Isometries for Ky Fan Norms Between Matrix Spaces

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Abstract

We characterize linear maps between different rectangular matrix spaces preserving Ky Fan norms.

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1 Introduction and statements of results

Let $M_{m,n}$ (M_n) be the linear space of $m \times n$ ($n \times n$) complex matrices. The *singular values* of $A \in M_{m,n}$ are the nonnegative square roots of the eigenvalues of A^*A, and they are denoted by $s_1(A) \geq \cdots \geq s_n(A)$. For $1 \leq k \leq \min\{m,n\}$, the *Ky Fan k-norm* on $M_{m,n}$ is defined and denoted by

$$\|A\|_k = s_1(A) + \cdots + s_k(A).$$

The Ky Fan 1-norm reduces to the *operator norm*; when $m = n$ the Ky Fan n-norm is also known as the *trace norm*.

Evidently, Ky Fan k-norms are unitarily invariant norms, i.e.,

$$\|UAV\|_k = \|A\|_k$$

for any $A \in M_{m,n}$, and unitary $U \in M_m$ and $V \in M_n$. Actually, they form an important class of unitarily invariant norms; see [1, Chapters 2 and 3]. For instance, given $A, B \in M_{m,n}$,

$$\|A\|_k \leq \|B\|_k \quad \text{for all } k = 1, \ldots, \min\{m,n\}$$

if and only if

$$\|A\| \leq \|B\| \quad \text{for all unitarily invariant norms } \|\cdot\|.$$

There has been considerable interest in studying isometries for Ky Fan norms on matrix spaces. For example, by a result of Kadison [5], one easily deduces that isometries for the operator norm on M_n has to have the from

$$A \mapsto UAV \quad \text{or} \quad A \mapsto UA^tV \tag{1}$$

for some unitary matrices $U, V \in M_n$. In [4], the authors showed that the same conclusion holds for Ky Fan k-norm isometries for any $k = 1, \ldots, \min\{m,n\}$, where the second form in (1) can occur only when $m = n$. In [8], the authors considered the problem on block

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triangular matrix algebras in M_n, and showed that the isometries essentially have the same structure except when $m = n$, in this case, the second form in (1) has to be replaced by

$$A \mapsto UA^+V,$$

where A^+ is the transpose taken about the anti-diagonal so as to maintain the block triangular structure. In [3], the authors studied isometries $\phi : (M_n, \|\cdot\|_1), \to (M_p, \|\cdot\|_1)$ for $n \neq p$, and obtained a complete characterization when $p \leq 2n-2$; moreover, examples were given to show that ϕ may have complicated structure for $p > 2n-2$. In view of these, one may think that isometries $\phi : (M_n, \|\cdot\|_k), \to (M_p, \|\cdot\|_k)$ also have complicated structure for $k > 1$. It turns out that it is not the case as shown in the corollary of our main theorem, which characterizes isometries $\phi : (M_{m,n}, \|\cdot\|_{k'}), \to (M_{p,q}, \|\cdot\|_k)$ provided $k' > 1$. We need some notations and definitions to describe our main result.

For two matrices A and B with $A = (a_{ij})$ denote by $A \otimes B = (a_{ij}B)$. An $r \times s$ matrix X is called a *partial isometry* if $X^*X = I_s$, i.e., X has orthonormal columns.

Theorem 1.1 *Let $1 < k' \leq \min\{m,n\}$ and $1 \leq k \leq \min\{p,q\}$. Suppose $\phi : M_{m,n} \to M_{p,q}$ satisfies*

$$\|\phi(A)\|_k = \|A\|_{k'} \quad \textit{for all} \quad A \in M_{m,n}. \tag{2}$$

Then there exist nonnegative integers c_1 and c_2 with $c_1 + c_2 > 0$, and partial isometries U and V of sizes $p \times (c_1m + c_2n)$ and $q \times (c_1n + c_2m)$, respectively, such that one of the following holds.

(a) *$k' < \min\{m,n\}$, $k = k'(c_1 + c_2)$, and ϕ has the form*

$$A \mapsto \frac{1}{c_1 + c_2} U[(I_{c_1} \otimes A) \oplus (I_{c_2} \otimes A)^t)]V^*.$$

(b) *$k' = \min\{m,n\}$, $k'(c_1 + c_2) \leq k$, and there are diagonal matrices $D_1 \in M_{c_1}$ and $D_2 \in M_{c_2}$ with positive diagonal entries with $\operatorname{tr} D_1 + \operatorname{tr} D_2 = 1$, such that ϕ has the form*

$$A \mapsto U[(D_1 \otimes A) \oplus (D_2 \otimes A^t)]V^*.$$

If $k' = k$, then either $(c_1, c_2) = (1, 0)$ or $(c_1, c_2) = (0, 1)$. By adding columns to U and V to form unitary matrices, we have the following corollary.

Corollary 1.2 *Let $1 < k \leq \min\{m,n\}$. Suppose $\phi : M_{m,n} \to M_{p,q}$ satisfies*

$$\|\phi(A)\|_k = \|A\|_k \quad \textit{for all} \quad A \in M_{m,n}.$$

Then there are unitary matrices $U \in M_p$ and $V \in M_q$ such that ϕ has the form

$$A \mapsto U[A \oplus 0_{p-m,q-n}]V \quad \textit{or} \quad A \mapsto U[A^t \oplus 0_{p-n,q-m}]V.$$

2 Auxiliary results and proofs

Replacing ϕ by the mapping(s) $A \mapsto \phi(A^t)$ and/or $A \mapsto [\phi(A)]^t$, we may assume that $m \leq n$ and $p \leq q$. Two nonzero matrices $A, B \in M_{m,n}$ are said to be *orthogonal* if $AB^* = 0$ and $A^*B = 0$, equivalently, there are unitary matrices U and V such that $UAV = \sum_{j=1}^{r} a_j E_{jj}$ and $UBV = \sum_{j=r+1}^{r+s} b_j E_{jj}$ with $a_1 \geq \cdots \geq a_r > 0$ and $b_1 \geq \cdots \geq b_s > 0$ for some r, s with $r + s \leq \min\{m, n\}$. The nonzero matrices $A_1, \cdots A_d \in M_{m,n}$ are said to be *pairwise orthogonal* $m \times n$ matrices if $A_i A_j^* = 0$ and $A_i^* A_j = 0$ for any distinct pair (i, j). In such case, there are unitary $U \in M_m$ and $V \in M_n$, $0 = r_0 < r_1 < \cdots < r_d \leq \min\{m, n\}$ and positive numbers $a_1, \cdots, a_{r_d}$ such that $UA_iV = \sum_{r_{i-1} < j \leq r_i} a_j E_{jj}$.

We begin with the following lemma from [8, Lemma 5].

Lemma 2.1 *Let $A, B \in M_{m,n}$ be nonzero. Then $\|aA + bB\|_k = |a|\|A\|_k + |b|\|B\|_k$ for every $a, b \in \mathbb{C}$ if and only if A and B are orthogonal and* $\operatorname{rank} A + \operatorname{rank} B \leq k$.

By Lemma 2.1 and a simple inductive argument, we have the following.

Lemma 2.2 *Let $\phi : M_{m,n} \to M_{p,q}$ be a map satisfying (2). Suppose the rank one matrices $A_1, \ldots, A_d \in M_{m,n}$, $d \leq \min\{m, n\}$, are pairwise orthogonal. Then $\phi(A_1), \ldots, \phi(A_d) \in M_{p,q}$ are nonzero and pairwise orthogonal. Furthermore, for any $1 \leq s_1 < \cdots < s_{k'} \leq d$, $\sum_{j=1}^{k'} \operatorname{rank} \phi(A_{s_j}) \leq k$.*

Proof of Theorem 1.1.

For the sufficiency part of the Theorem 1.1, one readily sees that singular values of $\phi(A)$ has $c = (c_1 + c_2)$ copies of $\dfrac{s_1(A)}{c}, \ldots, \dfrac{s_m(A)}{c}$, if ϕ has the form (a). On the other hand, if $k' = m$ and ϕ has the form (b), then $k \geq ck'$ and so the Ky Fan k-norm of $\phi(A)$ is just the sum of its singular values. Let $D_1 \oplus D_2 = \operatorname{diag}(d_1, \ldots, d_c)$. Then,

$$\|\phi(A)\|_k = d_1\|A\|_{k'} + \cdots + d_c\|A\|_{k'} = tr(D_1 \oplus D_2)\|A\|_{k'} = \|A\|_{k'}.$$

To prove the necessity part, let $(p', q') = (p - c_1 m - c_2 n, q - c_1 n - c_2 m)$. It suffices to prove that there are unitary matrices $U \in M_p$ and $V \in M_q$ such that ϕ has the form

$$(a) \qquad A \mapsto \frac{1}{c_1 + c_2} U[(I_{c_1} \otimes A) \oplus (I_{c_2} \otimes A^t) \oplus 0_{p',q'}]V^* \quad \text{if} \quad k' < m,$$

$$(b) \qquad A \mapsto U[(D_1 \otimes A) \oplus (D_2 \otimes A^t) \oplus 0_{p',q'}]V^* \quad \text{if} \quad k' = m.$$

We divide the proof into three cases:

$$\text{(I)}\ \ k' < m = n, \qquad \text{(II)}\ \ k' = m = n, \qquad \text{and} \qquad \text{(III)}\ \ m < n.$$

First consider case (I) : $k' < m = n$. For any $A \in M_{m,n}$ with singular values $1, 0, \ldots, 0$, there are unitary X and Y such that $A = XE_{11}Y$. Let $A_j = XE_{jj}Y$ for $j = 1, \ldots, m$.

Then $A_1, \ldots, A_m$ are pairwise orthogonal. By Lemma 2.2, $\phi(A_1), \ldots, \phi(A_m)$ are pairwise orthogonal. Thus, there exist unitary U and V, $0 = r_0 < r_1 < \cdots < r_d \leq m$ and positive numbers $a_1, \cdots, a_{r_d}$ such that

$$B_i = U\phi(A_i)V = \sum_{r_{i-1}<j\leq r_i} a_j E_{jj} \quad \text{for any} \quad i = 1, \ldots, m.$$

By Lemma 2.2 again, the sum of any k' matrices chosen from $B_1, \ldots, B_m$ has rank at most k. Let $1 \leq t_1 < \cdots < t_{k'} \leq m$. Then

$$s_\ell\left(\sum_{j=1}^{k'} B_{t_j}\right) = 0, \quad \text{for all } \ell > k\,. \tag{3}$$

Moreover, if $t \in \{1, \ldots, m\} \setminus \{t_1, \ldots, t_{k'}\}$, we claim that

$$s_1(B_t) \leq s_k\left(\sum_{j=1}^{k'} B_{t_j}\right). \tag{4}$$

If (4) does not hold, then $s_1(B_t) > s_k\left(\sum_{j=1}^{k'} B_{t_j}\right)$, which gives the following contradiction:

$$k' = \left\|A_t + \sum_{j=1}^{k'} A_{t_j}\right\|_{k'} = \left\|B_t + \sum_{j=1}^{k'} B_{t_j}\right\|_k > \left\|\sum_{j=1}^{k'} B_{t_j}\right\|_k = \left\|\sum_{j=1}^{k'} A_{t_j}\right\|_{k'} = k'\,.$$

Let $c = k/k'$. It follows from (2), (3) and (4) that for each $1 \leq j \leq m$, $s_i(B_j) = 1/c$ for $1 \leq i \leq c$ and $s_i(B_j) = 0$ for $c < i \leq p$. Thus, we see that

(i) every rank one matrix is mapped to a rank c matrix, and

(ii) every unitary matrix is mapped to a matrix with singular values $\underbrace{1/c, \ldots, 1/c}_{cm}, 0, \ldots, 0$.

Since (i) holds, by Theorem 2.5 in [7] ϕ has the form

$$A \mapsto R[(I_{c_1} \otimes A) \oplus (I_{c_2} \otimes A^t) \oplus 0_{p',q'}]S^*$$

for some invertible $R \in M_p$ and $S \in M_q$. Let R_1 (respectively, S_1) be obtained from R (respectively, S) by removing its last p' (respectively, q') columns. Then

$$R[(I_{c_1} \otimes A) \oplus (I_{c_2} \otimes A^t) \oplus 0_{p',q'}]S^* = R_1[(I_{c_1} \otimes A) \oplus (I_{c_2} \otimes A^t)]S_1^*.$$

By polar decomposition, there are unitary matrices $U \in M_p, V \in M_q$ and positive definite matrices $P \in M_{c_1m+c_2n}$ and $Q \in M_{c_1n+c_2m}$ such that

$$R_1 = U\begin{pmatrix} P \\ 0_{p',c_1m+c_2n} \end{pmatrix} \qquad \text{and} \qquad S_2 = V\begin{pmatrix} Q \\ 0_{q',c_1n+c_2m} \end{pmatrix}.$$

Thus,

$$\phi(X) = U\left\{P[(I_{c_1}\otimes A)\oplus(I_{c_2}\otimes A^t)]Q^*\oplus 0_{p',q'}\right\}V^*.$$

Define $\psi : M_m \to M_{cm}$ such that $\psi(X) = cP[(I_{c_1}\otimes A)\oplus(I_{c_2}\otimes A^t)]Q^*$. By (ii), we see that ψ maps unitary matrices to unitary matrices. By the result in [2], we see that $\psi(A) = W_1[(I_{c_1}\otimes A)\oplus(I_{c_2}\otimes A^t)]W_2$ for some unitary $W_1, W_2 \in M_{cm}$. Thus, condition (a) holds.

Next, we turn to case (II) : $k' = m = n$. From the first part of the proof in case (I), we can see that for any unitary $X, Y \in M_m$ and $\lambda_1,\dots,\lambda_m \in \mathbb{C}$, $\sum_{i=1}^m \lambda_i\phi(XE_{ii}Y)$ has rank at most k. Hence, $\phi(A)$ has rank at most k for all $A \in M_m$. We may assume that $p = q$ by appending $q-p$ zero rows to $\phi(A)$ for each $A \in M_m$. So, we assume that $\phi : M_m \to M_p$ and suppose $\phi(I_m) = D$ is a nonnegative diagonal matrix with diagonal entries arranged in descending order. For any Hermitian $X \in M_m$ with trace zero and spectrum in $[-1,1]$ and $t \in [-1,1]$,

$$\|\phi(I_m+tX)\|_k = \|I_m+tX\|_{k'} = k' = \|I_m\|_{k'} = \|\phi(I_m)\|_k = \operatorname{tr} D.$$

Let $Y = \phi(X)$. Then $\operatorname{tr} Y = 0$ because

$$|\operatorname{tr} D + t\operatorname{tr} Y| \le \|\phi(I_m+tX)\|_p = \|\phi(I_m+tX)\|_k = \operatorname{tr} D$$

for $t = \pm 1$. Moreover,

$$k' = \operatorname{tr}(D \pm Y) \le \|\phi(I_m+tX)\|_p = \|\phi(I_m+tX)\|_k = k'.$$

By [6, Corollary 3.2], we conclude that $D \pm Y$ is positive semi-definite. As a result, if $\phi(I_m) = D = \operatorname{diag}(d_1,\dots,d_r,0,\dots,0)$ with $d_1 \ge \cdots \ge d_r > 0$, then $\phi(X)$ has the form $Y \oplus 0_{p-r}$. We may now consider $\psi : M_m \to M_r$ such that $\phi(A) = \psi(A)\oplus 0_{p-r}$. It follows from the above argument that ψ maps Hermitian matrices to Hermitian matrices and $\|\psi(A)\|_r = \|\phi(A)\|_k = \|A\|_{k'}$. We claim that

(i) ψ maps positive semidefinite matrices to positive semidefinite matrices, and

(ii) ψ maps invertible Hermitian matrices to invertible Hermitian matrices.

To see (i), suppose $A \in M_m$ is positive semidefinite. Let $D_1 = \psi(I_m) = \operatorname{diag}(d_1,\dots,d_r)$. Choose $t > 0$ such that $D_1 + t\psi(A)$ is positive semidefinite. Then we have

$$\begin{aligned}\operatorname{tr}(D_1+t\psi(A)) &= \|D_1+t\psi(A)\|_r = \|I_m+tA\|_{k'} = \operatorname{tr}(I_m) + t\operatorname{tr}(A)\\ &= \|I_m\|_{k'} + t\|A\|_{k'} = \|\psi(I_m)\|_r + t\|\psi(A)\|_r = \operatorname{tr} D_1 + t\|\psi(A)\|_r.\end{aligned}$$

Thus, $\operatorname{tr}\psi(A) = \|\psi(A)\|_r$, and it follows form [6, Corollary 3.2] again that $\psi(A)$ is positive semidefinite.

To prove (ii), let

$$A = U^* \left(\sum_{j=1}^{m} \lambda_j E_{jj} \right) U$$

for some unitary U and $\lambda_j \in \mathrm{I\!R} \setminus \{0\}$ for $j = 1, \ldots, m$. Since $\phi(U^* E_{11} U), \ldots, \phi(U^* E_{mm} U)$ are pairwise orthogonal and $\phi(I_m) = D$, $\phi(U^* E_{jj} U) = V^* F_j V \oplus 0_{p-r}$ for $j = 1, \ldots, m$, such that $F_i = \sum_{r_{i-1} < s \le r_i} a_s E_{ss}$ for $0 = r_0 < \cdots < r_m = r$ and positive numbers $a_1, \ldots, a_{r_m}$. Therefore, $\psi(A) = V^* \left(\sum_{j=1}^{m} \lambda_j (\sum_{r_{i-1} < s \le r_i} a_s E_{ss}) \right) V$ is also invertible. Thus, condition (ii) holds.

Now, $\psi(I_m)$ is positive definite and ψ maps invertible Hermitian matrices to invertible Hermitian matrices. By (the proof of) [7, Proposition 3.4], we see that

$$\psi(X) = T^* [(I_{c_1} \otimes X) \oplus (I_{c_2} \otimes X^t)] T \tag{5}$$

for some invertible $T \in M_r$. In particular, we see that

(iii) ψ maps rank s matrices to rank cs matrices for $s = 1, \ldots, m$.

Next, we show that ψ has the form $X \mapsto U^*[(D_1 \otimes X) \oplus (D_2 \otimes X^t)]U$ for some unitary matrix U and diagonal matrices D_1 and D_2 with positive diagonal entries such that $\operatorname{tr} D_1 + \operatorname{tr} D_2 = 1$. Equivalently, we show that ψ has the form

$$A = (a_{uv}) \longmapsto V^* B V, \quad \text{where} \quad B = (B_{uv})_{1 \le u,v \le m} \quad \text{with} \quad B_{uv} = a_{uv} D_1 \oplus a_{vu} D_2$$

for some unitary V. First, by a suitable permutation, we can rewrite ψ in (5) as

$$A = (a_{uv}) \longmapsto S^* B S, \quad \text{where} \quad B = (B_{uv})_{1 \le u,v \le m} \quad \text{with} \quad B_{uv} = a_{uv} I_{c_1} \oplus a_{vu} I_{c_2} \tag{6}$$

for some nonsingular $S \in M_r$. By Lemma 2.2, we see that $\phi(E_{11}), \ldots, \phi(E_{mm})$ are pairwise orthogonal. Then for any distinct pair i and j,

$$[S^*(E_{ii} \otimes I_c) S]^* [S^*(E_{jj} \otimes I_c) S] = \psi(E_{ii})^* \psi(E_{jj}) = 0.$$

Thus, $(E_{ii} \otimes I_c) S S^* (E_{jj} \otimes I_c) = 0$ whenever $i \ne j$. It follows that $SS^* = S_1 \oplus \cdots \oplus S_n$ where $S_i \in M_c$.

Let $i > 1$, $X = E_{11} + E_{1i}$ and $Y = E_{i1} - E_{ii}$. From (6), $\psi(X) = S^*(B_{rs})S$ and $\psi(Y) = S^*(C_{rs})S$ so that

$$\tilde{B} = \begin{pmatrix} B_{11} & B_{1i} \\ B_{i1} & B_{ii} \end{pmatrix} = \begin{pmatrix} I_c & I_{c_1} \oplus 0_{c_2} \\ 0_{c_1} \oplus I_{c_2} & 0_c \end{pmatrix}, \quad \tilde{C} = \begin{pmatrix} C_{11} & C_{1i} \\ C_{i1} & C_{ii} \end{pmatrix} = \begin{pmatrix} 0_c & 0_{c_1} \oplus I_{c_2} \\ I_{c_1} \oplus 0_{c_2} & -I_c \end{pmatrix}$$

and all other B_{uv} and C_{uv} are 0_c. Let $J_1 = I_{c_1} \oplus 0_{c_2}$ and $J_2 = 0_{c_1} \oplus I_{c_2}$. Since X and Y are orthogonal, so are $\psi(X)$ and $\psi(Y)$. Hence $B^*(SS^*)C = 0$ and $B(SS^*)C^* = 0$. Thus,

$$\begin{pmatrix} J_2 S_i J_1 & S_1 J_2 - J_2 S_i \\ 0 & J_1 S_1 J_2 \end{pmatrix} = \tilde{B}^* (S_1 \oplus S_i) \tilde{C} = 0 = \tilde{B} (S_1 \oplus S_i) \tilde{C}^* = \begin{pmatrix} J_1 S_i J_2 & S_1 J_1 - J_1 S_i \\ 0 & J_2 S_1 J_1 \end{pmatrix}.$$

Since $J_2S_1J_1 = J_1S_1J_2 = J_2S_iJ_1 = J_1S_iJ_2 = 0$, each of the matrices S_1 and S_i is a direct sum of a matrix in M_{c_1} and a matrix in M_{c_2}. Furthermore, we can conclude that $S_1 = S_i = P_1 \oplus P_2$, where $P_1 \in M_{c_1}$ and $P_2 \in M_{c_2}$, from $S_1J_1 - J_1S_i = 0 = S_1J_2 - J_2S_i$. As i is arbitrary, $SS^* = I_m \otimes (P_1 \oplus P_2)$ with P_1 and P_2 are both positive definite. Thus there exist unitary $U_1 \in M_{c_1}$ and $U_2 \in M_{c_2}$ such that $U_1P_1U_1^* = D_1$ and $U_2P_2U_2^* = D_2$, where D_1 and D_2 are diagonal matrices with positive diagonal entries.

Let $U = I_m \otimes (U_1 \oplus U_2)$ and $\tilde{S} = US$. Then $\tilde{S}\tilde{S}^* = I_m \otimes (D_1 \oplus D_2)$. As the row vectors of $\tilde{S}$ form an orthogonal basis, we may write $\tilde{S} = DV$, where $D = I_m \otimes (D_1 \oplus D_2)^{1/2}$ and V is unitary.

On the other hand, we have $U^*BU = B$ for the block matrix B in (6), as

$$a_{uv}I_{c_1} \oplus a_{vu}I_{c_2} = (U_1 \oplus U_2)^*(a_{uv}I_{c_1} \oplus a_{vu}I_{c_2})(U_1 \oplus U_2).$$

Then $S^*BS = S^*U^*BUS = \tilde{S}^*B\tilde{S} = V^*D^*BDV$. In fact, the (i,j)-th block of D^*BD is equal to

$$(D_1 \oplus D_2)^{1/2}(a_{uv}I_{c_1} \oplus a_{vu}I_{c_2})(D_1 \oplus D_2)^{1/2} = a_{uv}D_1 \oplus a_{vu}D_2.$$

Thus, ϕ has the asserted form. Since $\|I_m \otimes (D_1 \oplus D_2)\|_{k'} = \|\psi(I_m)\|_r = \|I_m\|_{k'} = m$, it follows that $\operatorname{tr}(D_1 \oplus D_2) = \operatorname{tr} D_1 + \operatorname{tr} D_2 = 1$.

Finally, we consider case (III) : $m < n$. We prove the desired conclusion by induction on $n - m$ starting from $n - m = 0$, which follows from case (I) and (II). Suppose that $n - m = r > 0$ and the result holds for the cases when $n - m < r$. Applying the assumption on the restriction of ϕ on $M_{m,n}^0$, the subspace of $M_{m,n}$ which consists of matrices with zero n-th column, we conclude that for any $A \in M_{m,n}^0$,

$$\phi(A) = U[(D_1 \otimes \tilde{A}) \oplus (D_2 \otimes \tilde{A}^t) \oplus 0_{p',q'}]V$$

where $\tilde{A}$ denotes $m \times (n-1)$ matrices obtained by deleted the n-th column of A, $(p', q') = (p - c_1m - c_2(n-1), q - c_1(n-1) - c_2m)$, $U \in M_p$ and $V \in M_q$ are unitary and the following holds.

(a) If $k' < m$, $c = c_1 + c_2 = k/k'$, $D_1 = \frac{1}{c}I_{c_1}$ and $D_2 = \frac{1}{c}I_{c_2}$;

(b) If $k' = m$, $c = c_1 + c_2 \leq k/k'$, $D_1 \in M_{c_1}$ and $D_2 \in M_{c_2}$ are diagonal matrices with positive diagonal entries such that $\operatorname{tr} D_1 + \operatorname{tr} D_2 = 1$.

Now replacing ϕ by $X \mapsto U^*\phi(X)V^*$, we may assume that $U = I_p$ and $V = I_q$.

For any $\mathbf{x} \in M_{m,1}$, let A be the $m \times n$ matrix with $\mathbf{x}$ as the n-th column and zero in others, and $X = (X_{uv})_{1 \leq u,v \leq c+1} = \phi(A)$, where $X_{uu} \in M_{m,n-1}$ for $1 \leq u \leq c_1$, $X_{uu} \in M_{n-1,m}$ for $c_1 < u \leq c$ and $X_{c+1,c+1} \in M_{p',q'}$.

Take any nonzero $\mathbf{y} \in M_{m,1}$ such that $\mathbf{x}^*\mathbf{y} = 0$. (Note that $1 < k \leq m$ and hence $\mathbf{y}$ exists.) For any $l < n$, let B be the $m \times n$ matrix with $\mathbf{y}$ as the l-th column and zero in others. Then $Y = \phi(B) = (D_1 \otimes \tilde{B}) \oplus (D_2 \otimes (\tilde{B})^t) \oplus 0_{p',q'}$.

Since A and B are orthogonal, $X^*Y = 0_q$ and $XY^* = 0_p$. It follows from the structure of Y that

$$\begin{aligned}
X_{uv}^*\tilde{B} &= 0 \quad \text{when} \quad 1 \le u \le c_1 \text{ and } 1 \le v \le c+1, \\
X_{uv}^*\tilde{B}^t &= 0 \quad \text{when} \quad c_1 < u \le c \text{ and } 1 \le v \le c+1, \\
X_{uv}\tilde{B}^* &= 0 \quad \text{when} \quad 1 \le u \le c+1 \text{ and } 1 \le v \le c_1, \\
X_{uv}(\tilde{B}^t)^* &= 0 \quad \text{when} \quad 1 \le u \le c+1 \text{ and } c_1 < v \le c.
\end{aligned}$$

Since the l-th column of the $m \times (n-1)$ matrix $\tilde{B}$ is the nonzero vector y, if $X_{uv}\tilde{B}^* = 0$, then the l-th row of X_{uv} must be the zero. Furthermore, as l can be any integer in $\{1, \ldots, n-1\}$, we conclude that $X_{uv} = 0$. Similarly, X_{uv} must be the zero matrix if $X_{uv}^*\tilde{B}^t = 0$.

On the other hand, if $X_{uv}^*\tilde{B} = 0$, then all the columns of X_{uv} must be orthogonal to $\mathbf{y}$. Since $\mathbf{y}$ can be any vector orthogonal to $\mathbf{x}$, all columns of X_{uv} must be multiples of $\mathbf{x}$. Hence, $X_{uv} = \mathbf{x}\mathbf{w}^t$ for some vector $\mathbf{w}$ of suitable size. Similarly, since $X_{uv}(\tilde{B}^t)^* = 0$, we have $X_{uv} = \mathbf{z}\mathbf{x}^t$ for some $\mathbf{z}$.

By the arguments in the last two paragraphs, if $1 \le u \le c_1$ and $c_1 < v \le c$, then $\mathbf{x}\mathbf{w}^t = X_{uv} = \mathbf{z}\mathbf{x}^t$ for some $\mathbf{w}$ and $\mathbf{z}$ of suitable sizes. Thus, $\mathbf{w} = \lambda\mathbf{x}$ for some constant λ in $\mathbb{C}$. That is, $X_{uv} = \lambda\mathbf{x}\mathbf{x}^t$.

Combining the above analysis, we know that

$$\phi[0_{m,n-1} \mid \mathbf{x}] = \begin{pmatrix} 0_{c_1m,c_1n} & E(\mathbf{x}) & F(\mathbf{x}) \\ 0_{c_2n,c_1n} & 0_{c_2n,c_2m} & 0_{c_2n,q'} \\ 0_{p',c_1n} & G(\mathbf{x}) & H(\mathbf{x}) \end{pmatrix}$$

where $E(\mathbf{x}) = (\lambda_{uv}\mathbf{x}\mathbf{x}^t)_{1\le u\le c_1, 1\le v\le c_2}$, $F(\mathbf{x}) = \begin{pmatrix} \mathbf{x}\mathbf{w}_1^t \\ \vdots \\ \mathbf{x}\mathbf{w}_{c_1}^t \end{pmatrix}$, $G(\mathbf{x}) = \begin{pmatrix} \mathbf{z}_1\mathbf{w}^t & \cdots & \mathbf{z}_{c_2}\mathbf{x}^t \end{pmatrix}$, $H(\mathbf{x})$, λ_{uv}, $\mathbf{w}_u$ and $\mathbf{z}_v$ all depend on $\mathbf{x}$. By linearity of ϕ, λ_{uv}, $\mathbf{w}_u$ and $\mathbf{z}_v$ must be the same for all $\mathbf{x}$, and λ_{uv} must be zero. i.e., $E(\mathbf{x}) = 0_{c_1m,c_2m}$.

Now we consider the orthogonal pair $A = E_{11} + E_{1n}$ and $B = -E_{21} + E_{2n}$. Let $\mathbf{e}_i$ be the i-th column of I_m. Then

$$\phi(A) = \begin{pmatrix} D_1 \otimes \tilde{E}_{11} & 0_{c_1m,c_2m} & F(\mathbf{e}_1) \\ 0_{c_2n,c_1n} & D_2 \otimes \tilde{E}_{11}^t & 0_{c_2n,q'} \\ 0_{p',c_1n} & G(\mathbf{e}_1) & H(\mathbf{e}_1) \end{pmatrix}$$

and

$$\phi(B) = \begin{pmatrix} D_1 \otimes -\tilde{E}_{21} & 0_{c_1m,c_2m} & F(\mathbf{e}_2) \\ 0_{c_2n,c_1n} & D_2 \otimes -\tilde{E}_{21}^t & 0_{c_2n,q'} \\ 0_{p',c_1n} & G(\mathbf{e}_2) & H(\mathbf{e}_2) \end{pmatrix}.$$

Set $W = \begin{pmatrix} \mathbf{w}_1^t \\ \vdots \\ \mathbf{w}_{c_1}^t \end{pmatrix}$. Since $\phi(A)\phi(B)^* = 0$, the $(1,1)$-th block equals

$$\begin{aligned} 0_{c_1 m} &= (D_1 \otimes \tilde{E}_{11})(D_1 \otimes -\tilde{E}_{21})^* + F(\mathbf{e}_1)F(\mathbf{e}_2)^* \\ &= -(D_1^2 \otimes E_{12}) + (WW^* \otimes E_{12}) \\ &= (WW^* - D_1^2) \otimes E_{12}. \end{aligned}$$

Thus, $WW^* = D_1^2$. Let $D_1 = \operatorname{diag}(d_1, \ldots, d_{c_1})$. Hence, $\{\mathbf{w}_1/d_1, \ldots, \mathbf{w}_{c_1}/d_{c_1}\}$ is a set of orthonormal vectors. Let $U \in M_{q'}$ be a unitary matrix with $\mathbf{w}_1^t/d_1, \ldots, \mathbf{w}_{c_1}^t/d_{c_1}$ as the first c_1 rows. Then $F'(\mathbf{x}) = F(\mathbf{x})U^* = [D_1 \otimes \mathbf{x} \,|\, 0_{c_1 m, q'-c_1}]$.

Similarly, by considering $\phi(A)^*\phi(B) = 0$, we write $G'(\mathbf{x}) = V^* G(\mathbf{x}) = \begin{pmatrix} D_2 \otimes \mathbf{x}^t \\ 0_{p'-c_2, c_2 m} \end{pmatrix}$ for some unitary V. Now, we write

$$\phi[0_{m,n-1} \mid \mathbf{x}] \;=\; (I_{cn} \oplus V) \begin{pmatrix} 0_{c_1 m, c_1 n} & 0_{c_1 m, c_2 m} & F'(\mathbf{x}) \\ 0_{c_2 n, c_1 n} & 0_{c_2 n, c_2 m} & 0_{c_2 n, q'} \\ 0_{p', c_1 n} & G'(\mathbf{x}) & H'(\mathbf{x}) \end{pmatrix} (I_{cn} \oplus U).$$

On the other hand, by applying the assumption on the restriction of ϕ on the subspace of $M_{m,n}$ which consists of matrices with zero in the $(n-1)$-th column, we conclude that

$$\operatorname{rank} \phi[0_{m,n-1} \,|\, \mathbf{x}] = \operatorname{rank} \phi[\mathbf{x} \,|\, 0_{m,n-2} \mid \mathbf{x}] = \operatorname{rank} \phi[\mathbf{x} \,|\, 0_{m,n-1}] = c.$$

(Note that here we use that fact that $n > m \geq 2$ to ensure nontrivial consideration.) Therefore, $H'(\mathbf{x}) = 0$ for all $\mathbf{x}$. Finally, there exist permutation matrices P and Q such that for $A = [0_{m,n-1} \,|\, \mathbf{x}]$,

$$\phi(A) = (I_{cn} \oplus V)P[(D_1 \otimes A) \oplus (D_2 \otimes A^t) \oplus 0_{p'-c_2, q'-c_1}]Q(I_{cn} \oplus U).$$

The result follows. □

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