

GENERALIZED INTERLACING INEQUALITIES

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In memory of Professor Ky Fan

Abstract

We discuss some applications of generalized interlacing inequalities of Ky Fan to the study of (a) some classical matrix inequalities and (b) matrix problems in quantum information science.

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1 Introduction

Professor Ky Fan was a great scholar and educator. Not only did he obtain many interesting and useful results, he was also eager to help young colleagues and shared his insights and experience about mathematical research and education. The first author met him in several occasions and received valuable advice from him. For instance, he emphasized that “One should keep an open mind about research.” “Interesting mathematical results often build on simple assumptions but have deep implications.” Professor Fan had many results of this nature. For example, he and Pall [7] obtained the following generalized interlacing inequalities for eigenvalues of complex Hermitian (real symmetric) matrices that have many applications as shown in later sections.

Theorem 1.1 *Let $a_1 \geq \dots \geq a_n$ and $b_1 \geq \dots \geq b_k$ be real numbers and $1 \leq k < n$. There exists an $n \times n$ Hermitian matrix A with a $k \times k$ principal submatrix B having eigenvalues $a_1 \geq \dots \geq a_n$ and $b_1 \geq \dots \geq b_k$, respectively, if and only if*

$$a_j \geq b_j \geq a_{n-k+j}, \quad j = 1, \dots, k. \quad (1)$$

Here, we present a proof for completeness. Denote by M_n the set of $n \times n$ complex matrices and H_n the set of Hermitian matrices in M_n .

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Proof of Theorem 1.1. First consider the case when $k = n - 1$, and B is obtained from A by deleting its first row and first column. We need only address the case $b_j \in (a_{j+1}, a_j)$ for $j = 1, \dots, n - 1$, since the general case $b_j \in [a_{j+1}, a_j]$ follows by a continuity argument.

Let U be unitary such that $A = U^* \text{diag}(a_1, \dots, a_n)U$. If $u = (u_1, \dots, u_n)^t$ is the first column of U , then the $(1, 1)$ entry of $(tI - A)^{-1} = U^* \text{diag}(t - a_1, \dots, t - a_n)^{-1}U$ equals

$$f(t) = \sum_{j=1}^n |u_j|^2 (t - a_j)^{-1},$$

which is continuous in each interval (a_{j+1}, a_j) for $j = 1, \dots, n - 1$, and satisfies $\lim_{t \rightarrow a_{j+1}^+} f(t) = \infty$ and $\lim_{t \rightarrow a_j^-} f(t) = -\infty$. Thus, f has at least one zero in (a_{j+1}, a_j) for $j = 1, \dots, n - 1$. Now, by the adjoint formula for the inverse, it is the same as

$$f(t) = \det(tI - B) / \det(tI - A) = \left(\prod_{j=1}^{n-1} (t - b_j) \right) / \left(\prod_{j=1}^n (t - a_j) \right).$$

We conclude that b_j is the unique zero in (a_{j+1}, a_j) for $j = 1, \dots, n - 1$.

If $k < n - 1$, we can repeat the above argument $n - k$ times on a sequence of submatrices of A by removing one row and one column each time to get the desired inequalities.

To prove the converse, we first consider the case $k = n - 1$. We use the proof by Mirsky [19] described in Marshall and Olkin [18].

We aim at constructing $A = \begin{bmatrix} D & v \\ v^t & v_n \end{bmatrix}$, where $D = \text{diag}(b_1, \dots, b_{n-1})$ is a diagonal matrix and $v = (v_1, \dots, v_{n-1})^t$ is a real vector. Then for $f(t) = \det(tI - A) = \prod_{j=1}^n (t - a_j)$ and $g(t) = \det(tI - D) = \prod_{j=1}^{n-1} (t - b_j)$ we have

$$f(t) = g(t) \left(t - v_n - v^t (tI - D)^{-1} v \right) = g(t) \left(t - v_n - \sum_{j=1}^{n-1} \frac{v_j^2}{t - b_j} \right).$$

Let $v_n = \sum_{j=1}^n a_j - \sum_{j=1}^{n-1} b_j$, and for $j = 1, \dots, n - 1$, $v_j \geq 0$ such that

$$v_j^2 = -f(b_j) / g'(b_j) = -\frac{\prod_{r=1}^n (b_j - a_r)}{\prod_{r \neq j} (b_j - b_r)},$$

which is positive because $b_r - a_n > 0$, $b_r - a_r < 0$, and $(b_j - a_r) / (b_j - b_r) > 0$ for all $j \neq r$, by our assumption. Then the polynomial $f(t) - g(t) \left(t - v_n - \sum_{j=1}^{n-1} v_j^2 / (t - b_j) \right)$ must be the zero polynomial because it has degree $n - 2$, and has zeros at b_1, \dots, b_{n-1} . Thus, we get the desired matrix A .

Now, suppose $k < n - 1$. We can use induction on $n - k$. One can find $\tilde{a}_1 \geq \dots \geq \tilde{a}_{n-1}$ such that

$$\min\{a_j, b_{k-n+j+1}\} \geq \tilde{a}_j \geq \max\{a_{j+1}, b_j\}, \quad j = 1, \dots, n - 1,$$

here $b_{k-n+j+1} = a_j$ if $k - n + j + 1 \leq 0$. For instance, one can define

$$\tilde{a}_j = \begin{cases} \max\{a_{j+1}, b_j\} & 1 \leq j \leq k, \\ \min\{a_j, b_{k-n+j+1}\} & k < j < n. \end{cases}$$

Then

$$a_1 \geq \tilde{a}_1 \geq a_2 \geq \cdots \geq a_{n-1} \geq \tilde{a}_{n-1} \geq a_n,$$

and

$$\tilde{a}_j \geq b_j \geq \tilde{a}_{n-k-1+j}, \quad j = 1, \dots, k.$$

By the induction assumption, we can construct a matrix $\tilde{A} \in H_{n-1}$ with eigenvalues $\tilde{a}_1 \geq \cdots \geq \tilde{a}_{n-1}$, whose $k \times k$ leading principal submatrix has eigenvalues $b_1 \geq \cdots \geq b_k$, and \tilde{A} is a principal submatrix of $A \in H_n$ such that A has eigenvalues $a_1 \geq \cdots \geq a_n$. \square

2 Some applications to matrix inequalities

The generalized interlacing inequalities have many nice applications to matrix inequalities.

Theorem 2.1 *Suppose A is an $n \times n$ Hermitian matrix with eigenvalues $a_1 \geq \cdots \geq a_n$ and diagonal entries $d_1 \geq \cdots \geq d_n$. Then $(d_1, \dots, d_n) \prec (a_1, \dots, a_n)$, i.e., $\sum_{j=1}^k d_j \leq \sum_{j=1}^k a_j$ for $k = 1, \dots, n$, and the equality holds when $k = n$.*

Proof. Consider the $k \times k$ submatrix B of A with diagonal entries d_1, \dots, d_k . Suppose B has eigenvalues $b_1 \geq \cdots \geq b_k$. Then $\sum_{j=1}^k d_j = \sum_{j=1}^k b_j \leq \sum_{j=1}^k a_j$ by the interlacing inequalities. \square

Denote by $\lambda_1(A) \geq \cdots \geq \lambda_n(A)$ the eigenvalues of an $n \times n$ Hermitian matrix A . Suppose E is a small perturbation of A due to numerical errors and measuring errors, etc. We can compare the eigenvalues of $A + E$ and those of A . By the result of Liskii, Mirsky, and Wielandt, see e.g. [1, Theorem 9.4] or [22, Theorem IV.4.8], we have

$$\sum_{j=1}^k \lambda_{n-j+k}(E) \leq \sum_{j=1}^k (\lambda_{i_j}(A + E) - \lambda_{i_j}(A)) \leq \sum_{j=1}^k \lambda_{j_r}(E) \quad \text{for any } 1 \leq i_1 < \cdots < i_k \leq n.$$

One can see [13] for a simple proof, and [12] for results on perturbation on structured Hermitian matrices, where the generalized interlacing inequalities also play key roles in the proof. Here, we present the results and proofs of the Thompson's inequalities, which are important special cases for the complete set of inequalities by Horn, Klyascho, Fulton, etc., see e.g., [8].

Theorem 2.2 *Suppose A, B and $C = A + B$ are Hermitian matrices. Then for any $1 \leq i_1 < \cdots < i_k \leq n$ and $1 \leq j_1 < \cdots < j_k \leq n$ with $i_k + j_k - k \leq n$, we have*

$$\sum_{r=1}^k \lambda_{i_r}(A) + \sum_{r=1}^k \lambda_{j_r}(B) \geq \sum_{r=1}^k \lambda_{i_r+j_r-r}(C).$$

Proof. We prove by induction on n . Clearly, the result holds when $n = 1$. Assume $n > 1$. First the result is clear if $k = n$. Let p be the largest integer such that $i_t = t$ for $t = 1, \dots, p$, and let q be the largest integer such that $j_t = t$ for $t = 1, \dots, q$. Notice that $p \leq k$ and $q \leq k$. Exchanging A and B if necessary, we may assume that $q \leq p$. We can further assume that $i_k < n$. Suppose not.

Then $i_k = n$ and $i_k + j_k - k \leq n$ imply that $j_k = k$ and hence $p = q = k$. So, one can exchange A and B again so that $i_k = k < n$.

For simplicity, let $a_j = \lambda_j(A)$, $b_j = \lambda_j(B)$, and $c_j = \lambda_j(C)$ for $j = 1, \dots, n$. Let $\{u_1, \dots, u_n\}$ and $\{v_1, \dots, v_n\}$ be orthonormal sets of eigenvectors of B and C such that

$$Bu_j = b_j u_j \quad \text{and} \quad Cv_j = c_j v_j, \quad j = 1, \dots, n.$$

Let \mathcal{S} be an $(n-1)$ -dimensional subspace containing the vectors $v_1, \dots, v_q, u_{q+2}, \dots, u_n$ and P be the orthogonal projection of \mathbb{C}^n onto \mathcal{S} . Identify $\tilde{A} = PAP$, $\tilde{B} = PBP$ and $\tilde{C} = PCP$ as matrices in H_{n-1} . Then their eigenvalues $\tilde{a}_1 \geq \dots \geq \tilde{a}_{n-1}$, $\tilde{b}_1 \geq \dots \geq \tilde{b}_{n-1}$, $\tilde{c}_1 \geq \dots \geq \tilde{c}_{n-1}$, interlace, respectively, those of A, B and C . By the induction assumption on $\tilde{C} = \tilde{A} + \tilde{B}$ with the sequences $1 \leq i_1 < \dots < i_k \leq n-1$ and $1 \leq j_1 < \dots < j_q < j_{q+1} - 1 < \dots < j_k - 1 \leq n-1$, we have

$$\sum_{s=1}^q \tilde{c}_{i_s+j_s-s} + \sum_{s=q+1}^k \tilde{c}_{i_s+j_s-s-1} \leq \sum_{s=1}^k \tilde{a}_{i_s} + \sum_{s=1}^q \tilde{b}_{j_s} + \sum_{j=q+1}^k \tilde{b}_{j_s-1}. \quad (2)$$

By the interlacing inequality, we see that $\tilde{a}_{i_s} \leq a_{i_s}$ for $s = 1, \dots, k$; $\tilde{b}_{j_s} \leq b_{j_s}$ for $s = 1, \dots, q$; $\tilde{c}_{i_s+j_s-s-1} \geq c_{i_s+j_s-s}$ for $s = q+1, \dots, k$. Since \mathcal{S} contains v_1, \dots, v_q , it follows that c_1, \dots, c_q are eigenvalues of \tilde{C} , and by interlacing $c_t = \tilde{c}_t$ for $t = 1, \dots, q$. Moreover, for $s = 1, \dots, q$, we have $i_s = j_s = s$, and thus, $i_s + j_s - s = s$. It follows that

$$\sum_{s=1}^q \tilde{c}_{i_s+j_s-s} = \sum_{s=1}^q c_{i_s+j_s-s}.$$

Since \mathcal{S} contains u_{q+2}, \dots, u_n , it follows that b_{q+2}, \dots, b_n are eigenvalues of \tilde{B} , and by interlacing $b_t = \tilde{b}_{t-1}$ for $t = q+2, \dots, n$. As $j_{q+1} > q+1$, $b_{j_s} = \tilde{b}_{j_s-1}$ for $s \geq q+1$. Thus,

$$\sum_{s=q+1}^k \tilde{b}_{j_s-1} = \sum_{s=q+1}^k b_{j_s}.$$

Combining these facts with (2), we get the desired inequalities. \square

3 Applications in quantum information science

In this section, we describe some applications of the generalized interlacing inequalities to problems in quantum information science. Here, we briefly describe the background and refer the readers to [4] for details.

Quantum states are represented as density matrices in M_n , i.e., positive semidefinite matrices with trace one. A quantum channel is represented as a completely positive linear map on M_n with an operator sum representation

$$L(X) = \sum_{j=1}^r F_j X F_j^* \quad (3)$$

for some matrices F_1, \dots, F_r known as the error operators. A map L is said to be a completely positive map if the map $I_k \otimes L$ is a positive map for all k , see [3]. One would like to find a recovery channel T such that $T \circ L(X) = X$ whenever $PXP = X$ for some orthogonal projection P . The range space of P is known as a quantum error correction code of the channel L . One would like to find P with a maximum rank for the given channel L . For a given channel L with the operator sum representation described in (3), it is known that the range space of an orthogonal projection P is a quantum error correction code corresponding to L if and only if there are scalars $\gamma_{ij} \in \mathbb{C}$ such that

$$PF_i^* F_j P = \gamma_{ij} P \quad \text{for all } 1 \leq i, j \leq r,$$

see, e.g., [20, Chapter 10]. This leads to the study of the rank- k numerical range of $A \in M_n$ defined by

$$\Lambda_k(A) = \left\{ \gamma \in \mathbb{C} : U^* A U = \begin{bmatrix} \gamma I_k & * \\ * & * \end{bmatrix} \text{ for some unitary } U \right\}.$$

When $k = 1$, the definition reduces to the classical numerical range

$$W(A) = \{x^* A x : x \in \mathbb{C}^n, x^* x = 1\}$$

of $A \in M_n$, which is very useful in the study of matrices and operators; see [9, Chapter 1].

3.1 Higher rank numerical ranges

In [15], the following theorem was proved which confirmed some conjectures in [4].

Theorem 3.1 *Let $A \in M_n$. Then*

$$\Lambda_k(A) = \left\{ \gamma \in \mathbb{C} : e^{i\xi} \gamma + e^{-i\xi} \bar{\gamma} \leq \lambda_k(e^{i\xi} A + e^{-i\xi} A^*) \text{ for all } \xi \in [0, 2\pi) \right\}. \quad (4)$$

Consequently, $\Lambda_k(A)$ is always convex. If A is normal with eigenvalues a_1, \dots, a_n , then

$$\Lambda_k(A) = \bigcap_{1 \leq j_1 < \dots < j_{n-k+1} \leq n} \text{conv} \{a_{j_1}, \dots, a_{j_{n-k+1}}\}.$$

A key step of the proofs of (4) is the use of the generalized interlacing inequalities. To be precise, in the proof one has to show that zero is in the rank- k numerical range of a matrix B with eigenvalues $1 + i\mu_1, \dots, 1 + i\mu_k$ and $-1 + i\mu_{k+1}, \dots, -1 + i\mu_n$, where μ_1, \dots, μ_n are real numbers satisfying

$$-\mu_n > \mu_1, \quad -\mu_{n-1} > \mu_2, \quad \dots, \quad -\mu_{n-k+2} > \mu_{k-1}, \quad -\mu_{n-k+1} > \mu_k,$$

and

$$-\mu_{k+1} \leq \mu_k, \quad -\mu_{k+2} \leq \mu_{k-1}, \quad \dots, \quad -\mu_{2k-1} \leq \mu_2, \quad -\mu_{2k} \leq \mu_1.$$

To complete the proof, one has to apply Theorem 1.1 and conclude that there is a unitary U such that $U^* B U$ has a principal submatrix 0_k , and hence $0 \in \Lambda_k(B)$.

The following result gives the optimal bound on n for $\Lambda_k(A)$ to be non-empty.

Theorem 3.2 *The set $\Lambda_k(A)$ is non-empty if one of the following conditions is satisfied.*

- (I) $A \in H_n$, with $n \geq 2k - 1$. (II) $A \in M_n$, with $n \geq 3k - 2$.

Furthermore, the bounds in (I) and (II) are optimal.

Proof. The condition (I) was proved in [6]. Here we give a proof using Theorem 1.1. Suppose $A \in H_n$, with $n \geq 2k - 1$. We may assume that A has eigenvalues $a_1 \geq a_2 \geq \dots \geq a_n$. By Theorem 1.1, there is a unitary U such that U^*AU has a leading principal submatrix equal to $a_k I_k$.

Suppose $n < 2k - 1$. Let $A = \text{diag}(1, 2, \dots, n)$. If U^*AU has a leading principal submatrix equal to bI_k , then $a_j \geq b \geq a_{n-j+1}$ for $j = 1, \dots, k$, which is impossible. Thus, $\Lambda_k(A)$ is empty.

The condition (II) was proved in [14, Theorems 1 and 3]. \square

3.2 (p, k) numerical range

In [5], for a better error correction scheme, the authors consider a special decomposition of the matrix space. Given a quantum channel L with the operator sum representation in (3), they would like to find a recovery channel R such that for each $B \in M_k$,

$$R \circ L((I_p \otimes B) \oplus 0_{n-pk}) = (A_B \otimes B) \oplus 0_{n-pk} \quad \text{for some } A_B \in M_p.$$

Fixed an arbitrary orthonormal basis $\{e_1, \dots, e_p\}$ in \mathbb{C}^p . The authors showed that such recovery channel R exists if and only if there are scalars $\gamma_{ijrs} \in \mathbb{C}$ such that

$$P_{kk} F_i^* F_j P_{\ell\ell} = \gamma_{ijrs} P_{k\ell} \quad \text{for all } 1 \leq i, j \leq r \text{ and } 1 \leq k, \ell \leq p,$$

where $P_{k\ell} = (e_k e_\ell^* \otimes I_k) \oplus 0_{n-pk}$; see [11] and also [5]. This leads to an extension of rank- k numerical range to the (p, k) numerical range of $A \in M_n$ defined by

$$\Lambda_{p,k}(A) = \left\{ B \in M_p : U^*AU = \begin{bmatrix} B \otimes I_k & * \\ * & * \end{bmatrix} \text{ for some unitary } U \right\}.$$

It is easy to see that

$$\Lambda_{p,k}(A) = \{B \in M_p : \text{there is } X \text{ such that } X^*X = I_{pk} \text{ and } X^*AX = B \otimes I_k\}.$$

When $p = 1$, we get the rank- k numerical range; when $k = 1$, we get the p th matricial range (see [16]); when $p = k = 1$, we get the classical numerical range (see [9]).

The following theorem is easy to verify.

Theorem 3.3 *Let $A \in M_n$.*

(a) $\Lambda_{p,k}(\alpha A + \beta I_n) = \alpha \Lambda_{p,k}(A) + \beta I_p$ for any $\alpha, \beta \in \mathbb{C}$.

(b) $\Lambda_{p,k}(X^*AX) \subseteq \Lambda_{p,k}(A)$ for any $n \times m$ matrix X with $X^*X = I_m$, and the equality holds if $m = n$.

(c) A matrix $B \in M_p$ satisfies $B \in \Lambda_{p,k}(A)$ if and only if $V^*BV \in \Lambda_{p,k}(A)$ for any unitary $V \in M_p$. So, $\Lambda_{p,k}(A)$ is a union of unitary similarity orbits $\{V^*BV : V \text{ is unitary}\}$ in M_p .

The following theorem gives a necessary condition for $B \in \Lambda_{p,k}(A)$.

Theorem 3.4 *Let $A \in M_n$. If $B \in \Lambda_{p,k}(A)$, then*

$$\Lambda_{n-pk+1}(A) \subseteq \Lambda_1(B) \subseteq \Lambda_k(A).$$

Proof. Suppose there is a unitary U such that U^*AU has a principal submatrix of the form $\tilde{B} = B \otimes I_k$ with $B \in M_p$. Then for each $\xi \in [0, 2\pi)$, we can apply Theorem 1.1 to conclude that

$$\begin{aligned} \lambda_k(e^{i\xi}A + e^{-i\xi}A^*) &\geq \lambda_k(e^{i\xi}\tilde{B} + e^{-i\xi}\tilde{B}^*) = \lambda_1(e^{i\xi}B + e^{-i\xi}B^*) \\ &= \lambda_1(e^{i\xi}\tilde{B} + e^{-i\xi}\tilde{B}^*) \geq \lambda_{n-kp+1}(e^{i\xi}A + e^{-i\xi}A^*). \end{aligned}$$

Consequently, if we obtain convex sets $\Lambda_{n-kp+1}(A)$, $\Lambda_1(B)$, and $\Lambda_k(A)$ using the half spaces described in Theorem 3.1, then the half space containing $\Lambda_{n-kp+1}(A)$ will be a subset of the half space containing $\Lambda_1(B)$, which in turns will be a subset of the half space containing $\Lambda_k(A)$. Taking the intersection, we get the asserted inclusion relations. \square

For a general matrix $A \in M_n$, it is difficult to check whether a given matrix $B \in M_p$ belongs to $\Lambda_{p,k}(A)$. If A is Hermitian, we have a necessary and sufficient condition for $B \in \Lambda_{p,k}(A)$ in terms of the eigenvalues of A and those of B .

Theorem 3.5 *Let $A \in H_n$ and $pk \leq n$. Then $B \in \Lambda_{p,k}(A)$ if and only if the eigenvalues of $B \otimes I_k$ and A satisfy the generalized interlacing inequalities. In other words, $B \in \Lambda_{p,k}(A)$ if and only if*

$$\lambda_{n-(p-j+1)k+1}(A) \leq \lambda_j(B) \leq \lambda_{jk}(A) \quad \text{for all } j = 1, \dots, p.$$

Consequently, $\Lambda_{p,k}(A) \neq \emptyset$ if and only if

$$\lambda_{jk}(A) \geq \lambda_{n-(p-j+1)k+1}(A) \quad \text{for all } j = 1, \dots, p.$$

If $n \geq (p+1)k - 1$, then the above inequalities always hold and hence $\Lambda_{p,k}(A)$ is always non-empty. If $n < (p+1)k - 1$, then the above inequalities hold if and only if all of the inequalities become equalities so that $\Lambda_{p,k}(A)$ consists of matrices in H_p with eigenvalues $a_k, a_{2k}, \dots, a_{pk}$.

Proof. Notice that $B \in \Lambda_{p,k}(A)$ if and only if U^*AU has a principal submatrix with eigenvalues

$$\underbrace{\lambda_1(B), \dots, \lambda_1(B)}_k, \underbrace{\lambda_2(B), \dots, \lambda_2(B)}_k, \dots, \underbrace{\lambda_p(B), \dots, \lambda_p(B)}_k.$$

By Theorem 1.1, the first assertion follows. The last two assertions can be verified readily. \square

The following result provides bounds on n ensuring that $\Lambda_{p,k}(A)$ is non-empty for any $A \in M_n$.

Theorem 3.6 *If $n \geq 2(p+1)k - 3$, then $\Lambda_{p,k}(A)$ is non-empty for any $A \in M_n$.*

Proof. Suppose $n \geq 2(p+1)k - 3 = 2((p+1)k - 1) - 1$. Let $A = A_1 + iA_2$, with $A_1, A_2 \in H_n$. Then by Theorem 3.2, there exists a unitary $U \in M_n$ such that $U^*A_1U = \begin{bmatrix} aI_{(p+1)k-1} & * \\ * & * \end{bmatrix}$ for some $a \in \mathbb{R}$. Let $U^*A_2U = \begin{bmatrix} B & * \\ * & * \end{bmatrix}$ with $B \in H_{(p+1)k-1}$. By Theorem 3.5, $\Lambda_{p,k}(B) \neq \emptyset$. Therefore, $\Lambda_{p,k}(A) \neq \emptyset$. \square

One may improve the bound n if $A \in M_n$ has some special structure.

Theorem 3.7 *Suppose $A \in M_n$ is unitarily similar to $A_1 \oplus \dots \oplus A_p$ such that $\Lambda_k(A_j) \neq \emptyset$ for all j . Then $\Lambda_{p,k}(A)$ is nonempty. Consequently, if each A_j has dimension at least $3k - 2$, then $\Lambda_{p,k}(A)$ is nonempty. In particular, if $A \in M_n$ is normal with $n \geq (3k - 2)p$, then $\Lambda_{p,k}(A)$ is non-empty.*

Proof. The given assumption ensures that each A_j is unitarily similar to $\begin{bmatrix} b_j I_k & * \\ * & * \end{bmatrix}$. Thus, $\text{diag}(b_1, \dots, b_k) \in \Lambda_{p,k}(A)$. By Theorem 3.2, A_j has dimension at least $3k - 2$ ensuring that $\Lambda_k(A) \neq \emptyset$. The last assertion follows. \square

If $A \in H_n$ and $n \leq (p+1)k - 1$, it follows from Theorem 3.5 that $\Lambda_{p,k}(A)$ consists of matrices in H_p with eigenvalues $a_k, a_{2k}, \dots, a_{pk}$. Therefore, $\Lambda_{p,k}(A)$ is convex if and only if $a_k = a_{pk}$. If $n \geq (p+1)k$, we have the following.

Theorem 3.8 *Suppose $n \geq (p+1)k$ and $A \in H_n$. Then $\Lambda_{p,k}(A)$ is convex if and only if*

$$\lambda_k(A) = \lambda_{pk}(A) \quad \text{and} \quad \lambda_{n-pk+1}(A) = \lambda_{n-k+1}(A).$$

Proof. Suppose $\lambda_k(A) = \lambda_{pk}(A)$ and $\lambda_{n-pk+1}(A) = \lambda_{n-k+1}(A)$. Then by Theorem 3.5,

$$\Lambda_{p,k}(A) = \{B \in H_p : \lambda_{n-k+1}(A) \leq \lambda_j(B) \leq \lambda_k(A) \text{ for all } 1 \leq j \leq p\},$$

which is clearly convex.

We will prove the converse by contradiction. Suppose first that $\lambda_{pk}(A) < \lambda_k(A)$. Let $B = \text{diag}(\lambda_k(A), \lambda_{2k}(A), \dots, \lambda_{pk}(A))$ and $R = [r_{ij}]_{i,j=1}^p$ where $r_{ij} = 1$ if $i \equiv j + 1 \pmod{p}$, and $r_{ij} = 0$ otherwise. For $1 \leq j \leq p$, let $B_j = R^{j-1}B(R^t)^{j-1}$. Then $B_j \in \Lambda_{p,k}(A)$ for all $1 \leq j \leq p$ but $(\sum_{j=1}^p B_j)/p = \text{diag}(b, \dots, b)$, where $b = (\lambda_k(A) + \lambda_{2k}(A) + \dots + \lambda_{pk}(A))/p > \lambda_{pk}(A)$. Therefore, $\text{diag}(b, \dots, b) \notin \Lambda_{p,k}(A)$ and $\Lambda_{p,k}(A)$ is not convex. The proof for the case when $\lambda_{n-pk+1}(A) > \lambda_{n-k+1}(A)$ is similar. \square

We have obtained only some basic results for the (p, k) numerical range. There are many problems that deserve further study. We mention some of them in the next section.

4 Further research

In [7], Fan and Pall study the conditions on which a $k \times k$ normal matrix B can be imbedded into an $n \times n$ normal matrix with prescribed eigenvalues. They give a necessary and sufficient condition for the case when $k = n - 1$. Since then, the problem has attracted the attention of many researchers,

e.g. see [2, 10, 17, 21, 23] but, the question for $1 < k < n - 1$ remains open. More specifically, it would be interesting to solve the following.

Problem 4.1 Determine all possible $k \times k$ principal submatrices of U^*AU for a given $n \times n$ matrix A . Equivalently, we want to find all compressions of A .

When A is Hermitian, Theorem 1.1 provides the answer. We can also study the above problem for general $A \in M_n$. Suppose $B \in M_k$ is a principal submatrix of A . Let $A = A_1 + iA_2$ and $B = B_1 + iB_2$ be the Hermitian decomposition of A and B . Then $\cos tB_1 + \sin tB_2$ is a submatrix of $\cos tA_1 + \sin tA_2$ for all $t \in \mathbb{R}$ so that their eigenvalues satisfy the interlacing inequalities (1). Thus, we have an infinite family of inequalities. But the converse is not true as shown by the following example.

Example 4.2 Let $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{bmatrix}$ and $B = \begin{bmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix}$. Then direct computation shows that

for every $t \in \mathbb{R}$, the eigenvalues of $\cos tB_1 + \sin tB_2$ and $\cos tA_1 + \sin tA_2$ satisfy the interlacing inequalities (1). Since $\|B\| = \sqrt{2} > 1 = \|A\|$, B cannot be a principal submatrix of U^*AU for any unitary U .

A principal submatrix of a normal matrix may not be normal. Nevertheless, it would be interesting to determine all $k \times k$ normal principal submatrices of U^*AU for a given normal matrix $A \in M_n$.

The results in Section 3 are motivated by problems in quantum information science. In connection to Theorem 3.6, it would be interesting to answer the following.

Problem 4.3 Determine the optimal n so that $\Lambda_{p,k}(A)$ is non-empty for any $A \in M_n$.

It is known that $\Lambda_k(A)$ is always convex, and Theorem 3.8 gives the condition for $\Lambda_{p,k}(A)$ to be convex for Hermitian A . It is natural to ask the following.

Problem 4.4 Determine $A \in M_n$ such that $\Lambda_{p,k}(A)$ is convex.

Note that even for the special case when A is normal, the problems 4.3 and 4.4 are still open and highly nontrivial.

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