \ldots, a_{n'}$. So, the lower $s \times s$ principal submatrices of B_2 and $\tilde{B}_2$ are the same, say, equal to $X \in H_s$. Suppose X has eigenvalues $d_1 \geq \cdots \geq d_s$. Because B_2 has eigenvalues $b_{n'-k'+1} \geq \cdots \geq b_{n'}$, it follows from the interlacing inequalities (see [9]) that

$$b_{n'-k'+j} \geq d_j, \quad j = 1, \ldots, s. \tag{2.2}$$

Because $\tilde{B}_2$ has eigenvalues $b_1 \geq \cdots \geq b_{k'}$, by the interlacing inequalities again, we have

$$d_j \geq b_{k'-s+j}, \quad j = 1, \ldots, s. \tag{2.3}$$

Since $k' \leq n'/2$, $b_{k'-s+j} \geq b_{n'-k'+j}$ for $1 \leq j \leq s$. By (2.2) and (2.3), we see that

$$b_{k'-s+j} = d_j = b_{n'-k'+j}, \quad j = 1, \ldots, s.$$

Thus,

$$d_1 = \cdots = d_s = b_{k'-s+1} = \cdots = b_{n'-k'+s}. \tag{2.4}$$

Use the last s columns of V to form the matrix S. Then $S^*BS = d_1 I_s$.

By the above arguments, $S^*\phi(E_j)S = \gamma_j I_s$ for $j = 2, \ldots, n$, where $\gamma_j = \lambda_{k'-s+1}(\phi(E_j)) = \lambda_{n'-k'+s}(\phi(E_j))$. By Lemma 2.3,

$$\phi(E_1 + E_2 + \cdots + E_n) = \phi(I_n) = I_{n'}.$$

It follows that

$$S^*\phi(E_1)S = I_s - \sum_{j=2}^{n} \gamma_j I_s \tag{2.5}$$

is a scalar matrix, where $\gamma_1 = 1 - \sum_{j=2}^{n} \gamma_j$. Clearly, $\gamma_1 = a_{n'-s+1} = a_{n'}$.

(b) Note that

$$W_{k'}(\phi(E_1 + B)) = W_k(E_1 + B) = [0, 2/k].$$

Thus, $S_{k'}(\phi(E_1) + \phi(B)) = S_{k'}(\phi(E_1)) + S_{k'}(\phi(B))$. Then there is a unitary $W \in M_{n'}$ such that $W^*(\phi(E_1))W = Y_1 \oplus Y_2$ and $W^*(\phi(B))W = Z_1 \oplus Z_2$ with $Y_1, Z_1 \in M_{k'}$, where Y_1 has eigenvalues $a_1, \ldots, a_{k'}$ and Z_1 has eigenvalues $b_1, \ldots, b_{k'}$. Now,

$$W_{k'}(\phi(E_1 - B)) = W_k(E_1 - B) = [-1/k, 1/k].$$

We see that $S_{k'}(\phi(E_1) + \phi(-B)) = S_{k'}(\phi(E_1)) + S_{k'}(\phi(-B))$. So there exists a unitary $\tilde{W} \in M_{n'}$ such that $\tilde{W}^*\phi(E_1)\tilde{W} = \tilde{Y}_1 \oplus \tilde{Y}_2$ and $\tilde{W}^*\phi(B)\tilde{W} = \tilde{Z}_1 \oplus \tilde{Z}_2$, where $\tilde{Y}_1$ has eigenvalues $a_1, \ldots, a_{k'}$ and $\tilde{Z}_1$ has eigenvalues $b_{n'}, \ldots, b_{n'-k'+1}$. Since $a_r > a_{r+1}$, we may assume that the first r columns of W and $\tilde{W}$ are the eigenvectors of $\phi(E_1)$ corresponding to the eigenvalues $a_1, \ldots, a_r$. So, the leading $r \times r$ submatrices of Z_1 and $\tilde{Z}_1$ are the same, say, equal to $T \in H_r$. Suppose T has eigenvalues $t_1 \geq \cdots \geq t_r$. Since Z_1 has eigenvalues $b_1 \geq \cdots \geq b_{k'}$, by the interlacing inequalities

$$t_j \geq b_{k'-r+j} \quad j = 1, \ldots, r. \tag{2.6}$$

Since $\tilde{Z}_1$ has eigenvalues $b_{n'-k'+1} \geq \cdots \geq b_{n'}$, by the interlacing inequalities

$$b_{n'-k'+j} \geq t_j, \quad j = 1, \ldots, r. \tag{2.7}$$

Since $k' \leq n'/2$, $b_{k'-r+j} \geq b_{n'-k'+j}$ for $1 \leq j \leq r$. By (2.6) and (2.7), we see that

$$b_{k'-r+j} = t_j = b_{n'-k+j}, \quad j = 1, \ldots, r.$$

Since $k' \le n'/2$,

$$t_1 = \cdots = t_r = b_{k'-r+1} = \cdots = b_{n'-k'+r}.$$

Use the first r columns of W to form the matrix R. Then $R^*TR = t_1 I_r$. Consequently, $R^*\phi(E_j)R = \tilde{\gamma}_j I_r$ for $j = 2, \ldots, n$, as $\tilde{\gamma}_j = \lambda_{k'-r+1}(\phi(E_j)) = \lambda_{n'-k'+r}(\phi(E_j))$. Moreover, $R^*\phi(E_1)R = I_r - \sum_{j=2}^n \tilde{\gamma}_j I_r$, where $\tilde{\gamma}_1 = 1 - \sum_{j=2}^n \tilde{\gamma}_j$. Clearly, $\tilde{\gamma}_1 = a_1 = a_r$. ∎

Lemma 2.8 *Suppose $k = n/2$ and $k' \le n'/2$. If there is a rank one idempotent $E \in H_n$ such that $\phi(E)$ has negative eigenvalues, then $k' = n'/2$ and $\phi \circ \psi^{-1}(F)$ is positive semidefinite for any rank one idempotent $F \in H_n$.*

Proof. Suppose there is a rank one idempotent E such that $\phi(E)$ has negative eigenvalues. Assume that $\phi(E)$ has eigenvalues $a_1 \ge \cdots \ge a_{n'-s} \ge 0 > a_{n'-s+1} \ge \cdots \ge a_{n'}$. Since

$$W_{k'}(\phi(E)) = W_k(E) = [0, 1/k],$$

$a_{n'} + \cdots + a_{n'-k'+1} = 0$. Thus, $s < k'$.

Let $E_1, \ldots, E_n \in H_n$ be rank one idempotents such that $E_1 = E$ and $\sum_{j=1}^n E_j = I_n$. By Lemma 2.7 (a), there is an $n' \times s$ matrix S whose columns are orthonormal eigenvectors of the negative eigenvalues of $\phi(E_1)$ such that $S^*\phi(E_j)S = \gamma_j I_s$ with

$$\gamma_1 = 1 - \sum_{j=2}^n \gamma_j = a_{n'} = \cdots = a_{n'-s+1} \tag{2.8}$$

and

$$\gamma_j = \lambda_{k'-s+1}(\phi(E_j)) = \lambda_{n'-k'+s}(\phi(E_j)), \qquad j = 2, \ldots, n. \tag{2.9}$$

We must have

$$a_1 = \cdots = a_{k'}.$$

Otherwise, there is $r < k'$ such that $a_r > a_{r+1}$. By Lemma 2.7 (b), there is an $n' \times r$ matrix R whose columns are orthonormal eigenvectors of the r largest eigenvalues of $\phi(E_1)$ such that $R^*\phi(E_j)R = \tilde{\gamma}_j I_r$ with

$$\tilde{\gamma}_j = \lambda_{k'-r+1}(\phi(E_j)) = \lambda_{n'-k'+r}(\phi(E_j)), \qquad j = 2, \ldots, n. \tag{2.10}$$

By (2.9) and (2.10), we have $\tilde{\gamma}_j = \lambda_{k'}(\phi(E_j)) = \gamma_j$ for $j = 2, \ldots, n$. But then, we have $R^*\phi(E_1)R = I_r - \sum_{j=2}^n \gamma_j I_r$, where $a_1 = 1 - \sum_{j=2}^n \gamma_j = a_{n'}$, which is a contradiction. Since $W_k(E_1) = W_{k'}(\phi(E_1))$, we see that $a_1 = \cdots = a_{k'} = 1/k$.

Observe that $\sum_{j=1}^n \gamma_j = 1$ and $\gamma_1 < 0$. Thus, there exists $j \ge 2$ such that $\gamma_j > 0$. We may assume that $\gamma_2 > 0$. Since $s_{k'}(\phi(E_2)) = 0$ and $\lambda_{n'-k'+s}(\phi(E_2)) = \gamma_2 > 0$, we see that $\lambda_{n'}(\phi(E_2)) < 0$. Suppose $\phi(E_2)$ has t negative eigenvalues. Applying the arguments on $\phi(E_1)$ to $\phi(E_2)$, we see that the last t eigenvalues of $\phi(E_2)$ all equal to $1 - \sum_{j \ne 2} \lambda_{k'-s+1}(\phi(E_j))$ and

$$1/k = \lambda_l(\phi(E_2)) \qquad \text{for } l = 1, \ldots, k'.$$

By (2.9), we have

$$1/k = \gamma_2 = \lambda_{k'-s+1}(\phi(E_2)) = \cdots = \lambda_{n'-k'+s}(\phi(E_2)).$$

Interchanging the roles of E_1 and E_2, we see that

$$1/k = \lambda_1(\phi(E_1)) = \cdots = \lambda_{n'-k'+t}(\phi(E_1)).$$

Moreover, since $\lambda_{k'}(\phi(E_1)) = \lambda_{k'}(\phi(E_2)) = 1/k$, we see that

$$\lambda_{n'}(\phi(E_1)) = 1 - \sum_{j=3}^{n} \lambda_{k'}(\phi(E_j)) - 1/k = \lambda_{n'}(\phi(E_2)).$$

Suppose $j \geq 3$ is such that $\phi(E_j)$ has negative eigenvalues. We can apply the above arguments on $\phi(E_2)$ to $\phi(E_j)$ to conclude that $\lambda_{n'}(\phi(E_j)) = \gamma_1$, and

$$1/k = \lambda_l(\phi(E_j)) \qquad \text{for } l = 1, \ldots, n' - k' + s. \tag{2.11}$$

Suppose $j \geq 3$ and $\phi(E_j)$ is positive semidefinite. Then $\lambda_{n'-k'+1}(\phi(E_j)) = \cdots = \lambda_{n'}(\phi(E_j)) = 0$. By (2.9), we have

$$\lambda_{k'-s+1}(\phi(E_j)) = \lambda_{n'-k'+s}(\phi(E_j)) = 0. \tag{2.12}$$

Relabeling $E_1, \ldots, E_n$ if necessary, we can assume that $\phi(E_2), \ldots, \phi(E_m)$ have negative eigenvalues, and $\phi(E_j)$ is positive semidefinite for $j > m$. Then each one of $\phi(E_1), \ldots, \phi(E_m)$ has smallest eigenvalue

$$\gamma_1 = 1 - \sum_{j=2}^{n} \lambda_{k'}(\phi(E_j)) = 1 - (m-1)/k < 0.$$

If $m < n$, then $\phi(E_n)$ is positive semidefinite with fewer than k' positive eigenvalues. Since $W_{k'}(\phi(E_n)) = W_k(E_n) = [0, 1/k]$, we see that $\lambda_1(E_n) > 1/k$. By Lemma 2.7 (b), we have

$$1/k < \lambda_1(\phi(E_n)) = 1 - \sum_{j=1}^{n} \lambda_{k'}(\phi(E_j)) = 1 - m/k < 1 - (m-1)/k < 0,$$

which is a contradiction. So, we have $m = n$. By (2.11), and the fact that $W_{k'}(\phi(E_j)) = W_k(E_j) = [0, 1/k]$ for $j = 1, \ldots, n$, we have

$$\begin{aligned} n' &= \operatorname{tr}(I_{n'}) = \sum_{j=1}^{n} \operatorname{tr} \phi(E_j) \\ &= \sum_{j=1}^{n} S_{k'}(\phi(E_j)) + \sum_{l=k'+1}^{n'-k'} \sum_{j=1}^{n} \lambda_l(\phi(E_j)) + \sum_{j=1}^{n} s_{k'}(\phi(E_j)) \\ &= n(k'/k) + n(n' - 2k')(1/k) = 2k' + 2(n' - 2k') = 2n' - 2k'. \end{aligned}$$

Hence, $n' = 2k'$.

Suppose $F \in H_n$ is a rank one idempotent. We claim that $\lambda_1(F) = 1/k$. Since $n = 2k \geq 4$, there is a rank one idempotent $G \in H_n$ such that $EG = GE = 0$ and $FG = GF = 0$. Moreover, there exist rank one idempotents $G_3, \dots, G_n \in H_n$ such that $E+G+G_3+\cdots G_n = I_n$. Applying the previous argument with $(E, E_2, \dots, E_n)$ replaced by $(E, G, G_3, \dots, G_n)$, we see that G has negative eigenvalues. Now, there exist rank one idempotents $F_3, \dots, F_n \in H_n$ such that $G+F+F_3+\cdots+F_n = I_n$. Applying the previous arguments with $(E, E_2, \dots, E_n)$ replaced by $(G, F, F_3, \dots, F_n)$, we see that $\phi(F)$ has largest eigenvalue $1/k$.

Now, observe that $\psi^{-1}(F) = I_n/k - F$. Since $\phi(F)$ has largest eigenvalue $1/k$, we conclude that $\phi(\psi^{-1}(F)) = I_{n'}/k - \phi(F)$, is positive semidefinite as asserted. ∎

Lemma 2.9 *Suppose $k \leq n/2$, $2k' \leq n' < k' \cdot C(n,k)$, and $\phi(E)$ is positive semidefinite for any rank one idempotent $E \in H_n$. Then $\phi(F)$ and $\phi(G)$ are disjoint idempotents in $H_{n'}$ for any disjoint rank one idempotents $F, G \in H_n$.*

Proof. Let $E_1, \dots, E_n \in H_n$ be rank one idempotents such that $F = E_1$, $G = E_2$, and $E_1 + \cdots + E_n = I_n$. Then $Y_j = \phi(E_j) \in H_{n'}$ is positive semidefinite and

$$W_{k'}(Y_j) = [0, 1/k] = [s_{k'}(Y_j)/k', S_{k'}(Y_j)/k']$$

for all $j = 1, \dots, n$. We claim that there is $j \in \{1, \dots, n\}$ such that the largest eigenvalue of Y_j has multiplicity $r < k'$. If it is not true, then for $j = 1, \dots, n$,

$$\lambda_1(Y_j) = \cdots = \lambda_{k'}(Y_j) = 1/k,$$

as $S_{k'}(Y_j) = k'/k$. Now, for any $1 \leq j_1 < j_2 < \cdots < j_k \leq n$,

$$W_{k'}\left(\sum_{t=1}^{k} Y_{j_t}\right) = W_k\left(\sum_{t=1}^{k} E_{j_t}\right) = [0,1].$$

Thus, there exists an $n' \times k'$ matrix U such that

$$k' = S_{k'}\left(\sum_{t=1}^{k} Y_{j_t}\right) = \operatorname{tr}\left(U^*\left(\sum_{t=1}^{k} Y_{j_t}\right)U\right) = \sum_{t=1}^{k} \operatorname{tr}\left(U^* Y_{j_t} U\right) \leq \sum_{t=1}^{k} S_{k'}(Y_{j_t}) = k'.$$

It follows that $\operatorname{tr}(U^* Y_{j_t} U) = k'/k$ and hence $U^* Y_{j_t} U = (1/k) I_{k'}$ for $t = 1, \dots, k$. Since $Y_1 + \cdots + Y_n = I_{n'}$, we see that $U^* Y_t U = 0_{k'}$ for any $t \notin \{j_1, \dots, j_k\}$.

Now, for any other choice of $1 \leq \tilde{j}_1 < \tilde{j}_2 < \cdots < \tilde{j}_k \leq n$, there is a corresponding $n' \times k'$ matrix $\tilde{U}$ such that $\tilde{U}^*\tilde{U} = I_{k'}$ such that $\tilde{U}^* Y_{\tilde{j}_t} \tilde{U} = (1/k) I_{k'}$ for $t = 1, \dots, k$, and $\tilde{U}^* Y_t \tilde{U} = 0_{k'}$ for any $t \notin \{\tilde{j}_1, \dots, \tilde{j}_k\}$. Suppose $\tilde{j}_p \notin \{j_1, \dots, j_k\}$. Then $U^* Y_{\tilde{j}_p} U = 0_{k'}$ and $\tilde{U}^* Y_{\tilde{j}_p} \tilde{U} = (1/k) I_{k'}$. Thus, the columns of U belong to the kernel of $Y_{\tilde{j}_p}$ whereas the columns of $\tilde{U}$ belong to the kernel of $Y_{\tilde{j}_p} - (1/k) I_{k'}$. So, $U^* \tilde{U} = 0_{k'}$.

Combining the above arguments, we see that there are $C(n,k)$ matrices U's of size $n' \times k'$ such that $U^*U = I_{k'}$. Any two of such U have mutually orthogonal columns. So, there are $k' \cdot C(n,k)$ orthonormal columns. Hence $k' \cdot C(n,k) \le n'$, which contradicts our assumption.

By the above argument, we see that

$$\min\{r : \lambda_r(Y_t) > \lambda_{r+1}(Y_t) \text{ with } t \in \{1,\dots,n\}\} < k'.$$

Relabeling $Y_1,\dots,Y_n$ if necessary, we may assume that

$$\lambda_1(Y_1) = \cdots = \lambda_r(Y_1) > \lambda_{r+1}(Y_1)$$

and for $t = 2,\dots,n$,

$$\lambda_1(Y_t) = \cdots = \lambda_r(Y_t).$$

Note that the last k' eigenvalues of Y_j are all zeros. We claim that the first k' eigenvalues of Y_t can contain at most two distinct values. Otherwise, there are $1 \le s < s' < k'$ such that $\lambda_s(Y_t) > \lambda_{s+1}(Y_t)$ and $\lambda_{s'}(Y_t) > \lambda_{s'+1}(Y_t)$. But by Lemma 2.7 (b), $\lambda_1(Y_t) = \cdots = \lambda_{s'}(Y_t)$, which is impossible.

Note that the last k' eigenvalues of Y_t are all zeros. Applying Lemma 2.7(b) to Y_1, we have

$$\lambda_{k'-r+1}(Y_t) = \lambda_{n'-k'+r}(Y_t) = 0$$

for $t = 2,\dots,n$. Then there is $r_t < k' - r + 1 \le k'$ such that

$$\lambda_1(Y_t) = \lambda_{r_t}(Y_t) > \lambda_{r_t+1}(Y_t) = \lambda_{n'}(Y_t) = 0,$$

i.e., Y_t is unitarily similar to $\gamma_t I_{r_t} \oplus 0_{n'-r_t}$ for $t = 2,\dots,n$. Interchanging the role of Y_1 and Y_t, we conclude that Y_1 is unitarily similar to $\gamma_1 I_{r_1} \oplus 0_{n'-r_1}$. Since $W_{k'}(Y_t) = W_k(E_t) = [0, 1/k]$, $r_t\gamma_t = k'/k$.

Furthermore, we can see from Lemma 2.7 (b) that for $s \ne t$, all eigenvectors of Y_s corresponding to the eigenvalue γ_s are eigenvectors of Y_t corresponding to the eigenvalue 0. Hence, $Y_sY_t = 0$ for any $s \ne t$. Since $Y_1 + \cdots + Y_n = I_{n'}$, $\gamma_t = 1$ and $r_1 + \cdots + r_n = n'$. Hence, $r_t = k'/k = r$ for all $t = 1,\dots,n$ and $k'/k = n'/n$. This shows that every Y_t is unitarily similar to $I_r \oplus 0_{n'-r}$. Hence, $A \mapsto \phi(A)$ maps disjoint idempotents to disjoint idempotents.

∎

Proof of the necessity part of Theorem 1. Suppose $k < n/2$ and $k' \le n'/2$. By Lemmas 2.6 and 2.9, ϕ will map idempotents to idempotents. So, (see Corollary 4.3 in [10] and also [3, Theorem 2.1]), ϕ has the asserted form.

Suppose $k = n/2$ and $k' \le n'/2$. Apply Lemma 2.8; then apply Lemmas 2.6 and 2.9 to $\phi \circ \psi^{-1}$ to get the conclusion.

Suppose $k > n/2$ and $k' \le n'/2$. Then $\phi \circ \psi^{-1}$ satisfies $W_{n-k}(A) = W_{k'}(\phi \circ \psi^{-1}(A))$ for all $A \in M_n$. So, $\phi \circ \psi^{-1}$ has the desired form.

Suppose $k > n/2$ and $k' > n'/2$. Replace ϕ by $\Psi \circ \phi \circ \psi^{-1}$ with $\Psi : M_{n'} \to M_{n'}$ defined by $\Psi(X) = [(\operatorname{tr} X)I_{n'} - k'X]/(n'-k')$ for all $X \in M_{n'}$. Then $W_{n-k}(A) = W_{n'-k'}(\phi(A))$ for all $A \in M_n$. So, $\Psi \circ \phi \circ \psi^{-1}$ has the asserted form. It follows that ϕ has the same form as well. ∎

3 Results on Triangular Matrices

Let T_n be the set of $n \times n$ upper triangular matrices. In this section, we study those linear maps $\phi : T_n \to T_{n'}$ satisfying

$$W_{k'}(\phi(A)) = W_k(A) \quad \text{for all } A \in T_n. \tag{3.1}$$

Clearly, if a map ϕ has the form (W1) or (W2) in Theorem 2.1 for some unitary U such that $\phi(T_n) \subseteq T_{n'}$, then condition (3.1) holds. The following theorem shows that the converse of the above statement is also valid, and gives a condition on U to ensure that $\phi(T_n) \subseteq T_{n'}$.

Theorem 3.1 *Suppose $k \in \{2, \dots, n-1\}$, $k' \in \{1, \dots, n'\}$ and $n' < C(n,k) \min\{k', n'-k'\}$. There exists a linear map $\phi : T_n \to T_{n'}$ such that*

$$W_{k'}(\phi(A)) = W_k(A) \quad \textit{for all } A \in T_n$$

if and only if there is a unitary $U = (u_{ij}) \in M_{n'}$ and nonnegative integers p, q with $p + q = n'/n$ such that

$$\sum_{j=0}^{p-1} \overline{u}_{(jn+a),d}\ u_{(jn+b),c} + \sum_{j=p}^{p+q-1} \overline{u}_{(jn+b),d}\ u_{(jn+a),c} = 0 \tag{3.2}$$

for all $1 \le a \le b \le n$ and $1 \le c < d \le n'$, and one of the following holds:

(T1) *$n'/n = k'/k$ and ϕ has the form*

$$A \mapsto U^*[\underbrace{A \oplus \cdots \oplus A}_{p} \oplus \underbrace{A^t \oplus \cdots \oplus A^t}_{q}]U.$$

(T2) *$n'/n = k'/(n-k)$ and ϕ has the form*

$$A \mapsto U^*[\underbrace{\psi(A) \oplus \cdots \oplus \psi(A)}_{p} \oplus \underbrace{\psi(A)^t \oplus \cdots \oplus \psi(A)^t}_{q}]U,$$

where $\psi : T_n \to T_n$ is the mapping $A \mapsto [(\operatorname{tr} A)I_n - (n-k)A]/k$.

Let us further analyze condition (3.2) in the following. For any $1 \le a \le b \le n$ and $1 \le c \le n'$, define

$$u_c^{ba} = \begin{pmatrix} v_c^b \\ w_c^a \end{pmatrix} \quad \text{with} \quad v_c^b = \begin{pmatrix} u_{bc} \\ \vdots \\ u_{((p-1)n+b)c} \end{pmatrix} \quad \text{and} \quad w_c^a = \begin{pmatrix} u_{(pn+a)c} \\ \vdots \\ u_{((p+q-1)n+a)c} \end{pmatrix}.$$

Then (3.2) reduces to

$$(v_d^a, v_c^b) + (w_d^b, w_c^a) = (u_d^{ab}, u_c^{ba}) = 0 \quad \text{for all } 1 \le a \le b \le n \text{ and } 1 \le c < d \le n',$$

where $(\cdot,\cdot)$ denotes the usual inner product, i.e., $(x,y) = y^*x$. Suppose $U = (u_{ij})$ satisfies (3.2). Clearly, the set $\{u_1^{aa}, \ldots, u_{n'}^{aa}\}$ forms an orthogonal set on $\mathbb{C}^{n'/n}$. Then at most n'/n vectors of the set can be nonzero. Hence, at most $n(n'/n) = n'$ vectors of the set

$$\{u_1^{11}, \cdots u_{n'}^{11}\} \cup \cdots \cup \{u_1^{nn}, \cdots u_{n'}^{nn}\}$$

can be nonzero. As U is an $n' \times n'$ unitary matrix, exactly one vector in $\{u_c^{11}, \ldots u_c^{nn}\}$ can be nonzero. Otherwise, U has a zero column. Furthermore, if $a \neq b$, then at most one of v_c^b and w_c^a can be nonzero, as only one of u_c^{bb} and u_c^{aa} can be nonzero. Thus, we deduce from (3.2) that

$$(v_d^a, v_c^b) = 0 = (w_d^b, w_c^a) \quad \text{for all } 1 \leq a < b \leq n \text{ and } 1 \leq c < d \leq n'.$$

In conclusion, we have the following

Proposition 3.2 *A unitary matrix $U = (u_{ij}) \in M_{n'}$ satisfies (3.2) if and only if*

(i) *for each $1 \leq c \leq n'$, there is a $a \in \{1, \ldots, n\}$ such that*

$$(v_d^a, v_c^a) + (w_d^a, w_c^a) = 1$$

and v_c^b and w_c^b are zero vectors for all $b \neq a$; and

(ii) *for any $1 \leq a < b \leq n$ and $1 \leq c < d \leq n'$,*

$$(v_d^a, v_c^a) + (w_d^a, w_c^a) = (v_d^a, v_c^b) = (w_d^b, w_c^a) = 0.$$

Example If $n' = 6$, $n = p = 2$ and $q = 1$, then

$$U = \begin{pmatrix} 1/\sqrt{2} & 0 & 0 & 0 & 1/\sqrt{2} & 0 \\ 0 & 0 & 0 & 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 0 & 0 & -1/\sqrt{2} & 0 \\ 0 & 0 & 0 & 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

satisfies (3.2). But if U_1 is the matrix obtained from U by interchanging its $(4,4)$-th and $(4,6)$-th entries, then U_1 does not satisfy (3.2) as $(v_4^2, v_5^1) = 1$.

Proof of Theorem 3.1. Note that the left side of (3.2) equals the (d,c)-th entry of

$$U^*[\underbrace{E_{ab} \oplus \cdots \oplus E_{ab}}_{p} \oplus \underbrace{E_{ba} \oplus \cdots \oplus E_{ba}}_{q}]U.$$

Hence, (3.2) holds if and only if

$$U^*[\underbrace{A \oplus \cdots \oplus A}_{p} \oplus \underbrace{A^t \oplus \cdots \oplus A^t}_{q}]U \in T_{n'} \quad \text{for all } A \in T_n.$$

Therefore, if ϕ has the form (T1), $\phi(T_n) \subseteq T_{n'}$. If ϕ has the form (T2), since $\psi(T_n) = T_n$, we have the same conclusion too. Therefore the sufficiency part holds.

For the converse, take any diagonal matrix R with real diagonal entries. Then

$$W_{k'}(\phi(R)) = W_k(R) \subseteq \mathbb{R}.$$

By Lemma 2.2(d), $\phi(R)^* = \phi(R)$. As $\phi(R) \in T_{n'}$, $\phi(R)$ must be a diagonal matrix with real diagonal entries.

Now for any diagonal matrix $D \in D_n$, write $D_1 = (D + D^*)/2$ and $D_2 = (D - D^*)/(2i)$. Then D_1 and D_2 are diagonal matrices with real diagonal entries. It follows that $\phi(D_1)^* = \phi(D_1)$ and $\phi(D_2)^* = \phi(D_2)$. As $D = D_1 + iD_2$,

$$\phi(D^*)^* = (\phi(D_1) - i\phi(D_2))^* = \phi(D_1)^* + i\phi(D_2)^* = \phi(D_1) + i\phi(D_2) = \phi(D).$$

Every $A \in M_n$ can be expressed as $T_1 + T_2^*$ for some upper triangular matrices T_1 and T_2. Define $\Phi : M_n \to M_{n'}$ by

$$\Phi(A) = \phi(T_1) + \phi(T_2)^*.$$

Clearly, Φ is linear. Suppose A can be written as $U_1 + U_2^*$ for some $U_1, U_2 \in T_n$ distinct from T_1 and T_2. Let $D = T_1 - U_1 = U_2^* - T_2^*$. Then D is a diagonal matrix. Observe that

$$\begin{aligned} 0 &= \phi(D) - \phi(D^*)^* \\ &= \phi(T_1 - U_1) - \phi(U_2 - T_2)^* \\ &= \phi(T_1) + \phi(T_2)^* - \phi(U_1) - \phi(U_2)^* \\ &= \Phi(T_1 + T_2^*) - \Phi(U_1 + U_2^*). \end{aligned}$$

Hence, Φ is well defined. On the other hand, we see that for any $A \in M_m$ and $1 \le r \le m$,

$$\begin{aligned} W_r(A + A^*) &= \{\mathrm{tr}\,(X^*AX)/r + \mathrm{tr}\,(X^*A^*X)/r : X \text{ is } m \times r,\ X^*X = I_r\} \\ &= \{\mathrm{tr}\,(X^*AX)/r + \overline{\mathrm{tr}\,(X^*AX)/r} : X \text{ is } m \times r,\ X^*X = I_r\} \\ &= \{z + \overline{z} : z \in W_r(A)\}. \end{aligned}$$

Since every matrix $H \in H_n$ can be expressed as $H = T + T^*$ with $T \in T_n$,

$$\begin{aligned} W_{k'}(\Phi(H)) &= W_{k'}(\phi(T) + \phi(T)^*) = \{z + \overline{z} : z \in W_{k'}(\phi(T))\} \\ &= \{z + \overline{z} : z \in W_k(T)\} = W_k(T + T^*) = W_k(H). \end{aligned}$$

Hence, $\Phi : M_n \to M_{n'}$ is a linear map such that

$$W_{k'}(\Phi(H)) = W_k(H) \quad \text{for all } H \in H_n.$$

By Theorem 2.1, there exist a unitary $U \in M_{n'}$ and nonnegative integers p, q with $p+q = n'/n$ such that Φ satisfies (W1) or (W2) in Theorem 2.1. Since $\phi(A) = \Phi(A)$ for all $A \in T_n$, ϕ has the form (T1) or (T2). Finally, we check that U satisfies (3.2) as $\phi(E_{ab}) \in T_{n'}$ for all $a \le b$. ■

Remark 3.3 Denote by $T(n_1, \ldots, n_r)$ the algebra of upper block triangular matrices $A = (A_{ij})$ such that $A_{ii} \in M_{n_i}$ for $i = 1, \ldots, r$. One can extend Theorem 3.1 to linear map $\phi : T(n_1, \ldots, n_r) \to T(m_1, \ldots, m_s)$ for $n_1 + \cdots + n_r = n$, $m_1 + \cdots + m_s = n'$, and $n' < C(n,k)\min\{k', n' - k'\}$. The result and proofs are basically the same provided that U satisfies (3.2) for all $1 \le a, b \le n$, $1 \le c, d \le n'$ such that $E_{ab} \in T(n_1, \ldots, n_r)$ and $E_{cd} \notin T(m_1, \ldots, m_s)$. Since the corresponding statements are rather tedious, we omit the details.

Note also that if a linear map $\phi : T(n_1, \ldots, n_r) \to T(m_1, \ldots, m_s)$ satisfies $W_{k'}(\phi(A)) = W_k(A)$ for all $A \in T(n_1, \ldots, n_k)$, then one can replace $T(m_1, \ldots, m_s)$ by other block triangular matrix algebras such as $T(m_1 + m_2, m_3 \ldots, m_s)$ or $T(m_1 + m_2, m_3 + m_4, \ldots, m_s)$, etc.

4 k-Numerical Radius

Theorem 4.1 *Let* $(\mathcal{M}, \mathcal{M}') = (H_n, H_{n'})$, $(M_n, M_{n'})$ *or* $(T_n, T_{n'})$, $k \in \{2, \ldots, n-1\}$, $k' \in \{1, \ldots, n'\}$ *and* $n' < C(n,k)\min\{k', n' - k'\}$. *Then a linear operator* $\widetilde{\phi} : \mathcal{M} \to \mathcal{M}'$ *satisfies*

$$w_{k'}(\widetilde{\phi}(A)) = w_k(A) \quad \text{for all } A \in \mathcal{M}, \tag{4.1}$$

and $\widetilde{\phi}(X) = I_{n'}$ *for some* $X \in \mathcal{M}$ *if and only if there is a complex unit* μ *such that* $\phi = \mu\widetilde{\phi}$ *satisfies*

$$W_{k'}(\phi(A)) = W_k(A) \quad \text{for all } A \in \mathcal{M},$$

equivalently, ϕ *has the form in Theorem* 2.1 *or Theorem* 3.1.

Lemma 4.2 *For any* $T = (t_{ij}) \in T_n$, *if*

$$\frac{1}{k}\left|\sum_{i=1}^{k} t_{n_i n_i}\right| = w_k(T) \quad \text{for all } 1 \le n_1 < \cdots < n_k \le n, \tag{4.2}$$

then T *is a diagonal matrix.*

Proof. Suppose $t_{ij} \neq 0$ for some $i < j$. Denote by $X[i,j] \in M_2$ the submatrix of $X \in M_n$ lying in the rows and columns indexed by i and j. Then $W_1(T[i,j])$ is an elliptical disk with the length of minor axis equal to $|t_{ij}|$, and foci t_{ii} and t_{jj}; see [11]. Thus, there is a unitary $U \in M_2$ such that the $(1,1)$ entry of $U^*T[i,j]U$ equals $t_{ii} + z$ and

$$\left|z + t_{ii} + \sum_{i=2}^{k} t_{n_i n_i}\right| > \left|t_{ii} + \sum_{i=2}^{k} t_{n_i n_i}\right| = kw_k(T),$$

where $1 \le n_2 < \cdots < n_k \le n$ are chosen from $\{1, \ldots, n\} \setminus \{i, j\}$. Let $V \in M_n$ be obtained from I_n by replacing $I_n[i,j]$ with U, and $V^*TV = (\tilde{t}_{rs})$. Then

$$kw_k(T) = kw_k(V^*TV) \ge \left|\tilde{t}_{ii} + \sum_{i=2}^{k} \tilde{t}_{n_i n_i}\right| = \left|z + t_{ii} + \sum_{i=2}^{k} t_{n_i n_i}\right| > kw_k(T),$$

which is a contradiction. ■

Lemma 4.3 *Let $\mathcal{M} = H_n, M_n$ or T_n. Suppose $k \in \{1, \ldots, n-1\}$. Given any matrix $A \in \mathcal{M}$, $A = \mu I_n$ with $|\mu| = 1$ if and only if for any $B \in M$, there is θ (depending on B) with $|\theta| = 1$ such that*

$$w_k(A + \theta B) = w_k(A) + w_k(B) = 1 + w_k(B),$$

i.e., there is an $n \times k$ matrix U (depending on B) with $U^*U = I_k$ such that

$$w_k(A) = |\mathrm{tr}\,(U^*AU)|/k = 1 \quad \text{and} \quad w_k(B) = |\mathrm{tr}\,(U^*BU)|/k. \tag{4.3}$$

Proof. Suppose $A = \mu I_n$ for some $|\mu| = 1$. For any $B \in \mathcal{M}$, if $w_k(B) = |\mathrm{tr}\,(U^*BU)|/k$ for some $n \times k$ matrix U with $U^*U = I_k$, then

$$|\mathrm{tr}\,(U^*AU)|/k = |\mathrm{tr}\,(U^*(\mu I_n)U)|/k = 1 = w_k(A).$$

For the converse, suppose for any $B \in \mathcal{M}$, there is an $n \times k$ matrix U with $U^*U = I_k$ such that (4.3) holds.

Let $K = I_k \oplus 0_{n-k}$. For any $n \times k$ matrix X with $X^*X = I_k$, we write $X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$ with $X_1 \in M_k$. Clearly, $\mathrm{tr}\,(X_1^*X_1 + X_2^*X_2) = \mathrm{tr}\,(X^*X) = k$. Then

$$(k - \mathrm{tr}\,(X_2^*X_2))/k = \mathrm{tr}\,(X_1^*X_1)/k = \mathrm{tr}\,(X^*KX)/k \in W_k(K) = [0, 1].$$

It follows that $\mathrm{tr}\,(X^*KX)/k = 1$ if and only if $\mathrm{tr}\,(X_2^*X_2) = 0$. Since $X_2^*X_2$ is positive semidefinite, $\mathrm{tr}\,(X_2^*X_2) = 0$ if and only if $X_2^*X_2 = 0_k$. Thus, X_1 must be unitary.

Suppose $\mathcal{M} = H_n$ or M_n. Take any $n \times k$ matrix V with $V^*V = I_k$, we extend V to an $n \times n$ unitary matrix $W = (\,V \quad V'\,)$ with some suitable $n \times (n-k)$ matrix V'. Choose $B = WKW^*$. Then there is an $n \times k$ matrix U with $U^*U = I_k$ such that

$$1 = w_k(A) = |\mathrm{tr}\,(U^*AU)|/k \quad \text{and} \quad w_k(K) = w_k(WKW^*) = |\mathrm{tr}\,(U^*WKW^*U)|/k.$$

By the above argument, $W^*U = X = \begin{pmatrix} X_1 \\ 0 \end{pmatrix}$ for some unitary matrix $X_1 \in M_k$. Thus, $U = VX_1$ and

$$|\mathrm{tr}\,(V^*AV)|/k = |\mathrm{tr}\,(X_1^*V^*AVX_1)|/k = |\mathrm{tr}\,(U^*AU)| = w_k(A) = 1.$$

It follows that all elements of $W_k(A)$ lie on the unit circle. Since $W_k(A)$ is convex, $W_k(A)$ must be a singleton set. By Lemma 2.2(e), $A = \mu I_n$ for some $|\mu| = 1$.

It remains to show the case for $\mathcal{M} = T_n$. For any $1 \le n_1 < \cdots < n_k \le n$, let $P = (p_{ij})$ be the $n \times n$ permutation matrix with $p_{n_i i} = 1$ for $i = 1, \ldots, k$ and $B = PKP^* \in T_n$. Then there is an $n \times k$ matrix U with $U^*U = I_k$ such that

$$1 = w_k(T) = |\mathrm{tr}\,(U^*TU)|/k \quad \text{and} \quad w_k(K) = w_k(PKP^*) = |\mathrm{tr}\,(U^*PKP^*U)|/k.$$

By the above argument, $P^*U = X = \begin{pmatrix} X_1 \\ 0 \end{pmatrix}$ for some unitary matrix $X_1 \in M_k$. Thus,

$$kw_k(T) = |\mathrm{tr}\,(U^*TU)| = |\mathrm{tr}\,(X^*P^*TPX)| = |\mathrm{tr}\,(X_1^*T_1X_1)| = |\mathrm{tr}\,T_1| = \left|\sum_{i=1}^{k} t_{n_in_i}\right|,$$

where T_1 is the $k \times k$ principal submatrix of P^*TP. As $n_1, \ldots, n_k$ are arbitrary, T satisfies (4.2). By Lemma 4.2, we conclude that T is a diagonal matrix.

Finally we show that the diagonal entries of T are the same. Suppose $t_{ii} \neq t_{jj}$ for some $i \neq j$. For simplicity, we assume that $t_{11} \neq t_{22}$. Take $B = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \oplus I_{k-1} \oplus 0_{n-k-1}$. Then $w_k(B) = 1$ and $|\mathrm{tr}\,(X^*BX)|/k = w_k(B)$ if and only if the $n \times k$ matrix X has the form $\begin{pmatrix} X_1 & 0 \\ 0 & X_2 \\ 0 & 0 \end{pmatrix}$ with $X_1 = \begin{pmatrix} \alpha/\sqrt{2} \\ \alpha/\sqrt{2} \end{pmatrix}$ for some $|\alpha| = 1$ and unitary $X_2 \in M_{k-1}$. In this case,

$$\left|\frac{1}{2}(t_{11} + t_{22}) + \sum_{i=3}^{k+1} t_{ii}\right| = |\mathrm{tr}\,(X^*TX)| = k.$$

Let $\alpha = t_{11} + \sum_{i=3}^{k+1} t_{ii}$ and $\beta = t_{22} + \sum_{i=3}^{k+1} t_{ii}$. Since T satisfies (4.2), we see that

$$|(\alpha + \beta)/2| = k = |\alpha| = |\beta|,$$

and hence $t_{11} = t_{22}$, which is the desired contradiction. ■

The following lemma is a modification of [15, Lemma 2], we give the proof here for the sake of completeness.

Lemma 4.4 *Let* $(\mathcal{M}, \mathcal{M}') = (H_n, H_{n'})$, $(M_n, M_{n'})$ *or* $(T_n, T_{n'})$, $k \in \{1, \ldots, n-1\}$ *and* $k' \in \{1, \ldots, n'\}$. *If* $\widetilde{\phi} : \mathcal{M} \to \mathcal{M}'$ *is a linear map satisfying (4.1) and* $\widetilde{\phi}(I_n) = I_{n'}$, *then*

$$W_{k'}(\widetilde{\phi}(A)) = W_k(A) \quad \text{for all } A \in \mathcal{M}.$$

Proof. Suppose $W_k(A) \not\subseteq W_{k'}(\widetilde{\phi}(A))$. Let $z \in W_k(A) \backslash W_{k'}(\widetilde{\phi}(A))$. Since $W_{k'}(\widetilde{\phi}(A))$ is compact, there exists some $\lambda \in \mathbb{C}$ such that

$$|z + \lambda| > |z' + \lambda|$$

for all $z' \in W_{k'}(\widetilde{\phi}(A))$. Here,

$$w_k(A + \lambda I_n) > w_{k'}(\widetilde{\phi}(A) + \lambda I_{n'}) = w_{k'}(\widetilde{\phi}(A + \lambda I_n)) = w_k(A + \lambda I_n)$$

which is impossible. Therefore, $W_k(A) \subseteq W_{k'}(\widetilde{\phi}(A))$. Similarly, we have $W_{k'}(\widetilde{\phi}(A)) \subseteq W_k(A)$. The result follows. ■

Proof of Theorem 4.1. The sufficiency part is clear. For the necessity part, suppose there is $X \in \mathcal{M}$ such that $\widetilde{\phi}(X) = I_{n'}$. For any $B \in \mathcal{M}$, there exists some $\theta \in \mathbb{C}$ with $|\theta| = 1$ such that

$$w_k(X + \theta B) = w_{k'}(I_{n'} + \theta\widetilde{\phi}(B)) = w_{k'}(I_{n'}) + w_{k'}(\widetilde{\phi}(B)) = w_k(X) + w_k(B).$$

By Lemma 4.3, $X = \mu I_n$ for some $\mu \in \mathbb{C}$ with $|\mu| = 1$. We see that the map $A \mapsto \mu\widetilde{\phi}(A)$ maps I_n to $I_{n'}$ and satisfies (4.3). Then the result follows by Lemma 4.4. ■

5 Open problems

There are many open problems deserved further study. We mention a few of them in the following.

1. If $n' = C(n,k)\min\{k', n' - k'\}$, there are exceptional maps for the range preservers have the form
$$A \mapsto U^*\Delta_k(A)U \text{ or } A \mapsto U^*\Delta_k(A)^t U$$
with $k' = 1$. Are there other exceptional maps?

2. If $n' \le 2C(n,k)\min\{k', n'-k'\} - 2$, there are exceptional maps for the range preservers have the form
$$A \mapsto U^*[\Delta_k(A) \oplus f(\Delta_k(A))]U \text{ or } A \mapsto U^*[\Delta_k(A)^t \oplus f(\Delta_k(A))]U$$
for some unital positive linear map $f : M_{C(n,k)} \to M_{n'-C(n,k)}$, here $k' = 1$. Are there other exceptional maps?

3. In Theorem 4.1, an assumption that $\widetilde{\phi}(X) = I_{n'}$ for some $X \in \mathcal{M}$ is needed. For $k' = 1$, since $w_1(A) = w_1(A \oplus 0)$, the condition is clearly necessary. Can this assumption be removed when $k' > 1$?

4. How about extending the results to infinite dimensional operators, nest algebras, etc.?

5. What about other types of generalized numerical ranges and radii?

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