On The Inverse Mean First Passage Matrix Problem And The Inverse M–Matrix Problem

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Abstract

The *inverse mean first passage time problem* is given a positive matrix $M \in \mathbb{R}^{n,n}$, then when does there exist an n–state discrete–time homogeneous ergodic Markov chain $\mathcal{C}$, whose mean first passage matrix is M? The *inverse M–matrix problem* is given a nonnegative matrix A, then when is A an inverse of an M–matrix. The main thrust of this paper is to show that the existence of a solution to one of the problems can be characterized by the existence of a solution to the other. In so doing we extend earlier results of Tetali and Fiedler.

Keywords: Markov chains, stationary distribution, mean first passage times, nonnegative matrices, diagonally dominant M–matrices, inverse M–matrices.
AMS Classification: 60J10, 60J20, 15A51, 15A48.

1 Introduction

In this paper a *Markov chain*[1] shall always be taken to mean *a finite–state discrete–time homogeneous ergodic Markov chain.*

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[1]For more background material on Markov chains see the books by Feller [6], by Kemeny and Snell [12], and by Seneta [21].

Suppose that $\mathcal{C}$ is a Markov chain on n states. For $1 \leq i, j \leq n$, the *mean first passage* (MFP) *time from state i to state j*, denoted by $m_{i,j}$, is the expected number of time steps for reaching state j for the first time, when initially the chain was in state i. The matrix $M = (m_{i,j})$ is called the *MFP matrix of the chain*. We define the *inverse mean first passage matrix problem* as follows: Given an $n \times n$ matrix $M = (m_{i,j})$ whose entries are all positive numbers, then when does there exist a Markov chain $\mathcal{C}$ on n states such that M is its MFP matrix?

We comment that for Markov chains, MFP times give us an idea about the short range behavior of the chain. For example, if we arrive at a holiday destination and the weather is rainy, we are less interested in the average number of days per year which are rainy or sunny, respectively, but rather we are interested in the expected time that it will take the weather to turn sunny, given that it is now rainy. In the context of random walks, MFP times are sometimes called *mean hitting times*, see Karlin, Lindqvist, and Yao [11] and Tetali [22]. In a recent article in the journal of Nature, [4], Condamin, Bńichou, Tejedor, Voituriez, and Klafter, explain that MFP times can answer such questions as *how long will it take a random walker to reach a given target*?[2] In view of the aforementioned applications of MFP times, it is interesting to note that MFP times can be used for other purposes too, such as in connection with *condition numbers for Markov chains* which are uses in the estimation of the error in computing the stationary distribution vector of the chain, see Cho and Meyer [3].

The *inverse M–matrix problem* is defined as follows: Suppose that A is an $n \times n$ nonnegative matrix[3]. Then when is A an inverse of an M–matrix, that is when does there exist a nonnegative matrix B and a scalar $s > \rho(B)$, the spectral radius of B, such that $A = (sI - B)^{-1}$? We mention that there are many papers which study the inverse M–matrix problem. Here we give a very partial list: Elsner, Neumann, and Nabben [5], Hogben [8, 9], Johnson [10], Koltracht and Neumann [14], Lewin and Neumann [15], Markham [16], Martinez, Michon, and Zhang [17], McDonald, Neumann, Schneider,

[2]We mention that methods for computing MFP times for random walks whose underlying graph is a tree were developed by Kirkland and Neumann in [13].

[3]For more background material on nonnegative matrices and M–matrices see the book by Berman and Plemmons [1].

and Tsatsomeros [18], and Nabben and Varga [20].

There are three papers from the 1990s by Tetali [22], by Fiedler [7], and by Xue [24], that hint directly or indirectly on a connection between the inverse MFP matrix problem and the inverse M–matrix problem, when the M–matrix in question is diagonally dominant. In this paper we shall make that connection more explicit and we shall show that the existence of a solution to one can be characterized by the existence of a solution to the other.

A word about our notation. For an $n \times n$ matrix B, B_k will denote the $(n-1) \times (n-1)$ principal submatrix of B obtained by deleting its k–th row and column. The matrices I and J will denote, respectively, the identity matrix and the matrix of all 1's. Their dimensions will only be indicated when they are not clear from the context. Finally, for a matrix $X \in \mathbb{R}^{n,n}$, $X_{\rm diag}$ is the diagonal matrix whose diagonal entries are the corresponding diagonal entries of X.

The most explicit paper of the three papers hinting upon the connection is the one by Tetali. Tetali's most relevant result to our work here is the following:

Theorem 1.1 (Tetali [22, Theorem 2.1]) *Let $T = (t_{i,j}) \in \mathbb{R}^{n,n}$ be a transition matrix for a Markov chain $\mathcal{C}$ whose diagonal entries are all 0, let $A = I - T$, and suppose that $\Pi = \mathrm{diag}(\pi_1, \ldots, \pi_n)$, where $\pi = (\pi_1, \ldots, \pi_n)^t$ is the stationary distribution of the chain, that is $\pi^t T = \pi^t$ and $\|\pi\|_1 = 1$. Let $M = (m_{i,j})$ be the MFP matrix of the chain. Then*

$$(\Pi_n A_n) H^{(n)} \;=\; I_{n-1}, \tag{1.1}$$

where $H^{(n)} = (h_{i,j}) \in \mathbb{R}^{n-1,n-1}$ is the matrix whose elements are given by:

$$h_{i,j} \;=\; \begin{cases} m_{i,n} + m_{n,i}, & \text{if } i = j, \\ \\ m_{i,n} + m_{n,j} - m_{i,j}, & \text{if } i \neq j. \end{cases} \tag{1.2}$$

We immediately note that as A_n is a row diagonally dominant M–matrix (of order $(n-1) \times (n-1)$), then so is $\Pi_n A_n$. Thus $H^{(n)}$ is an inverse of a diagonally dominant M–matrix and we see that Tetali's Theorem 1.1 suggests a link between MFP matrices and inverses of row diagonally dominant M–matrices and hence a connection between the inverse MFP matrix problem and the inverse M–matrix problem.

Before we proceed, let us observe that the matrix $H^{(n)}$ satisfying condition (1.1) is not only the inverse of a row diagonally dominant M–matrix, but actually, **also**, the *inverse of a column diagonally dominant* M–matrix. To see this partition the stationary vector π as $\pi = [\bar{\pi}^t \ \pi_n]^t$, where $\bar{\pi} \in \mathbb{R}^{n-1}$, and observe first that because $\pi^t A = 0$ and A is an M–matrix, we can write that: $\bar{\pi}^t A_n = -\pi_n[a_{n,1} \dots a_{n,n-1}] \geq 0$. Next, note that from (1.1), $\Pi_n A_n = \left(H^{(n)}\right)^{-1}$ and so we can write that:

$$e^t \left(H^{(n)}\right)^{-1} \ = \ e^t \Pi_n A_n \ = \ \bar{\pi}^t A_n \ \geq \ 0. \tag{1.3}$$

In Section 2 we study the matrix $(M - M_{\text{diag}})^{-1}$ which arises when considering a necessary and sufficient condition, essentially due to Kemeny and Snell, for an $n \times n$ positive matrix M to solve the inverse MFP problem. Our main results are developed in Section 3. We begin by generalizing Tetali's Theorem 1.1 to any Markov chain $\mathcal{C}$, not just one possessing a transition matrix with a zero diagonal. We continue by finding two sets of equivalent conditions for a matrix H to satisfy (1.2) for some Markov chain $\mathcal{C}$. These results lead us to a corollary giving necessary and sufficient conditions for a nonnegative matrix to be the inverse of a row and column diagonally dominant M–matrix. In Section 4, we show that our results here extend also Fiedler's characterization for a symmetric nonnegative matrix arising in resistive networks to be the inverse of a diagonally dominant M–matrix.

For historical reasons the authors want to recall here Varga's well known paper [23] on diagonal dominance which itself pays tribute to the earlier contributions to the subject by Ostrowski and by Olga Taussky Todd.

2 Necessary and Sufficient Conditions for the Solution to the Inverse MFP Problem

In this section, we consider a matrix, which will be denoted by N, which occurs in a necessary and sufficient condition, essentially due to Kemeny and

Snell, for a positive matrix M to be the MFP matrix of some Markov chain $\mathcal{C}$ with a transition matrix T.

Suppose now that $M \in \mathbb{R}^{n,n}$ is a positive matrix. Let $N := M - M_{\text{diag}}$. If M is the MFP matrix for some Markov chain $\mathcal{C}$, then according to Kemeny and Snell [12, pp.81], N is an invertible matrix. Kemeny and Snell show further [12, Theorem 4.4.12] that in this case the transition matrix for the chain is given by:

$$\hat{T} \;=\; I + (M_{\text{diag}} - J)N^{-1}. \tag{2.4}$$

Indeed, Meyer [19] shows that the MFP matrix is the unique solution to the matrix equation

$$(I - T)X \;=\; J - TX_{\text{diag}}\,, \tag{2.5}$$

where X is in $\mathbb{R}^{n,n}$, and it is easy to check that the matrix $\hat{T}$ given in (2.4) satisfies equation (2.5).

The following equivalence is implicit in the book of Kemeny and Snell:

Theorem 2.1 *Let $M \in \mathbb{R}^{n,n}$ be a positive matrix, set $N := M - M_{\text{diag}}$. Then M is the MFP matrix for some Markov chain $\mathcal{C}$ whose transition matrix is T if and only if N is invertible and the matrix $\hat{T}$, given in* (2.4)*, is nonnegative, irreducible, and stochastic. In this case, $T = \hat{T}$.*

Proof. The "if" part of the theorem is just the result of Kemeny and Snell, for if M is an MFP matrix for some Markov chain $\mathcal{C}$. then its transition matrix is given by (2.4).

To prove the "only if" part suppose that M is a positive matrix, $N := M - M_{\text{diag}}$ is an invertible matrix, and the matrix $\hat{T}$, given in (2.4), is nonnegative, irreducible, and stochastic. Let $\hat{M}$ be the MFP matrix induced via transition matrix $\hat{T}$. Then $\hat{M}$ is the unique matrix in $\mathbb{R}^{n,n}$ satisfying the matrix equation

$$(I - \hat{T})X \;=\; J - \hat{T}X_{\text{diag}}.$$

However, as can be readily checked, M too satisfies this equation and hence $M = \hat{M}$, showing that M is an MFP matrix and the proof is complete. $\square$

An immediate, but interesting, corollary of the above theorem is the following:

Corollary 2.2 *Suppose that* $N \in \mathbb{R}^{n,n}$ *is a nonnegative invertible matrix with zero diagonal entries. Let* $N^{-1} = (p_{i,j})$*. Then* $N = M - M_{\rm diag}$ *for some MFP matrix* M *of a Markov chain* $\mathcal{C}$ *on* n *states if and only if*

$$\begin{cases} \sum_{k=1}^n p_{i,k} \ > \ 0, \quad \text{for all } i = 1, \ldots, n, \\ p_{i,j} \ \geq \ \dfrac{\sum_{k=1}^n p_{i,k} \sum_{k=1}^n p_{k,j}}{\sum_{1 \leq k,\ell \leq n} p_{k,\ell}}, \quad \text{for all } i \neq j, \ i,j = 1, \ldots, n, \\ p_{i,i} \dfrac{\sum_{1 \leq k,\ell \leq n} p_{k,\ell}}{\sum_{k=1}^n p_{i,k}} - \displaystyle\sum_{k=1}^n p_{k,i} \ \geq \ -1, \quad \text{for all } i = 1, \ldots, n, \end{cases} \tag{2.6}$$

and the (any) $n \times n$ *matrix* $G = (g_{i,j})$ *whose off–diagonal entries are given by:*

$$g_{i,j} \ = \ p_{i,j} - \frac{\sum_{k=1}^n p_{i,k} \sum_{k=1}^n p_{k,j}}{\sum_{1 \leq k,\ell \leq n} p_{k,\ell}}, \quad \text{for all } i \neq j, \ i,j = 1, \ldots, n, \tag{2.7}$$

is irreducible.

Proof. Suppose that (2.6) holds. Set

$$\pi_i \ := \ \frac{\sum_{k=1}^n p_{i,k}}{\sum_{1 \leq k,\ell \leq n} p_{k,\ell}}, \quad \text{for} \quad i = 1, \ldots, n. \tag{2.8}$$

Then $\sum_{i=1}^n \pi_i = 1$. Now let $D = \text{diag}\,(\pi_1^{-1}, \ldots, \pi_n^{-1})$. From (2.6) and (2.8) it readily follows that the off–diagonal entries of $(D - J)N^{-1}$ are nonnegative while the row sums of $(D-J)N^{-1}$ are all 0. Furthermore, from the third condition in (2.6) and from (2.8) it follows that the diagonal entries of $(D-J)N^{-1}$ are bounded below by -1. Hence the matrix $T := I + (D - J)N^{-1}$ is nonnegative and its row sums are all 1 and so it is stochastic. Moreover, from the definition of T we readily observe that $t_{i,j} = g_{i,j}$, for all $i, j = 1, \ldots, n$ with $i \neq j$, showing that T is an irreducible matrix. Thus, by Theorem 2.1, $D + N$ is the MFP matrix induced by T.

Now suppose that $N = M - M_{\text{diag}}$ for some MFP matrix M. Again denote the entries of N^{-1} by $p_{i,j}$, $i,j = 1,\ldots,n$. Then, by Theorem 2.1, $T = I + (M_{\text{diag}} - J)N^{-1}$ is an irreducible transition matrix for some Markov chain $\mathcal{C}$ so that, in particular, the off–diagonal entries of T which are given by

$$m_{i,i}p_{i,j} - \sum_{k=1}^{n} p_{k,j} \;=\; \frac{1}{\pi_i}p_{i,j} - \sum_{k=1}^{n} p_{k,j}, \quad \text{for all } i \neq j, \tag{2.9}$$

are nonnegative. Now, from the equality $Te = e$ we have at once that

$$(M_{\text{diag}} - J)N^{-1}e \;=\; 0. \tag{2.10}$$

A careful analysis of (2.10) together with the representation of the off–diagonal entries of the matrix $(M_{\text{diag}} - J)N^{-1}$ as given in (2.9) now yield the second inequality in (2.6) and they imply the fact that the entries the stationary distribution vector π satisfy the equalities given in (2.8).

Next, the diagonal entries of $(M_{\text{diag}} - J)N^{-1}$ are given by:

$$p_{i,i}\frac{1}{\pi_i} - \sum_{k=1}^{n} p_{k,i} \;=\; m_{i,i}p_{i,i} - \sum_{k=1}^{n} p_{k,i}, \quad i = 1,\ldots,n.$$

Thus, as $T = I + (M_{\text{diag}} - J)N^{-1}$ is nonnegative and stochastic, we get at once, using the fact that the entries of π satisfy the equalities in (2.8), that the third inequality in (2.6) holds. Moreover, as T is irreducible, the matrix G whose off–diagonal entries are given in (2.7) must also be irreducible.

Finally, from (2.10) it is also possible to deduce that:

$$(e^t N^{-1} e) N M_{\text{diag}}^{-1} e \;=\; e \;> 0.$$

Thus, in particular, $e^t N^{-1} e > 0$ and so

$$N^{-1}e \;=\; (e^t N^{-1} e) M_{\text{diag}}^{-1} e \;>\; 0,$$

from which the first inequality in (2.6) follows. $\square$

Remark 2.3 The column sums of N^{-1} need not be nonnegative as shown by the following example: Let

$$T = \begin{pmatrix} 0.2451 & 0.06011 & 0.0218 & 0.108 & 0.5256 & 0.03944 \\ 0.2038 & 0.02206 & 0.1188 & 0.0373 & 0.124 & 0.494 \\ 0.09678 & 0.1925 & 0.1872 & 0.1904 & 0.1184 & 0.2148 \\ 0.09864 & 0.003722 & 0.02683 & 0.01902 & 0.7317 & 0.1201 \\ 0.181 & 0.1096 & 0.1145 & 0.09044 & 0.3715 & 0.1329 \\ 0.03331 & 0.05538 & 0.08807 & 0.314 & 0.505 & 0.004243 \end{pmatrix}.$$

Then

$$N^{-1} = \begin{pmatrix} -0.1036 & 0.02709 & 0.02377 & 0.02534 & 0.05179 & 0.007414 \\ 0.02385 & -0.07107 & 0.02062 & 0.007651 & -0.005539 & 0.04137 \\ 0.01701 & 0.02862 & -0.06344 & 0.02291 & -0.006784 & 0.02079 \\ 0.02216 & 0.01422 & 0.01903 & -0.1111 & 0.0649 & 0.01543 \\ 0.1102 & 0.09263 & 0.1017 & 0.06031 & -0.3385 & 0.05835 \\ 0.01629 & 0.02325 & 0.03003 & 0.05052 & 0.04287 & -0.1349 \end{pmatrix},$$

in which case $\sum_{i=1}^{6} p_{i,5} = -0.1912$.

3 Connection to the Inverses of Row and Column Diagonally Dominant M–matrices

In this section we generalize Tetali's Theorem 1.1 to transition matrices T not constrained to have zero diagonal entries and connect the entries of the inverse of $(I-T)_n$ to the entries of the MFP matrix induced by T. In terms of notation, for an $n \times n$ matrix B, we shall continue in this section with the notation, introduced in Section 1, that for each $k = 1, \ldots, n$, B_k is the $(n-1) \times (n-1)$ principal submatrix of B obtained by deleting its k–th row and column.

Given a positive matrix $M = (m_{i,j}) \in \mathbb{R}^{n \times n}$, define the $(n-1) \times (n-1)$

matrix $H^{(n)} = (h_{i,j})$ by

$$h_{i,j} \;=\; \begin{cases} m_{i,n} + m_{n,i}, & \text{if } i = j, \\ m_{i,n} + m_{n,j} - m_{i,j}, & \text{if } i \neq j, \end{cases} \qquad \text{for all} \quad i, j = 1, \ldots, n-1. \tag{3.11}$$

Next, for each $k = 1, \ldots, n$, define the $(n-1) \times n$ matrix by

$$P^{(k)} \;=\; \begin{bmatrix} e_1 & \cdots & e_{k-1} & -e & e_k & \cdots & e_{n-1} \end{bmatrix}.$$

Notice that $P^{(k)}e = 0$, for all $k = 1, \ldots, n$. In particular, we have that $P^{(n)} = \begin{bmatrix} I_{n-1} & -e \end{bmatrix}$ and it can be checked that the equality in (3.11) is equivalent to

$$H^{(n)} \;=\; -P^{(n)}(M - M_{\text{diag}})P^{(n)t}. \tag{3.12}$$

In [22, Theorem 2.1] Tetali shows that if $M = (m_{i,j})$ is the MFP matrix for a transition matrix T having zero diagonal entries with stationary vector $\pi = (\pi_1, \ldots, \pi_n)^t$, then

$$\Pi_n A_n H^{(n)} \;=\; I,$$

where $A = I - T$ and $\Pi = \text{diag}\,(\pi_1, \ldots, \pi_n)$. The following theorem generalizes Telatli's Theorem 1.1 to an arbitrary transition matrix.

Theorem 3.1 *Suppose that T is the transition matrix of a Markov chain $\mathcal{C}$ on n states with the MFP matrix M and the stationary distribution vector $\pi = (\pi_1, \ldots, \pi_n)^t$. Let $A = I - T$ and set $\Pi = \text{diag}\,(\pi_1, \ldots, \pi_n)$. For $k = 1, \ldots, n$, define*

$$H^{(k)} \;=\; -P^{(k)}(M - M_{\text{diag}})P^{(k)t}.$$

Then

$$\Pi_k A_k H^{(k)} \;=\; I.$$

Proof. It suffices to prove the result for the case when $k = n$. Let $A = I - T$. By (2.5) we have that

$$\Pi A(M - M_{\text{diag}}) \;=\; \Pi J - I.$$

If in the above matrix equality we consider the first $n-1$ rows and use the fact that the first $n-1$ rows of ΠA are equal to

$$\begin{bmatrix}\Pi_n A_n & -\Pi_n A_n e\end{bmatrix} = \Pi_n A_n P^{(n)},$$

then we obtain that

$$\begin{aligned} & \Pi_n A_n P^{(n)}(M - M_{\text{diag}}) = \Pi_n J_{n-1,n} - \begin{bmatrix} I & 0 \end{bmatrix} \\ \Rightarrow \quad & \Pi_n A_n P^{(n)}(M - M_{\text{diag}}) P^{(n)t} = \Pi^{(n)} J_{n-1,n} P^{(n)t} - \begin{bmatrix} I & 0 \end{bmatrix} P^{(n)t} \\ \Rightarrow \quad & \Pi_n A_n(-H^{(n)}) = -I. \end{aligned}$$

Noting that $P^{(n)} J_{n,n-1} = 0$, the result now follows. □

We remark that from Theorem 3.1 we have that

$$A_k^{-1} = H^{(k)}\Pi_k = P^{(k)}(M - M_{\text{diag}})P^{(k)t}\Pi^{(k)}, \quad \text{for all } k = 1, \ldots, n. \tag{3.13}$$

Now for each $k = 1, \ldots, n-1$, define the $(n-1) \times (n-1)$ matrix:

$$Q^{(k)} = \begin{bmatrix} e_1 & \cdots & e_{k-1} & -e & e_k & \cdots & e_{n-2} \end{bmatrix}.$$

Said otherwise, $Q^{(k)}$ is the matrix obtained from $P^{(k)}$ by deleting its n–th column. Furthermore, one can check that $P^{(k)} = Q^{(k)}P^{(n)}$, for all $k = 1, \ldots, n-1$. But then (3.13) implies that

$$A_k^{-1} = Q^{(k)}P^{(n)}(M - M_{\text{diag}})P^{(n)t}Q^{(k)t}\Pi_k = Q^{(k)}A_n^{-1}\Pi_n^{-1}Q^{(k)t}\Pi_k.$$

Put $R^{(k)} := \Pi_k Q^{(k)} \Pi_n^{-1}$. Then:

$$R^{(k)} = \begin{bmatrix} e_1 & \cdots & e_{k-1} & -\pi_k^{-1}\Pi_k e & e_k & \cdots & e_{n-2} \end{bmatrix}.$$

Furthermore we have that:

$$A_k^{-1} = Q^{(k)}A_n^{-1}R^{(k)t}, \quad \text{for all} \quad k = 1, \ldots, n-1. \tag{3.14}$$

We comment that case $k = 1$ in (3.14) is a result of Xue in [24] and, in essence, the entire observation (3.14) should be attributed to Xue.

We shall next find a necessary and sufficient condition for a matrix $H \in \mathbb{R}^{n-1,n-1}$ to satisfy that for some positive matrix M, condition (3.12) holds.

Theorem 3.2 *Let $H \in \mathbb{R}^{(n-1),(n-1)}$. Then the following are equivalent:*

(a) *H is invertible, H^{-1} is a row and column diagonally dominant M–matrix, and*

$$\operatorname{tr}(I+J)H^{-1} \;\leq\; 1.$$

(b) *There exists a Markov chain $\mathcal{C}$ on n states with a transition matrix $T \in \mathbb{R}^{n,n}$ and a stationary vector $\pi = (\pi_1, \ldots, \pi_n)^t$ such that*

$$\Pi_n A_n H \;=\; I, \tag{3.15}$$

where $A = I - T$.

(c) *There exists an MFP matrix M of a Markov chain $\mathcal{C}$ such that*

$$H \;=\; -P^{(n)}(M - M_{\text{diag}})P^{(n)t}.$$

is a positive matrix.

Proof. The equivalence for (b) and (c) can be easily deduced from Theorem 3.1. It remains to show that (a) and (b) are equivalent.

Suppose first that (b) holds so that, by (3.15), $H^{-1} = \Pi_n A_n = \Pi_n(I-T)_n$. Now by arguments similar to the ones presented following Tetali's Theorem 1.1 through (3.15) , we see that again H^{-1} is a row and column diagonally dominant M–matrix. Furthermore,

$$\operatorname{tr} H^{-1} \;=\; \operatorname{tr}\Pi_n - \operatorname{tr}\Pi_n T_n \;\leq\; \operatorname{tr}\Pi_n \;=\; 1 - \pi_n$$

and hence

$$\operatorname{tr} JH^{-1} \;=\; e^t H^{-1} e \;=\; e^t \Pi_n (I - T_n) e \;=\; \pi_n(1 - t_{n,n}) \;\leq\; \pi_n.$$

Thus, $\operatorname{tr}(I+J)H^{-1} \leq 1$.

Suppose next that H^{-1} is a row and column diagonally dominant M–matrix satisfying the trace inequality in (a). Note that as H^{-1} is column diagonally dominant, trace$(JH^{-1}) \geq 0$. In fact, the nonsingularity of H^{-1}

implies that at least one diagonal entry of JH^{-1} must be positive and so we can write that:

$$\operatorname{trace}\left(H^{-1}\right) \;<\; \operatorname{trace}\left(H^{-1}\right) + \operatorname{trace}\left(JH^{-1}\right) \;\leq\; \operatorname{trace}\left((I+J)H^{-1}\right).$$

Let $d_1, \ldots, d_{n-1}$ be the diagonal entries of H^{-1}. Then, due to our assumption that $\operatorname{trace}((I+J)H^{-1}) \leq 1$, we see that $\sum_{i=1}^{n-1} d_i < 1$. We can now choose positive numbers $\pi_1, \ldots, \pi_n$, with $\sum_{j=1}^{n} \pi_j = 1$, such that

$$\pi_j \;\geq\; d_j, \quad \text{for } j = 1, \ldots, n-1. \tag{3.16}$$

Set $\pi = (\pi_1, \ldots, \pi_n)^t$, $\Pi := \operatorname{diag}(\pi_1, \ldots, \pi_n)$, and

$$T \;:=\; I - \Pi^{-1} P^{(n)t} H^{-1} P^{(n)} \;=\; I - \Pi^{-1} \begin{bmatrix} H^{-1} & -H^{-1}e \\ -e^t H^{-1} & e^t H^{-1} e \end{bmatrix}. \tag{3.17}$$

One can readily check that T is nonnegative, $Te = e$, and $\pi^t T = \pi^t$. To see that T is irreducible, note first that since H nonsingular, the vectors $e^t H^{-1}$ and $H^{-1}e$, which are nonnegative by virtue of H^{-1}s being a row and column diagonally dominant M–matrix, are (also) nonzero. Hence, for some $1 \leq k \leq n-1$, $t_{n,k} = (e^t H^{-1})_k > 0$ and, for some $1 \leq j \leq n-1$, $t_{j,n} = (H^{-1}e)_j > 0$. Suppose that T is reducible. Then one can find a permutation matrix of the form $P = Q \oplus [1]$ such that PTP^t has the form $\left[\begin{smallmatrix} \tilde{T}_{1,1} & \tilde{T}_{1,2} \\ \tilde{T}_{2,1} & \tilde{T}_{2,2} \end{smallmatrix}\right]$, where at least one of $\tilde{T}_{1,2}$ or $\tilde{T}_{1,2}$ is a zero block and $\tilde{T}_{2,2}$ has dimension $k \times k$, with $k \geq 2$ due to the non–zeroness of the last row and column of T. Without loss of generality we can assume that P is the identity matrix. Partition $QH^{-1}Q^t = H^{-1}$ in the form $\left[\begin{smallmatrix} (H^{-1})_{1,1} & (H^{-1})_{1,2} \\ (H^{-1})_{2,1} & (H^{-1})_{2,2} \end{smallmatrix}\right]$, where $(H^{-1})_{1,1} \in \mathbb{R}^{n-k,n-k}$. Then the zero pattern of $PTP^t = T$ now implies that either (i) $(H^{-1})_{1,2} = 0$ and $(H^{-1})_{1,1}e = 0$ or (ii) $(H^{-1})_{2,1} = 0$ and $e^t(H^{-1})_{1,1} = 0$. Suppose that (i) holds. Then as $(H^{-1})_{1,1}$ is invertible, $(H^{-1})_{1,1}e \neq 0$, which is not possible. The same argument follows if case (ii) holds. Hence T must be irreducible. Thus T is a transition matrix for some Markov chain $\mathcal{C}$ whose stationary distribution is the vector π. Furthermore, we have that $\Pi_n A_n = H^{-1}$, where $A = I - T$.

□

Remark 3.3 A few comments on Theorem 3.2 are in place.

a) The equivalences in the theorem continues to hold if we replace A_n and $P^{(n)}$ by A_k and $P^{(k)}$, for all $k = 1, \ldots, n-1$, respectively.

b) Given a transition matrix T for a Markov chain $\mathcal{C}$, then T uniquely determines an MFP matrix M and, through (3.12), M, in turn, uniquely determines $H^{(n)}$ and hence it also uniquely determines $(H^n)^{-1}$.

   We can therefore ask the converse question: *Given a matrix* $H = H^{(n)}$ *satisfying the conditions* (a) *in Theorem* 3.2*, to what extent does it determine uniquely a transition matrix* T *for a Markov chain* $\mathcal{C}$?

   From the proof that (a) implies (b) in the above theorem it can be readily seen that if $\mathrm{tr}\,(I+J)H^{-1} \leq 1$, so that the sum of the diagonal entries of H^{-1}, namely, $\sum_{i=1}^{n-1} d_i < 1$, then each choice of $\pi_1, \ldots, \pi_n$, satisfying $\pi_j \geq d_j$, $j = 1, \ldots, n-1$, and $\sum_{j=1}^{n} \pi_j = 1$, generates, via (3.17), a different transition matrix T. As an example let us take:

$$H = \begin{pmatrix} 19.18 & 10.17 & 12.28 & 14.42 \\ 13.01 & 17.61 & 16.19 & 14.65 \\ 10.12 & 10.69 & 23.7 & 14.15 \\ 9.737 & 7.639 & 11.14 & 16.44 \end{pmatrix}.$$

   Then as can be checked that

$$H^{-1} = \begin{pmatrix} 0.1082 & -0.033 & -0.004525 & -0.06159 \\ -0.04286 & 0.1245 & -0.04768 & -0.03233 \\ -0.0008424 & -0.03229 & 0.08538 & -0.04396 \\ -0.04358 & -0.01642 & -0.03303 & 0.1421 \end{pmatrix}$$

   is an invertible row and column diagonally dominant matrix with $\mathrm{trace}((I+J)H^{-1}) = 0.5283$.

   Next let $\pi_1 = [0.2, 0.2, 0.2, 0.18, 0.22]$ and $\pi_2 = [0.2, 0.2, 0.2, 0.25, 0.15]$ and we see that both vectors satisfy the condition (3.16) with respect to the diagonal entries of H^{-1}. Put $\Pi_1 = \mathrm{diag}(\pi_1)$ and $\Pi_2 = \mathrm{diag}(\pi_2)$. Then using (3.17) we obtain two stochastic matrices:

$$T_1 = \begin{pmatrix} 0.4591 & 0.165 & 0.02262 & 0.308 & 0.0453 \\ 0.2143 & 0.3774 & 0.2384 & 0.1617 & 0.008151 \\ 0.004212 & 0.1614 & 0.5731 & 0.2198 & 0.04145 \\ 0.2421 & 0.09125 & 0.1835 & 0.2104 & 0.2727 \\ 0.09496 & 0.1945 & 0.0006288 & 0.01926 & 0.6906 \end{pmatrix}$$

and

$$T_2 = \begin{pmatrix} 0.4591 & 0.165 & 0.02262 & 0.308 & 0.0453 \\ 0.2143 & 0.3774 & 0.2384 & 0.1617 & 0.008151 \\ 0.004212 & 0.1614 & 0.5731 & 0.2198 & 0.04145 \\ 0.1743 & 0.0657 & 0.1321 & 0.4315 & 0.1963 \\ 0.1393 & 0.2853 & 0.0009222 & 0.02825 & 0.5463 \end{pmatrix},$$

corresponding to Π_1 and Π_2, respectively. On computing the MFP matrices induced by the transition matrices T_1 and T_2 we obtain that:

$$M_1 = \begin{pmatrix} 5.0 & 6.363 & 10.89 & 4.276 & 10.92 \\ 7.251 & 5.0 & 8.052 & 5.125 & 12.0 \\ 9.586 & 6.366 & 5.0 & 5.078 & 11.45 \\ 7.184 & 6.635 & 9.767 & 5.556 & 8.663 \\ 8.258 & 5.611 & 12.25 & 7.777 & 4.545 \end{pmatrix}$$

and

$$M_2 = \begin{pmatrix} 5.0 & 6.347 & 10.91 & 4.135 & 11.93 \\ 7.267 & 5.0 & 8.088 & 5.0 & 13.03 \\ 9.566 & 6.331 & 5.0 & 4.918 & 12.44 \\ 7.326 & 6.76 & 9.927 & 4.0 & 9.814 \\ 7.249 & 4.586 & 11.26 & 6.627 & 6.667 \end{pmatrix},$$

respectively. A computation now shows that

$$\begin{aligned} -P^{(5)}(M_1 - (M_1)_{\mathrm{diag}})P^{(5)t} &= -P^{(5)}(M_2 - (M_2)_{\mathrm{diag}})P^{(5)t} \\ &= \begin{pmatrix} 19.18 & 10.17 & 12.28 & 14.42 \\ 13.01 & 17.61 & 16.19 & 14.65 \\ 10.12 & 10.69 & 23.7 & 14.15 \\ 9.737 & 7.639 & 11.14 & 16.44 \end{pmatrix} = H. \end{aligned}$$

We further note that if $\text{trace}(H^{-1})+e^tH^{-1}e = 1$ and we choose $\pi_i = d_i$, for $i = 1, \ldots, n-1$, so the necessarily $\pi_n = e^tH^{-1}e$, then we readily see from (3.17), the defining equation for T, that the diagonal entries of T must be all 0.

c) The transition matrix T in Theorem 3.2(b) comes from a reversible Markov chain if and only if the matrix H is symmetric.

The above results lead us to a corollary providing necessary and sufficient conditions for an $n \times n$ nonnegative matrix to be an inverse of a row and column diagonally dominant M–matrix, thus adding to the known classes of inverse M–matrices:

Corollary 3.4 *Suppose that $A = (a_{i,j}) \in \mathbb{R}^{n,n}$. Then the following conditions are equivalent:*

(a) *A is invertible and A^{-1} is a row and column diagonally dominant M–matrix.*

(b) *A is a matrix whose entries are determined as follows: there exists a Markov chain $\mathcal{C}$ on $n+1$ states, whose MFP matrix is $M = (m_{i,j}) \in \mathbb{R}^{n+1,n+1}$, and a constant $k > 0$ such that*

$$a_{i,j} = \begin{cases} k\left(m_{i,n+1} + m_{n+1,j} - m_{i,j}\right), & \text{if} \quad i \neq j, \\ \\ k\left(m_{i,n+1} + m_{n+1,j}\right), & \text{if} \quad i = j, \end{cases} \tag{3.18}$$

for all $i, j = 1, \ldots, n$.

4 Resistive Electrical Networks and the Results of Fiedler

Given a connected undirected graph $\mathcal{G} = (V, E)$ on $n+1$ nodes (vertices), i.e., $|V| = n+1$, an *electrical network* $\mathcal{N}(\mathcal{G})$ can be induced by $\mathcal{G}$ as follows. For any edge $(i,j) \in E$, let $r_{i,j}$ be the resistance between the corresponding nodes. When nodes i and j are not adjacent, the resistance between the nodes is taken to be infinite. The *graph* $\mathcal{G}$ can then be regarded as a **weighted**

graph with the weight of the edge given by the *conductance* $c_{i,j} = \frac{1}{r_{i,j}}$, if nodes i and j are adjacent, and $c_{i,j} = 0$, if nodes i and j are not adjacent. For any two nodes $i, j \in V$, $R_{i,j}$ will denote the **effective resistance between the corresponding nodes** in $\mathcal{N}(\mathcal{G})$, namely, the potential difference we need to impose between nodes i and j to get a current flow of 1 Volt from i to j. Notice that in this setting, $R_{i,i} = 0$.

Next, the transition probabilities of a random walk on $\mathcal{G}$ are usually set as follow:

$$t_{i,j} \;=\; \frac{c_{i,j}}{\sum_{k \in V} c_{i,k}}.$$

It is easy to check that the matrix $T := (t_{i,j})$ is an $(n+1) \times (n+1)$ stochastic matrix. Let $m_{i,j}$ be, in the language of *randomized algorithms*, see Tetali [22], the *expected cost of a random walk that starts at i and ends upon first reaching j*. This is another way of saying that $m_{i,j}$ is the MFP time from node i to node j.

In [7], Fiedler provided a connection between effective resistances $R_{i,j}$ and the inverses of irreducible diagonally dominant symmetric M–matrices, see [7, Theorem 2.1(i) & (iii)]. We shall now provide an alternative proof to Fiedler's theorm.

Theorem 4.1 (Fiedler [7, Theorem 2.1, (i) & (iii)]) *Let $A = (a_{i,j}) \in \mathbb{R}^{n,n}$. Then the following are equivalent.*

(a) *A is invertible and A^{-1} is an irreducible diagonally dominant symmetric M–matrix.*

(b) *There is a connected resistive network $\mathcal{N}(\mathcal{G})$ with $n+1$ nodes, labeled $1, \ldots, n+1$, such that the effective resistances $R_{i,j}$ satisfy*

$$a_{i,j} \;=\; \frac{1}{2}\left(R_{i,n+1} + R_{n+1,j} - R_{i,j}\right), \quad \text{for } i, j = 1, \ldots, n. \tag{4.19}$$

To give our alternative proof to Fieldler's Theorem we need the following result which can be readily deduced from a result of Chandra, Raghavan, Ruzzo, Smolensky, and Tiwari. [2, Theorem 2.2]:

Proposition 4.2 ([2, Theorem 2.2]) *Let* $\hat{C} := \sum_{(i,j)\in V\times V} c_{i,j}$. *Then for any two distinct nodes* $i, j \in V$,

$$2m_{i,j} \;=\; \hat{C}R_{i,j}.$$

Proof of Theorem 4.1. To prove the theorem, it suffices to show that Fiedler's assertion in Theorem 4.1 (b) is equivalent to part (b) of our Corollary 3.4.

Suppose now that $\mathcal{N}(\mathcal{G})$ is a connected electrical network with $n+1$ nodes and that $A = (a_{i,j}) \in \mathbb{R}^{n,n}$ satisfies (4.19). Set $T := (t_{i,j})$, with $t_{i,j} = c_{i,j}/\sum_{1\le k\le n} c_{i,k}$, $i, j = 1, \ldots, n+1$. Then T is a $(n+1)\times(n+1)$ transition matrix for some chain. By Proposition 4.2, if $M = (m_{i,j})$ is the MFP matrix obtained from T, then $R_{i,j} = 2m_{i,j}/\hat{C}$, for $i \neq j$, where $\hat{C} = \sum_{i,j\in V\times V} c_{i,j}$. Note also that $R_{i,i} = 0$ for all i. Thus A satisfies (3.18) and hence Corollary 3.4(b) follows with $k = 1/\hat{C}$.

Conversely, suppose that M is an $(n+1)\times(n+1)$ MFP matrix induced by a transition matrix T and $A = (a_{i,j}) \in \mathbb{R}^{n,n}$ satisfies (3.18). We now proceed to construct an resistive electrical network $\mathcal{N}(\mathcal{G})$, where $\mathcal{G}$ is a weighted graph with the conductances $c_{i,j} = t_{i,j}/k(n+1)$. By proposition 4.2, for any $i \neq j$,

$$km_{i,j} \;=\; \frac{1}{2}k\hat{C}R_{i,j} \;=\; \frac{1}{2}R_{i,j}\left(\sum_{r,s\in V} kc_{r,s}\right) \;=\; \frac{1}{2}R_{i,j}\left(\sum_{r,s\in V} \frac{t_{r,s}}{n+1}\right) \;=\; \frac{1}{2}R_{i,j}.$$

Thus A satisfies (4.19) and so assertion (b) of Theorem 4.1 holds. Our proof is done. □

ACKNOWLEDGEMENT The authors wish to thank Professors Iddo Ben–Ari, Stephen J. Kirkland, Olga Pryporova, and Jianghong Xu for useful discussions about the inverse MFP matrix problem.

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