Maps preserving the nilpotency of products of operators

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This paper is dedicated to Professor Roger Horn on the occasion of his 65th birthday.

Abstract

Let $\mathcal{B}(X)$ be the algebra of all bounded linear operators on the Banach space X, and let $\mathcal{N}(X)$ be the set of nilpotent operators in $\mathcal{B}(X)$. Suppose $\phi : \mathcal{B}(X) \to \mathcal{B}(X)$ is a surjective map such that $A, B \in \mathcal{B}(X)$ satisfy $AB \in \mathcal{N}(X)$ if and only if $\phi(A)\phi(B) \in \mathcal{N}(X)$. If X is infinite dimensional, then there exists a map $f : \mathcal{B}(X) \to \mathbb{C} \setminus \{0\}$ such that one of the following holds:

(a) There is a bijective bounded linear or conjugate-linear operator $S : X \to X$ such that ϕ has the form $A \mapsto S[f(A)A]S^{-1}$.

(b) The space X is reflexive, and there exists a bijective bounded linear or conjugate-linear operator $S : X' \to X$ such that ϕ has the form $A \mapsto S[f(A)A']S^{-1}$.

If X has dimension n with $3 \leq n < \infty$, and $\mathcal{B}(X)$ is identified with the algebra M_n of $n \times n$ complex matrices, then there exist a map $f : M_n \to \mathbb{C} \setminus \{0\}$, a field automorphism $\xi : \mathbb{C} \to \mathbb{C}$, and an invertible $S \in M_n$ such that ϕ has one of the following forms:

$$A = [a_{ij}] \mapsto f(A)S[\xi(a_{ij})]S^{-1} \qquad \text{or} \qquad A = [a_{ij}] \mapsto f(A)S[\xi(a_{ij})]^t S^{-1},$$

where A^t denotes the transpose of A.

The results are extended to the product of more than two operators and to other types of products on $\mathcal{B}(X)$ including the Jordan triple product $A * B = ABA$. Furthermore, the results in the finite dimensional case are used to characterize surjective maps on matrices preserving the spectral radius of products of matrices.

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1 Introduction

Let $\mathcal{B}(X)$ be the algebra of all bounded linear operators on the Banach space X, and let $\mathcal{N}(X)$ be the subset of all nilpotent operators. We are interested in determining the structure of surjective maps $\phi : \mathcal{B}(X) \to \mathcal{B}(X)$ having the property that for every pair $A, B \in \mathcal{B}(X)$,

$$AB \in \mathcal{N}(X) \iff \phi(A)\phi(B) \in \mathcal{N}(X).$$

There has been considerable interest in studying maps on operators or matrices mapping the set of nilpotents into or onto itself, and maps on operators or matrices preserving the spectral radius or the spectrum of operators or matrices. We call such maps nilpotent preservers, spectral radius preservers, and spectrum preservers, respectively. The structure of linear nilpotent preservers was described in [2] and [13]. In the finite dimensional case the assumption of preserving nilpotents can be reformulated as the assumption of preserving matrices with the zero spectral radius. So, from the structural result for linear nilpotent preservers we get immediately the general form of linear spectral radius preservers on matrix algebras. In the infinite dimensional case the situation is more complicated because of many quasinilpotents that are not nilpotents (see [3]).

If X has dimension n with $n < \infty$, then $\mathcal{B}(X)$ is identified with the algebra M_n of $n \times n$ complex matrices, and $\mathcal{N}(X)$ becomes the set N_n of nilpotent matrices in M_n. In [7] (see also [8]), multiplicative maps on matrices leaving invariant various functions and subsets of matrices were characterized. In particular, it was shown that a nonzero multiplicative map on M_n mapping the set of nilpotent matrices into itself has the form $A \mapsto SA_\xi S^{-1}$ for some invertible matrix S and some field endomorphism ξ of $\mathbb{C}$. Here, A_ξ denotes the matrix obtained from A by applying ξ entrywise.

Clearly, maps on matrices preserving nilpotent matrices or the spectral radius can be quite arbitrary on individual matrices. Hence, if one does not impose any algebraic condition like linearity, additivity, or multiplicativity on preservers of nilpotents or spectral radius, one needs to include some related conditions connecting different matrices in order to get a reasonable structural result. In [1], surjective maps ϕ on the algebra of all $n \times n$ complex matrices preserving the spectral radius of a difference of matrices were characterized.

Motivated by problems concerning local automorphisms Molnár [11, 12] studied maps preserving the spectrum of operator or matrix products (for related results see [5, 6]). If the spectrum of matrix products is preserved, then in particular, the nilpotency of matrix products is preserved.

In this paper, we determine the structure of surjective maps $\phi : \mathcal{B}(X) \to \mathcal{B}(X)$ preserving the nilpotency of operator products. Specifically, our results describe the structure of surjective maps $\phi : \mathcal{B}(X) \to \mathcal{B}(X)$ such that for any $A_1, \ldots, A_k \in \mathcal{B}(X)$

$$A_1 * \cdots * A_k \in \mathcal{N}(X) \qquad \iff \qquad \phi(A_1) * \cdots * \phi(A_k) \in \mathcal{N}(X)$$

for various types of products including the usual product $A_1 * \cdots * A_k = A_1 \cdots A_k$ and the Jordan triple product $A_1 * A_2 = A_1 A_2 A_1$.

In Section 2, we present the results for the usual product and the Jordan triple product of two operators. Extension of the results to other types of products are presented in Section 3.

We conclude this section by fixing some notation. For every nonzero $x \in X$ and $f \in X'$ the symbol $x \otimes f$ stands for the rank one bounded linear operator on X defined by $(x \otimes f)y = f(y)x$,

$y \in X$. Note that every rank one element of $\mathcal{B}(X)$ can be written in this way. The rank one operator $x \otimes f$ is nilpotent if and only if $f(x) = 0$. It is an idempotent if and only if $f(x) = 1$. Let $x \otimes f$ and $y \otimes g$ be two rank one operators. We will write $x \otimes f \sim y \otimes g$ if x and y are linearly dependent or f and g are linearly dependent.

Let U be any vector space. Denote by $[u]$ the subspace (of dimension 0 or 1) spanned by $u \in U$, and denote by $\mathbb{P}U$ the projective space over U, i.e.,

$$\mathbb{P}U = \{[u] \,:\, u \in U \setminus \{0\}\}.$$

Let $\mathbb{C}^*$ be the set of all nonzero complex numbers. For $A = [a_{ij}] \in M_n$, let A^t be the transpose of A. Also let $\overline{A} = [\overline{a_{ij}}]$ and $A^* = \overline{A}^t$. The spectral radius of A will be denoted by $\rho(A)$.

2 The usual product and the Jordan triple product of two operators

In this section, we always assume that $A * B = AB$ or $A * B = ABA$. Let us first present the main theorems.

Theorem 2.1 *Let X be an infinite dimensional Banach space. Then a surjective map $\phi : \mathcal{B}(X) \to \mathcal{B}(X)$ satisfies*

$$A * B \in \mathcal{N}(X) \iff \phi(A) * \phi(B) \in \mathcal{N}(X), \quad A, B \in \mathcal{B}(X), \tag{1}$$

if and only if

(a) *there is a bijective bounded linear or conjugate-linear operator $S : X \to X$ such that ϕ has the form $A \mapsto S[f(A)A]S^{-1}$, or*

(b) *the space X is reflexive, and there exists a bijective bounded linear or conjugate-linear operator $S : X' \to X$ such that ϕ has the form $A \mapsto S[f(A)A']S^{-1}$,*

where $f : \mathcal{B}(X) \to \mathbb{C}^*$ is a map such that for every nonzero $A \in \mathcal{B}(X)$ the map $\lambda \mapsto \lambda f(\lambda A)$ is surjective on $\mathbb{C}$.

Next, we state our main result for the finite dimensional case. Note that the identity function and the complex conjugation $\lambda \mapsto \overline{\lambda}$ are continuous automorphisms of the complex field. It is known that there exist non-continuous automorphisms of the complex field [9]. If $\xi : \mathbb{C} \to \mathbb{C}$ is an automorphism of the complex field and $A \in M_n$ then we denote by A_ξ the matrix obtained from A by applying ξ entrywise, $A_\xi = [a_{ij}]_\xi = [\xi(a_{ij})]$.

Theorem 2.2 *Let $n \geq 3$. Then a surjective map $\phi : M_n \to M_n$ satisfies*

$$A * B \in N_n \iff \phi(A) * \phi(B) \in N_n, \quad A, B \in M_n,$$

if and only if ϕ has the form

$$\text{(a)}\;\; A \mapsto f(A)SA_\xi S^{-1} \qquad \textit{or} \qquad \text{(b)}\;\; A \mapsto f(A)SA_\xi^t S^{-1},$$

where $\xi : \mathbb{C} \to \mathbb{C}$ is a field automorphism, $S \in M_n$ is an invertible matrix, and $f : M_n \to \mathbb{C}^*$ is a map such that for every nonzero $A \in M_n$ the map $\lambda \mapsto \xi(\lambda)f(\lambda A)$ is surjective on $\mathbb{C}$.

Using Theorem 2.2 we can characterize maps preserving the spectral radius of $A * B$ on M_n.

Corollary 2.3 *Let $n \geq 3$. A surjective map $\phi : M_n \to M_n$ satisfies*

$$\rho(A * B) = \rho(\phi(A) * \phi(B)), \qquad A, B \in M_n,$$

if and only if ϕ has one of the following forms:

$$\text{(a)}\ A \mapsto f(A)SAS^{-1}, \qquad \text{(b)}\ A \mapsto \overline{f(A)}S\overline{A}S^{-1},$$

$$\text{(c)}\ A \mapsto f(A)SA^tS^{-1}, \qquad \text{(d)}\ A \mapsto \overline{f(A)}SA^*S^{-1},$$

where $S \in M_n$ is an invertible matrix, and $f : M_n \to \{z \in \mathbb{C} : |z| = 1\}$ is a map such that for any nonzero $A \in M_n$ the map $\lambda \mapsto \lambda f(\lambda A)$ is surjective on $\mathbb{C}$.

We establish some preliminary results in the next subsection, and give the proofs of the above theorems and corollary in another subsection.

2.1 Preliminary results

Let X have dimension at least three. Consider a surjective map $\phi : \mathcal{B}(X) \to \mathcal{B}(X)$ such that for every pair $A, B \in \mathcal{B}(X)$ the product AB is nilpotent if and only if $\phi(A)\phi(B)$ is nilpotent. Obviously, $A \in \mathcal{B}(X)$ is nilpotent if and only if A^2 is. Thus, $\phi(\mathcal{N}(X)) = \mathcal{N}(X)$. Next, observe that $\phi(0) = 0$ and $\phi(A) \neq 0$ for every nonzero $A \in \mathcal{B}(X)$. This follows from the simple fact that for every $A \in \mathcal{B}(X)$ the following two statements are equivalent:

- $A = 0$,
- AT is nilpotent for every $T \in \mathcal{B}(X)$.

Further, we have $\phi(\lambda I) \in \mathbb{C}^* I$ for every $\lambda \in \mathbb{C}^*$. Moreover, if $\phi(A) = \mu I$ for some $\mu \in \mathbb{C}^*$, then A is a nonzero scalar operator, i.e., $A = \delta I$ for some $\delta \in \mathbb{C}^*$. This is a consequence of the observation that for every nonzero $A \in \mathcal{B}(X)$ the following two statements are equivalent:

- A is a scalar operator,
- AN is nilpotent for every $N \in \mathcal{N}(X)$.

Let $A, B \in \mathcal{B}(X)$ be any nonzero operators. We will show that A and B are linearly dependent if and only if $\phi(A)$ and $\phi(B)$ are linearly dependent. To check this we have to show that for every pair of nonzero operators $A, B \in \mathcal{B}(X)$ the following two statements are equivalent:

- A and B are linearly dependent,
- for every $T \in \mathcal{B}(X)$ the operator AT is nilpotent if and only if BT is nilpotent.

We will show even more. Namely, these two statements are equivalent to

- for every $T \in \mathcal{B}(X)$ the operator AT is nilpotent whenever BT is nilpotent.

Clearly, the first condition implies the second one, and the second one implies the third one. So, assume that for every $T \in \mathcal{B}(X)$ we have $BT \in \mathcal{N}(X) \Rightarrow AT \in \mathcal{N}(X)$. Set $T = x \otimes f$ to observe that for every pair $x \in X$, $f \in X'$, we have $f(Bx) = 0 \Rightarrow f(Ax) = 0$. It follows that for every $x \in X$ the vector Ax belongs to the linear span of Bx. By [4, Theorem 2.3], either A and B are linearly dependent, or they are both of rank one with the same image. In the first case we are done, while in the second case we have $A = u \otimes g$ and $B = u \otimes k$ for some nonzero $u \in X$ and some nonzero $g, k \in X'$. We must prove that g and k are linearly dependent. Assume the contrary. Then we can find $x \in X$ such that $g(x) = 1$ and $k(x) = 0$. Choose $f \in X'$ with $f(u) = 1$. Then $f(Bx) = 0$ and $f(Ax) = 1$, a contradiction.

Hence, ϕ induces a bijective map $\Phi : \mathbb{P}\mathcal{B}(X) \to \mathbb{P}\mathcal{B}(X)$ defined by

$$\Phi([A]) = [\phi(A)], \quad A \in \mathcal{B}(X) \setminus \{0\}.$$

Let A and B be again nonzero operators in $\mathcal{B}(X)$. We will now consider the following condition

$$\text{for every } N \in \mathcal{N}(X) \text{ we have } AN \in \mathcal{N}(X) \Rightarrow BN \in \mathcal{N}(X). \tag{2}$$

In studying this condition we will need the following lemma.

Lemma 2.4 *Let $T, S \in \mathcal{B}(X)$. Assume that for every $x \in X$ the vector Tx belongs to the linear span of x and Sx. Then $T = \lambda I + \mu S$ for some $\lambda, \mu \in \mathbb{C}$.*

Proof. Assume first that the operators T, S, and I are linearly dependent. Then $\alpha T + \beta S + \gamma I = 0$ for some scalars α, β, and γ that are not all zero. If $\alpha \neq 0$, then T is a linear combination of S and I, as desired. In the case when $\alpha = 0$, we have $\beta \neq 0$. Thus, S is a scalar operator. This further yields that Tx belongs to the linear span of x for every $x \in X$. It follows that $T = \lambda I$ for some $\lambda \in \mathbb{C}$ and we are done.

In order to complete the proof we have to show that the assumption that T, S, and I are linearly independent leads to a contradiction. Assume that they are linearly independent. Because $\dim X \geq 3$, the identity has rank at least 3. With this observation and the assumptions on T, S, I, we can apply [10, Theorem 2.4] to conclude that there exist $\alpha, \beta, \gamma \in \mathbb{C}$ such that

$$\alpha T + \beta S + \gamma I = R = x \otimes f$$

for some rank one operator $R \in \mathcal{B}(X)$.

First, consider the case when $\alpha \neq 0$. Then T is a linear combination of S, I, and $x \otimes f$. It follows that $(x \otimes f)z = f(z)x$ belongs to the linear span of z and Sz for every $z \in X$. We will show that for every $u \in X$ the vectors u, Su, x are linearly dependent. Assume the contrary. Then we can find $u \in X$ such that u, Su, x are linearly independent. If $f(u) \neq 0$, then $f(u)x$ does not belong to the linear span of u and Su, a contradiction. Hence, $f(u) = 0$. Choose $v \in X$ with $f(v) \neq 0$. Applying [4, Lemma 2.1] we can find a nonzero $\mu \in \mathbb{C}$ such that $S(u + \mu v)$, $u + \mu v, x$ are linearly independent. But now $f(u + \mu v) \neq 0$ and we arrive at a contradiction in the same way as above.

Hence, the vectors u, Su, x are linearly dependent for every $u \in X$. Denote by Q the canonical quotient map of X onto $X/[x]$. The operators $Q = QI$ and QS are locally linearly dependent, that is, Qw and QSw are linearly dependent for every $w \in X$. By [4, Theorem 2.3], Q and QS are linearly dependent. This yields that $S = \delta I + x \otimes g$ for some complex δ and some functional $g \in X'$. If $g = 0$, then S is a scalar operator. This contradicts our assumption that T, S, and I are linearly independent. So, we are done in this special case. Thus, we will assume from now on that $g \neq 0$.

It follows that $f(z)x$ belongs to the linear span of z and $g(z)x$ for every $z \in X$. We will complete the proof of our first case by showing that g and f are linearly dependent. Indeed, assume for a moment that we have already proved that $R = x \otimes f = \tau x \otimes g$ for some complex τ. Then R is a linear combination of S and I, which further yields that T is a linear combination of S and I, contradicting our assumption that T, S, and I are linearly independent.

Assume on the contrary that f and g are linearly independent. Then we can find $v \in X$ linearly independent of x such that $f(v) \neq 0$ and $g(v) = 0$. We know that $f(v)x$ belongs to the linear span of v and $g(v)x = 0$. This contradiction completes the proof of our first case.

It remains to consider the case when $\alpha = 0$. Then clearly, $\beta \neq 0$, and therefore, S is a linear combination of R and I. If S is a scalar operator, we are done. If not, then after replacing S by $\eta S - \nu I$ for appropriate complex numbers η, ν, we may, and we will assume that $S = R = x \otimes f$. Hence, Tz belongs to the linear span of z and $f(z)x$ for every $z \in X$. In particular, Tz belongs to the linear span of z for every z from the kernel of f. This yields that the restriction of T to the kernel of f is a scalar operator. From here we conclude that $T = \eta I + v \otimes f$ for some $\eta \in \mathbb{C}$ and some nonzero $v \in X$. Hence, our assumption now reads as follows: for every $z \in X$ the vector $f(z)v$ belongs to the linear span of z and $f(z)x$. It follows that v and x are linearly dependent. But then $T = \eta I + \kappa x \otimes f$ is a linear combination of I and S. This contradiction completes the proof. □

We are now ready to deal with condition (2). Assume that A and B are nonzero operators such that for every $N \in \mathcal{N}(X)$ we have $AN \in \mathcal{N}(X) \Rightarrow BN \in \mathcal{N}(X)$. Take any $x \in X$ and $f \in X'$ with $f(x) = 0$. Set $N = x \otimes f$. Then the above condition reads as follows. For every pair $x \in X$ and $f \in X'$ we have

$$f(x) = 0 \quad \text{and} \quad f(Ax) = 0 \Rightarrow f(Bx) = 0. \tag{3}$$

Assume that there exists $x \in X$ such that Bx does not belong to the linear span of x and Ax. Then we can find $f \in X'$ such that $f(x) = f(Ax) = 0$ and $f(Bx) \neq 0$, contradicting (3). Hence, by Lemma 2.4, condition (2) implies that $B = \lambda I + \mu A$ for some $\lambda, \mu \in \mathbb{C}$.

Denote by $\mathcal{F}_1(X)$ the set of all rank one bounded linear operators on X. We have the following lemma.

Lemma 2.5 *Let $N \in \mathcal{N}(X)$ and $T \in \mathcal{F}_1(X)$. Assume that $N + T \in \mathcal{N}(X)$. Then $T^2 = 0$.*

Proof. Let m be a positive integer such that $N^m = 0$. We can write X as a direct sum of closed subspaces

$$X = \text{span}\,\{x, y\} \oplus Y$$

such that the restriction of T to Y is zero operator and the image of T is contained in span $\{x, y\}$. It is enough to show that the restriction of T to the subspace span $\{x, y\}$ is a square-zero operator. The linear span of vectors $x, Nx, \ldots, N^{m-1}x, y, Ny, \ldots, N^{m-1}y$ is invariant under both T and N. The restrictions of these two operators to this finite dimensional space can be identified with matrices. So, we can calculate their traces. As both N and $N + T$ are nilpotents, the traces of their restrictions must be zero. By linearity, the trace of the restriction of T is zero. We complete the proof by recalling the fact that every rank one trace-zero matrix is a square-zero matrix. □

Corollary 2.6 *Assume that $B \in \mathcal{B}(X)$ is of the form scalar plus rank one, that is, $B \in \mathbb{C}I + \mathcal{F}_1(X)$. Then there exists $A \in \mathbb{C}I + \mathcal{F}_1(X)$ linearly independent of B such that for every $N \in \mathcal{N}(X)$ we have $AN \in \mathcal{N}(X) \Rightarrow BN \in \mathcal{N}(X)$.*

Proof. Let $B = R + \lambda I$ for some $\lambda \in \mathbb{C}$ and some $R \in \mathcal{F}_1(X)$. Set $A = R + \mu I$, where μ is a nonzero complex number chosen in such a way that A and B are linearly independent. Choose further any nilpotent operator N such that $AN = \mu N + RN$ is nilpotent. Applying Lemma 2.5 we get that RN is nilpotent of rank at most one. If $RN \neq 0$, then by [13, Proposition 2.1], the operator $N + \alpha RN$ is nilpotent for every nonzero $\alpha \in \mathbb{C}$. Clearly, we have $N + \alpha RN \in \mathcal{N}(X)$, $\alpha \in \mathbb{C}$, also in the case when $RN = 0$. We have to show that $BN = RN + \lambda N$ is nilpotent. This is certainly true when $\lambda = 0$. If λ is nonzero, then we observe that BN is nilpotent if and only if $N + \frac{1}{\lambda} RN$ is. And this is indeed the case by what we have already proved. □

Lemma 2.7 *Let $A \in \mathcal{B}(X)$, $A \notin \mathbb{C}I$. Then the following are equivalent:*

- $A \notin \mathbb{C}I + \mathcal{F}_1(X)$,
- *there exists an idempotent $P \in \mathcal{B}(X)$ of rank 3 such that PAP is not of the form $\lambda P + R$, where $\lambda \in \mathbb{C}$ and $R \in \mathcal{F}_1(X) \cup \{0\}$.*

Proof. If A is of the form scalar plus rank one then obviously, PAP is of the form scalar times P plus rank at most one for every idempotent P of rank 3. To prove the nontrivial direction assume that $A \notin \mathbb{C}I + \mathcal{F}_1(X)$. Let us first consider the case that there exists $x \in X$ such that x, Ax, and A^2x are linearly independent. Let P be any idempotent of rank 3 whose image is spanned by x, Ax, A^2x. With respect to the direct sum decomposition $X = \operatorname{Im} P \oplus \operatorname{Ker} P$ the matrix representations of the operators P and PAP are

$$P = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad PAP = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Choosing the basis x, Ax, A^2x for the subspace $\operatorname{Im} P$ we get the following matrix representation of A_1

$$\begin{bmatrix} 0 & 0 & \mu \\ 1 & 0 & \tau \\ 0 & 1 & \eta \end{bmatrix},$$

where μ, τ, and η are some complex numbers. It is then clear that $PAP - \lambda P$ has rank at least 2 for every complex number λ, as desired.

It remains to consider the case when x, Ax, A^2x are linearly dependent for every $x \in X$. Then $p(A) = 0$ for some complex polynomial of degree at most two. As A is not a scalar plus rank one operator there exists an idempotent Q of rank 4, such that QAQ has one of the following two matrix representations:

$$QAQ = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix},$$

where

$$A_1 = \begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & \mu \end{bmatrix} \quad \text{or} \quad A_1 = \begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{bmatrix}.$$

Here, $\lambda \neq \mu$. In the first case A_1 is similar to

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 \\ 0 & \frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & \mu \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & \frac{\lambda+\mu}{2} & 0 & \frac{\lambda-\mu}{2} \\ 0 & 0 & \mu & 0 \\ 0 & \frac{\lambda-\mu}{2} & 0 & \frac{\lambda+\mu}{2} \end{bmatrix}.$$

Obviously, the upper left 3×3 corner is not of the form 3×3 scalar matrix plus a matrix of rank at most one. This completes the proof in the first case.

In the second case we observe that A_1 is similar to

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda + \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \lambda - \frac{1}{2} \end{bmatrix}.$$

The same argument as before completes the proof. □

Lemma 2.8 *Let C be a non-scalar 3×3 matrix that is not of the form a scalar matrix plus rank one matrix. Then C is similar to a matrix of the form*

$$\begin{bmatrix} * & \lambda & \lambda \\ * & -\lambda & -\lambda \\ * & * & * \end{bmatrix},$$

where λ is a nonzero complex number.

Proof. With no loss of generality we may assume that C has the Jordan canonical form. As it is not a scalar plus rank one it has to be of one of the forms

$$\begin{bmatrix} \tau & 0 & 0 \\ 0 & \eta & 0 \\ 0 & 0 & \nu \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \tau & 1 & 0 \\ 0 & \tau & 0 \\ 0 & 0 & \nu \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \tau & 1 & 0 \\ 0 & \tau & 1 \\ 0 & 0 & \tau \end{bmatrix},$$

where τ, η, and ν are pairwise distinct complex numbers. In the first case we may assume that $\eta \neq 0$, since otherwise we can permute the diagonal elements by a similarity transformation induced by a permutation matrix. The matrix

$$\begin{bmatrix} \tau & -\eta & -\eta \\ 0 & \eta & \eta \\ 0 & 0 & \nu \end{bmatrix}$$

has three different eigenvalues, and is therefore similar to the first of the above matrices. The matrix

$$\begin{bmatrix} \tau & -\tau & -\tau \\ 0 & \tau & \tau \\ 0 & 0 & \nu \end{bmatrix}$$

has the desired form when $\tau \neq 0$. The eigenspace corresponding to the eigenvalue τ is one-dimensional, and therefore, this matrix is similar to the second of the above matrices. In the case when $\tau = 0$ the second matrix above is

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \nu \end{bmatrix}$$

with $\nu \neq 0$. We complete the proof in this special case by observing that it is similar to

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 0 & 0 & \nu \end{bmatrix}.$$

Indeed, the eigenvalue 0 has algebraic multiplicity two and geometric multiplicity one. We similarly check that the last of the above matrices is similar to

$$\begin{bmatrix} \tau & -\tau & -\tau \\ 0 & \tau & \tau \\ 0 & 0 & \tau \end{bmatrix}$$

when $\tau \neq 0$, and to

$$\begin{bmatrix} 2 & 4 & 4 \\ -1 & -4 & -4 \\ 0 & 2 & 2 \end{bmatrix}$$

when $\tau = 0$. This completes the proof. $\square$

Corollary 2.9 *Let $B \in \mathcal{B}(X)$ be a non-scalar operator. Then the following are equivalent:*

- $B \in \mathbb{C}I + \mathcal{F}_1(X)$,
- *there exists an operator $A \in \mathcal{B}(X)$ such that A and B are linearly independent and for every $N \in \mathcal{N}(X)$ we have $AN \in \mathcal{N}(X) \Rightarrow BN \in \mathcal{N}(X)$.*

Proof. One direction is the statement of Corollary 2.6. So, assume that a non-scalar operator $B \in \mathcal{B}(X)$ is given and that there exists $A \in \mathcal{B}(X)$ satisfying the second condition. We already know that this condition implies that $B = \lambda I + \mu A$. We have $\mu \neq 0$. As A and B are linearly independent, we necessarily have $\lambda \neq 0$.

We have to show that $B \in \mathbb{C}I + \mathcal{F}_1(X)$. Assume on the contrary that $B \notin \mathbb{C}I + \mathcal{F}_1(X)$. Then $A = \frac{1}{\mu}B - \frac{\lambda}{\mu}I \notin \mathbb{C}I + \mathcal{F}_1(X)$. By the previous two lemmas we know that there exist a direct sum decomposition $X = X_1 \oplus X_2$ with $\dim X_1 = 3$ and a basis of X_1 such that with respect to the chosen direct sum decomposition the operator A has matrix representation

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$$

and the matrix representation of A_1 corresponding to the chosen basis is of the form

$$\begin{bmatrix} * & \eta & \eta \\ * & -\eta & -\eta \\ * & * & * \end{bmatrix},$$

where η is a nonzero complex number. Choose

$$N = \begin{bmatrix} N_1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{with} \quad N_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

and observe that for every $T \in \mathcal{B}(X)$,

$$T = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix},$$

the product TN is nilpotent if and only if $T_1 N_1$ is nilpotent.

Now, the upper left 3×3 corners of AN and BN are

$$\begin{bmatrix} \eta & \eta & 0 \\ -\eta & -\eta & 0 \\ * & * & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \mu\eta & \mu\eta & 0 \\ -\mu\eta + \lambda & -\mu\eta & 0 \\ * & * & 0 \end{bmatrix},$$

respectively. The second one is not nilpotent (the upper left 2×2 corner is not of rank one, and is therefore not nilpotent), while the first one is nilpotent. This contradiction completes the proof. □

Lemma 2.10 *Assume that $A \in \mathbb{C}I + \mathcal{F}_1(X)$. Then the following are equivalent:*

- *$A \in \mathcal{F}_1(X)$,*
- *every $C \in \mathbb{C}I + \mathcal{F}_1(X)$ with the property that for every $N \in \mathcal{N}(X)$ we have $AN \in \mathcal{N}(X) \Rightarrow CN \in \mathcal{N}(X)$ belongs to the linear span of A.*

Proof. Assume that A is of rank one and that an operator $C \in \mathbb{C}I + \mathcal{F}_1(X)$ has the property that for every $N \in \mathcal{N}(X)$ we have $AN \in \mathcal{N}(X) \Rightarrow CN \in \mathcal{N}(X)$. Then we know that $C = \alpha I + \beta A$ with $\beta \neq 0$. We have to show that $\alpha = 0$. Assume on the contrary that $\alpha \neq 0$. We can find a direct sum decomposition of X, $X = X_1 \oplus X_2$, with $\dim X_1 = 3$, and a basis of X_1 such that the corresponding matrix representation of A is

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix},$$

where either

$$A_1 = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ with } \lambda \neq 0, \quad \text{or } A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Set

$$N = \begin{bmatrix} N_1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{with} \quad N_1 = \begin{bmatrix} 0 & 2 & -2 \\ 0 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Clearly, $N \in \mathcal{N}(X)$. Obviously, both products AN and CN have nonzero entries only in the upper left 3×3 corner. It is easy to see that this corner of AN is nilpotent. The upper left 3×3 corner of CN is either

$$\begin{bmatrix} 0 & 2(\alpha + \beta\lambda) & -2(\alpha + \beta\lambda) \\ 0 & -\alpha & \alpha \\ \alpha & \alpha & \alpha \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & 2\alpha - \beta & -2\alpha + \beta \\ 0 & -\alpha & \alpha \\ \alpha & \alpha & \alpha \end{bmatrix}.$$

Let δ be any complex number. Then

$$\begin{bmatrix} 0 & \delta & -\delta \\ 0 & -\alpha & \alpha \\ \alpha & \alpha & \alpha \end{bmatrix}^2 = \begin{bmatrix} -\delta\alpha & -2\delta\alpha & 0 \\ \alpha^2 & 2\alpha^2 & 0 \\ \alpha^2 & \delta\alpha & -\delta\alpha + 2\alpha^2 \end{bmatrix}.$$

Set first $\delta = 2(\alpha + \beta\lambda)$ and then $\delta = 2\alpha - \beta$. In both cases we have $-\delta\alpha + 2\alpha^2 \neq 0$. So, none of the above two matrices is nilpotent. Thus, $CN \notin \mathcal{N}(X)$, a contradiction.

To prove the converse assume that $A = \alpha I + R$, where R is a rank one operator and α is a nonzero complex number. Set $C = \frac{\alpha}{2} I + R$. Then C does not belong to the linear span of A. But if N is any nilpotent such that $AN = \alpha N + RN$ is nilpotent, then by Lemma 2.5, the operator RN is nilpotent. Hence, by [13, Proposition 2.1], $\alpha N + 2RN$ is nilpotent, which further yields that CN is nilpotent. This completes the proof. □

Lemma 2.11 *Let A_1 and A_2 be linearly independent rank one operators. Then the following are equivalent:*

- $A_1 \sim A_2$,
- *there exists a rank one operator B such that B is linearly independent of A_1, B is linearly independent of A_2, and for every $T \in \mathcal{B}(X)$ we have $A_i T \in \mathcal{N}(X)$, $i = 1, 2$, $\Rightarrow BT \in \mathcal{N}(X)$.*

Proof. Let $A_1 = x \otimes f$ and $A_2 = y \otimes g$. If $A_1 \sim A_2$, then $y = \lambda x$ or $g = \lambda f$ for some nonzero complex number λ. We will consider only the second possibility. After absorbing the constant in the tensor product we may assume that $A_2 = y \otimes f$. Define B to be $B = A_1 + A_2$. Since A_1 and A_2 are linearly independent, B is linearly independent of A_i, $i = 1, 2$. If $A_i T \in \mathcal{N}(X)$, $i = 1, 2$, then $f(Tx) = f(Ty) = 0$. This yields that $f(T(x + y)) = 0$, or equivalently, $BT \in \mathcal{N}(X)$.

To prove the other direction assume that $A_1 = x \otimes f$ and $A_2 = y \otimes g$ are rank one operators such that x and y as well as f and g are linearly independent. Suppose also that there exists $B = u \otimes k$ satisfying the second condition. We will show that k is a linear combination of f and g. Assume on the contrary that this is not the case. Then we can find a vector $z \in X$ such that $k(z) \neq 0$, while $f(z) = g(z) = 0$. We can further find $T \in \mathcal{B}(X)$ such that $Tu \neq 0$ and all vectors Tx, Ty, Tu belong to the linear span of z. This implies that $f(Tx) = g(Ty) = 0$ and $k(Tu) \neq 0$, which gives $A_i T \in \mathcal{N}(X)$, $i = 1, 2$, but $BT \notin \mathcal{N}(X)$, a contradiction. In a similar way we show that u is a linear combination of x and y. Hence, $B = (\lambda x + \mu y) \otimes (\alpha f + \beta g)$. Let η and ν be any complex numbers. As f and g are linearly independent we can find $w_1 \in X$ such that $f(w_1) = 0$ and $g(w_1) = \eta$, and $w_2 \in X$ such that $f(w_2) = \nu$ and $g(w_2) = 0$. Since x and y are linearly independent we can further find $T \in \mathcal{B}(X)$ satisfying $Tx = w_1$ and $Ty = w_2$. Then $f(Tx) = 0 = g(Ty)$, and thus,

$$0 = k(Tu) = (\alpha f + \beta g)(\lambda Tx + \mu Ty) = \alpha\mu\nu + \beta\lambda\eta.$$

It follows that $\alpha\mu = \beta\lambda = 0$, which further yields that B is a multiple of either A_1, or A_2, a contradiction. □

We say that two rank one idempotents P and Q are orthogonal if $PQ = QP = 0$.

Lemma 2.12 *Let P and Q, $P \neq Q$, be rank one idempotents. Then the following are equivalent:*

- *P and Q are orthogonal,*

- *there exist rank one nilpotents M and N such that $P \sim N$, $P \sim M$, $Q \sim N$, $Q \sim M$, and $N \not\sim M$.*

Proof. Let $P = x \otimes f$ and $Q = y \otimes g$ be orthogonal rank one idempotents. Then $f(x) = g(y) = 1$ and $f(y) = g(x) = 0$. Set $N = x \otimes g$ and $M = y \otimes f$. It is easy to verify that the second condition is satisfied.

So, assume now that there exist M and N satisfying the second condition. Let $P = x \otimes f$ for some $x \in X$ and $f \in X'$ with $f(x) = 1$. Then either $N = x \otimes g$ for some nonzero $g \in X'$ with $g(x) = 0$, or $N = z \otimes f$ for some nonzero $z \in X$ with $f(z) = 0$. We will consider only the first case. Because $M \sim P$ and $N \not\sim M$ we have necessarily $M = y \otimes f$ for some nonzero $y \in X$ satisfying $f(y) = 0$. From $f(x) = 1$ and $g(x) = f(y) = 0$ we conclude that x and y are linearly independent and f and g are linearly independent. Because Q is an idempotent and $Q \sim N$ and $Q \sim M$ we have either $Q = P$, or $Q = \tau y \otimes g$ for some $\tau \in \mathbb{C}^*$. The first possibility cannot occur, and therefore, $PQ = QP = 0$. □

Lemma 2.13 *Let $A, B \in \mathcal{B}(X) \setminus \{0\}$ and assume that for every idempotent P of rank one we have $AP \in \mathcal{N}(X)$ if and only if $BP \in \mathcal{N}(X)$. Then A and B are linearly dependent.*

Proof. Let $x \in X$ be any vector. If Ax and Bx are linearly independent, then at least one of them, say Ax, is linearly independent of x. But then we can find $f \in X'$ such that $f(x) = 1$, $f(Ax) = 0$, and $f(Bx) \neq 0$. As a consequence, $Ax \otimes f \in \mathcal{N}(X)$, while $Bx \otimes f \notin \mathcal{N}(X)$, a contradiction. So, for every $x \in X$ the vectors Ax and Bx are linearly dependent. By [4, Theorem 2.3], either the operators A and B are linearly dependent, or there exists a nonzero $y \in X$ such that $A = y \otimes f$ and $B = y \otimes g$ for some linearly independent functionals f and g. In the second case there exists $u \in X$ such that $f(u) = 1$ and $g(u) = 0$. We can further find a functional $k \in X'$ with $k(u) = 1$ and $k(y) \neq 0$. Set $P = u \otimes k$. Then $AP = y \otimes k \notin \mathcal{N}(X)$, while $BP = 0$, a contradiction. □

2.2 Proofs of the theorems and corollary

We continue to assume that X is a Banach space with $\dim X \geq 3$ and $\phi : \mathcal{B}(X) \to \mathcal{B}(X)$ is a surjective map satisfying (1). Clearly, $A \in \mathcal{B}(X)$ is nilpotent if and only if A^2 or A^3 is. Thus, $\phi(\mathcal{N}(X)) = \mathcal{N}(X)$. Note that for any $A, B \in \mathcal{B}(X)$, $ABA \in \mathcal{N}(X)$ if and only if $A^2B \in \mathcal{N}(X)$. So condition (1) can be described as

$$A^r B \in \mathcal{N}(X) \iff \phi(A)^r \phi(B) \in \mathcal{N}(X), \quad A, B \in \mathcal{B}(X),$$

where $r = 1$ or 2.

To prove our theorems we establish the following proposition, which is of independent interest and will be used in the next section as well.

Proposition 2.14 *Suppose $\mathcal{S}$ is a subset of $\mathcal{B}(X)$ containing the sets $\mathbb{C}I + \mathcal{F}_1(X)$ and $\mathbb{C}I$. Let $\phi : \mathcal{B}(X) \to \mathcal{B}(X)$ and $\psi : \mathcal{S} \to \mathcal{S}$ be surjective maps satisfying $\phi(\mathcal{N}(X)) = \mathcal{N}(X)$, $\psi(\mathcal{N}(X) \cap \mathcal{S}) = \mathcal{N}(X) \cap \mathcal{S}$ and*

$$AB \in \mathcal{N}(X) \iff \psi(A)\phi(B) \in \mathcal{N}(X), \quad (A, B) \in \mathcal{S} \times \mathcal{B}(X).$$

Then ϕ satisfies (a)-(b) in Theorems 2.1 or 2.2, and ψ has the form $A \mapsto g(A)\phi(A)$ for some $\mathbb{C}^*$-valued map g on $\mathcal{S}$.

Proof. Suppose $\psi : \mathcal{S} \to \mathcal{S}$ and $\phi : \mathcal{B}(X) \to \mathcal{B}(X)$ are surjective maps such that $\phi(\mathcal{N}(X)) = \mathcal{N}(X)$, $\psi(\mathcal{N}(X) \cap \mathcal{S}) = \mathcal{N}(X) \cap \mathcal{S}$, and for any $(A, B) \in \mathcal{S} \times B(X)$, $AB \in \mathcal{N}(X)$ if and only if $\psi(A)\phi(B) \in \mathcal{N}(X)$. Using the observations in the beginning of subsection 2.1, we can show that ψ maps the set of nonzero operators onto itself, and maps the set of nonzero scalar operators onto itself; two nonzero operators are linearly independent if and only if their ψ-images are linearly independent. By Corollaries 2.6 and 2.9, ψ maps the set of all scalar plus rank one operators onto itself. It then follows from Lemma 2.10 that ψ maps the set of rank one operators onto itself. We know that a rank one operator is nilpotent if and only if its ψ-image is. For every non-nilpotent rank one operator R there exists exactly one idempotent P that belongs to the linear span of R. Thus, ψ induces in a natural way a bijective map Ψ from the set of all rank one idempotents onto itself. Moreover, by Lemmas 2.11 and 2.12, this map preserves orthogonality in both directions, that is, two idempotents P and Q are orthogonal if and only if $\Psi(P)$ and $\Psi(Q)$ are.

Consider the infinite dimensional case. By [14, Theorem 2.4], either there exists a bounded invertible linear or conjugate-linear operator $S : X \to X$ such that

$$\Psi(P) = SPS^{-1}$$

for every rank one idempotent P, or X is reflexive and there exists a bounded invertible linear or conjugate-linear operator $S : X' \to X$ such that

$$\Psi(P) = SP'S^{-1}$$

for every rank one idempotent P. Let us consider just the second case. We will show that for every $A \in \mathcal{B}(X)$ there exists a nonzero scalar λ such that $\phi(A) = \lambda SA'S^{-1}$. Let $A \in \mathcal{B}(X)$ be any operator. For every rank one idempotent P we have

$$SA'S^{-1}\,SP'S^{-1} \in \mathcal{N}(X) \iff PA \in \mathcal{N}(X) \iff \psi(P)\phi(A) \in \mathcal{N}(X)$$

$$\iff SP'S^{-1}\phi(A) \in \mathcal{N}(X) \iff \phi(A)SP'S^{-1} \in \mathcal{N}(X).$$

The map $B \mapsto SB'S^{-1}$ is an anti-automorphism of $\mathcal{B}(X)$ mapping the set of all rank one idempotents onto itself. Thus, for every rank one idempotent Q we have

$$SA'S^{-1}Q \in \mathcal{N}(X) \iff \phi(A)Q \in \mathcal{N}(X).$$

The desired conclusion follows now directly from Lemma 2.13. Once ϕ is known, we can interchange the role of ψ and ϕ, and show that ψ has the same desired form by Lemma 2.13.

In the finite dimensional case we apply [14, Theorem 2.3] to conclude that there exist a nonsingular matrix $S \in M_n$ and an automorphism ξ of the complex field such that either

$$\Psi(P) = SP_\xi S^{-1}$$

for every rank one idempotent matrix P, or

$$\Psi(P) = SP_\xi^t S^{-1}$$

for every rank one idempotent matrix P. Now we complete the proof as in the infinite dimensional case. □

Proof of Theorems 2.1 and 2.2. The sufficiency parts are clear. Applying Proposition 2.14 with $\psi = \phi$, we obtain the result if ϕ satisfies (1) for $A * B = AB$.

Suppose $\phi : \mathcal{B}(X) \to \mathcal{B}(X)$ is a surjective map satisfying (1) for $A * B = ABA$. Note that ABA is nilpotent if and only if A^2B is so. Thus, we may assume that ϕ satisfies

$$A^2B \in \mathcal{N}(X) \iff \phi(A)^2\phi(B) \in \mathcal{N}(X), \quad A, B \in \mathcal{B}(X). \tag{4}$$

Define an equivalence relation on $\mathcal{B}(X)$ by $A \approx C$ if $[A^2] = [C^2]$. Then $A \approx C$ if and only if the following condition holds.

- For every $T \in \mathcal{B}(X)$, A^2T is nilpotent if and only if C^2T is nilpotent.

By (4), we see that $A \approx C$ if and only if $\phi(A) \approx \phi(C)$.

Let

$$\mathcal{B}^2(X) = \{T^2 : T \in \mathcal{B}(X)\}.$$

Note that $\mathbb{C}I + \mathcal{F}_1(X) \subseteq \mathcal{B}^2(X)$ as $\dim(X) \geq 3$. Let $\mathcal{R} \cup \{0\}$ be a set of distinct representatives of the equivalence relation $\approx$ on $\mathcal{B}(X)$. Then every nonzero $T \in \mathcal{B}^2(X)$ has a unique representation of the form $T = aA^2$ with $(a, A) \in \mathbb{C}^* \times \mathcal{R}$, and every $(a, A) \in \mathbb{C}^* \times \mathcal{R}$ gives rise to a nonzero element $T = aA^2 \in \mathcal{B}^2(X)$.

Define $\psi : \mathcal{B}^2(X) \to \mathcal{B}^2(X)$ such that $\psi(0) = 0$ and $\psi(aA^2) = a\phi(A)^2$ for any $(a, A) \in \mathbb{C}^* \times \mathcal{R}$. Then ψ is surjective. To see this, let $T = B^2 \neq 0$ with $B \in \mathcal{B}(X)$. Then $B \not\approx 0$. Since ϕ is surjective, there is $C \in \mathcal{B}(X)$ such that $\phi(C) = B$. Note that $B \not\approx 0$ implies $C \not\approx 0$. So, $A \approx C$ for some $A \in \mathcal{R}$ and hence $\phi(A) \approx \phi(C) = B$. Thus, $T = B^2 = a\phi(A)^2$ for some $a \in \mathbb{C}^*$ such that $\psi(aA^2) = a\phi(A)^2 = T$.

Evidently, we have $\psi(\mathcal{N}(X) \cap \mathcal{B}^2(X)) = \mathcal{N}(X) \cap \mathcal{B}^2(X)$. Since

$$[\phi(A)^2] = [\psi(A^2)] \qquad \text{for any nonzero } A \in \mathcal{B}(X),$$

condition (4) implies that

$$AB \in \mathcal{N}(X) \iff \psi(A)\phi(B) \in \mathcal{N}(X), \qquad (A, B) \in \mathcal{B}^2(X) \times \mathcal{B}(X).$$

Thus, the result follows from Proposition 2.14. □

Proof of Corollary 2.3. We consider only the case for the Jordan triple product. The proof for the usual product is similar and simpler. Because the map ϕ preserves the spectral radius of products of matrices, it preserves the nilpotency of products of matrices. Thus, we can apply Theorem 2.2. After composing ϕ with a similarity transformation and the transposition, if necessary, we may, and we will assume that the map ϕ is of the form

$$A \mapsto f(A)A_\xi, \quad A \in M_n,$$

for some $\mathbb{C}^*$-valued map f on M_n and some automorphism ξ of $\mathbb{C}$. In particular, $\phi(I_n) = \lambda I_n$ for some nonzero complex number λ. Since $1 = \rho(I_n^3) = \rho(\phi(I_n)^3)$, we have $|\lambda| = 1$. After multiplying ϕ by λ^{-1} we may assume with no loss of generality that $\phi(I_n) = I_n$. It follows that

$\rho(A) = \rho(\phi(A)) = \rho(f(A)A_\xi)$ for every $A \in M_n$. Hence, if A is not nilpotent, then A_ξ is not nilpotent as well and in this case

$$|f(A)| = \frac{\rho(A)}{\rho(A_\xi)}.$$

By this fact and the assumption that $\rho(ABA) = \rho(\phi(A)\phi(B)\phi(A))$ we get

$$\frac{\rho(ABA)}{\rho((ABA)_\xi)} = \frac{\rho(A)\rho(B)\rho(A)}{\rho(A_\xi)\rho(B_\xi)\rho(A_\xi)}$$

for every pair of matrices A, B such that none of A, B, and ABA is nilpotent. Indeed,

$$\rho(ABA) = \rho(f(A)A_\xi f(B)B_\xi f(A)A_\xi) = |f(A)|\,|f(B)|\,|f(A)|\rho((ABA)_\xi)$$

$$= \frac{\rho(A)}{\rho(A_\xi)}\,\frac{\rho(B)}{\rho(B_\xi)}\,\frac{\rho(A)}{\rho(A_\xi)}\rho((ABA)_\xi).$$

Choose $A = E_{11} + (\lambda - \mu)E_{12}$ and $B = \mu E_{11} + E_{21}$ with $\lambda, \mu \neq 0$ to get

$$\frac{|\lambda|}{|\xi(\lambda)|} = \frac{|\mu|}{|\xi(\mu)|},$$

which yields the existence of a complex constant c such that $|\xi(\lambda)| = c|\lambda|$, $\lambda \in \mathbb{C}$. It is well-known that every bounded automorphism of the complex field is either the identity, or the complex conjugation. Thus, ϕ has the form

$$A \mapsto f(A)A \qquad \text{or} \qquad A \mapsto f(A)\overline{A}$$

on M_n. It is now trivial to complete the proof. □

3 Extension to other types of products

In this section, we extend the results in Section 2 to other types of products on $\mathcal{B}(X)$. We introduce the following definition.

Definition 3.1 *Let $k \geq 2$ be a positive integer, and let $(i_1, \ldots, i_m)$ be a sequence with terms chosen from $\{1, \ldots, k\}$. Define a product of k operators $A_1, \ldots, A_k \in \mathcal{B}(X)$ by*

$$A_1 * \cdots * A_k = A_{i_1}A_{i_2}\cdots A_{i_m}.$$

We have the following result.

Theorem 3.2 *Let X be an infinite dimensional Banach space, and consider a product defined as in Definition 3.1 such that there is a term i_p in the sequence $(i_1, \ldots, i_m)$ different from all other terms. Then a surjective map $\phi : \mathcal{B}(X) \to \mathcal{B}(X)$ satisfies*

$$A_1 * \cdots * A_k \in \mathcal{N}(X) \iff \phi(A_1) * \cdots * \phi(A_k) \in \mathcal{N}(X), \quad A_1, \ldots, A_k \in \mathcal{B}(X),$$

if and only if

(a) *there is a bijective bounded linear or conjugate-linear operator* $S : X \to X$ *such that* ϕ *has the form* $A \mapsto S[f(A)A]S^{-1}$, *or*

(b) *the space* X *is reflexive,* $(i_{p+1}, \ldots, i_m, i_1, \ldots, i_{p-1}) = (i_{p-1}, \ldots, i_1, i_m, \ldots, i_{p+1})$*, and there exists a bijective bounded linear or conjugate-linear operator* $S : X' \to X$ *such that* ϕ *has the form* $A \mapsto S[f(A)A']S^{-1}$,

where $f : \mathcal{B}(X) \to \mathbb{C}^*$ *is a map such that for every nonzero* $A \in \mathcal{B}(X)$ *the map* $\lambda \mapsto \lambda f(\lambda A)$ *is surjective on* $\mathbb{C}$.

The assumption that there is i_p appearing only once in the terms of the sequence $(i_1, \ldots, i_m)$ is clearly necessary. For instance, if $A * B = A^2B^2$, then any map $\phi : \mathcal{B}(X) \to \mathcal{B}(X)$ which permute the elements in $\mathcal{T} = \{C \in \mathcal{B}(X) : C^2 = 0\}$ and fix all elements in $\mathcal{B}(X) \setminus \mathcal{T}$ will be a surjective map satisfying $A * B \in \mathcal{N}(X) \iff \phi(A) * \phi(B) \in \mathcal{N}(X)$.

We have the following result for the finite dimensional case.

Theorem 3.3 *Let* $n \geq 3$. *Consider a product on* M_n *defined as in Definition 3.1 such that there is a term* i_p *in* $(i_1, \ldots, i_m)$ *different from all other terms and there is another term* i_q *appearing at most* $n - 1$ *times in* $(i_1, \ldots, i_m)$. *Then a surjective map* $\phi : M_n \to M_n$ *satisfies*

$$A_1 * \cdots * A_k \in N_n \iff \phi(A_1) * \cdots * \phi(A_k) \in N_n, \qquad A_1, \ldots, A_k \in M_n,$$

if and only if

(a) ϕ *has the form* $A \mapsto f(A)SA_\xi S^{-1}$, *or*

(b) $(i_{p+1}, \ldots, i_m, i_1, \ldots, i_{p-1}) = (i_{p-1}, \ldots, i_1, i_m, \ldots, i_{p+1})$ *and* ϕ *has the form* $A \mapsto f(A)SA_\xi^t S^{-1}$,

where $\xi : \mathbb{C} \to \mathbb{C}$ *is a field automorphism,* $S \in M_n$ *is an invertible matrix, and* $f : M_n \to \mathbb{C}^*$ *such that for every nonzero* $A \in M_n$ *the map* $\lambda \mapsto \xi(\lambda)f(\lambda A)$ *is surjective on* $\mathbb{C}$.

Similar to the infinite dimensional case, the assumption of the existence of i_p appearing once in the terms of $(i_1, \ldots, i_m)$ is necessary. However, it is unclear whether the assumption on i_q is essential. Nonetheless, these assumptions will be trivially satisfied if we consider the usual product $A_1 * \cdots * A_k = A_1 \cdots A_k$ and the Jordan triple product $A_1 * A_2 = A_1A_2A_1$.

Similar to Corollary 2.3, we can prove the following result about the spectral radius of products.

Corollary 3.4 *Let* $n \geq 3$. *Consider a product on* M_n *satisfying the hypothesis of Theorem 3.3. A surjective map* $\phi : M_n \to M_n$ *satisfies*

$$\rho(A_1 * \cdots * A_k) = \rho(\phi(A_1) * \cdots * \phi(A_k)), \qquad A_1, \ldots, A_k \in M_n,$$

if and only if ϕ *has one of the following holds:*

(a) ϕ *has the form*

$$A \mapsto f(A)SAS^{-1} \qquad \text{or} \qquad A \mapsto \overline{f(A)}S\overline{A}S^{-1};$$

(b) $(i_{p+1}, \ldots, i_m, i_1, \ldots, i_{p-1}) = (i_{p-1}, \ldots, i_1, i_m, \ldots, i_{p+1})$ *and* ϕ *has the form*

$$A \mapsto f(A)SA^tS^{-1} \qquad \text{or} \qquad A \mapsto \overline{f(A)}SA^*S^{-1};$$

where $S \in M_n$ is an invertible matrix, and $f : M_n \to \{z \in \mathbb{C} : |z| = 1\}$ is a map such that for any nonzero $A \in M_n$ the map $\lambda \mapsto \lambda f(\lambda A)$ is surjective on $\mathbb{C}$.

Proof of Theorems 3.2 and 3.3. The sufficiency parts are clear. Assume that ϕ is surjective and preserves nilpotency of the product $A_1 * \cdots * A_k$. We may set $A_{i_p} = B$ and all other $A_{i_j} = A$. Then ϕ preserves nilpotency of $A * B = A^u B A^v$ for some nonnegative integers u and v such that $u + v = m - 1$. Clearly, $A^u B A^v$ is nilpotent if and only if $A^{u+v} B$ is nilpotent. Thus, ϕ preserves the nilpotency of the product $A * B = A^r B$ with $r = u + v$.

In the finite dimensional case, we will show that ϕ preserves the nilpotency of the product $A * B = A^r B$ for some integer r less than n. Our claim holds if $u + v < n$. Assume that it is not the case. We note that ϕ sends the set of (nonzero) scalar matrices onto itself. This follows from the observation that for any nonzero $B \in M_n$, the following two statements are equivalent.

- B is a scalar matrix,
- BT^{u+v} is nilpotent if and only if T is nilpotent.

Clearly, if B is a scalar matrix, then the second statement holds trivially. If B is not a scalar matrix, then there is an invertible $R \in M_n$ such that $RBR^{-1} = [b_{ij}]$ with $b_{11} = 0$. Let $T = RE_{11}R^{-1}$ be a rank one idempotent. Then BT^{u+v} is a nilpotent of rank at most one. But $T \notin N_n$.

Since ϕ sends the set of scalar matrices onto itself, we can choose $A_{i_p} = B$, $A_{i_q} = A$ and $A_{i_j} = I_n$ for other i_j, and conclude that ϕ will preserve the nilpotency of the product $A * B = A^r B$ where r is the number of times that i_q appears in $\{i_1, \ldots, i_m\}$ and is less than n.

Consequently, ϕ satisfies the following.

$$A^r B \in \mathcal{N}(X) \iff \phi(A)^r \phi(B) \in \mathcal{N}(X), \quad A, B \in \mathcal{B}(X). \tag{5}$$

Now we use the same idea as in the proof of Theorems 2.1 and 2.2, namely, determine a subset $\mathcal{R}$ of $\mathcal{B}(X)$ so that every nonzero element in $\mathcal{B}^r(X) = \{T^r : T \in \mathcal{B}(X)\}$ admits a unique representation aA^r with $(a, A) \in \mathbb{C}^* \times \mathcal{R}$, and define the surjective map $\psi : \mathcal{B}^r(X) \to \mathcal{B}^r(X)$ such that $\psi(0) = 0$ and $\psi(aA^r) = a\phi(A)^r$ for $(a, A) \in \mathbb{C}^* \times \mathcal{R}$. Then (5) implies that

$$AB \in \mathcal{N}(X) \iff \psi(A)\phi(B) \in \mathcal{N}(X), \quad (A, B) \in \mathcal{B}^r(X) \times \mathcal{B}(X).$$

Since $r < \dim(X)$, $\mathcal{B}^r(X)$ contains $\mathbb{C}I + \mathcal{F}_1(X)$. Thus, Proposition 2.14 applies with $\mathcal{S} = \mathcal{B}^r(X)$.

It remains to show that

$$(i_{p+1}, \ldots, i_m, i_1, \ldots, i_{p-1}) = (i_{p-1}, \ldots, i_1, i_m, \ldots, i_{p+1})$$

if ϕ has the form (b) in Theorem 3.2 or 3.3.

Consider the finite dimensional case. After composing ϕ with the map $A \mapsto f(A)^{-1}A$ and $A \mapsto A_{\xi^{-1}}$, we may assume that the map ϕ has the form $A \mapsto SA^tS^{-1}$. As $\phi(A_{i_1}) \cdots \phi(A_{i_m}) = S(A_{i_1}^t \cdots A_{i_m}^t)S^{-1}$, we have

$$A_{i_1} \cdots A_{i_m} \in N_n \iff A_{i_m} \cdots A_{i_1} \in N_n.$$

Evidently, the result holds for $k = 2$. Suppose $k \geq 3$. Note that we may assume $i_p = i_m$ as $A_{i_1} \cdots A_{i_m}$ is nilpotent if and only if $A_{i_{p+1}} \cdots A_{i_m} A_{i_1} \cdots A_{i_p}$ is so. Thus, we need to show

$(i_1, \ldots, i_{m-1}) = (i_{m-1}, \ldots, i_1)$. Assume the contrary, and let t be the smallest integer such that $i_t \neq i_{m-t}$.

Let U and V be matrices of the form

$$\begin{bmatrix} U_1 & 0 \\ 0 & I \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} V_1 & 0 \\ 0 & I \end{bmatrix} \quad \text{with} \quad U_1 = \begin{bmatrix} \lambda & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad V_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

for some nonzero λ. Take $A_{i_t} = U$ and $A_{i_j} = V$ for all $i_j \neq i_t$ and $i_j \neq i_m$. Then

$$A_{i_t} \cdots A_{i_{m-t}} = U^{d_1} V^{e_1} \cdots U^{d_p} V^{e_p}$$

for some positive $d_1, \ldots, d_p, e_1, \ldots, e_p$. Note that

$$U_1^{d_1} V_1^{e_1} \cdots U_1^{d_p} V_1^{e_p} = \begin{bmatrix} \lambda^{d_1+\cdots+d_p} & f(\lambda) \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad V_1^{e_p} U_1^{d_p} \cdots V_1^{e_1} U_1^{d_1} = \begin{bmatrix} \lambda^{d_1+\cdots+d_p} & g(\lambda) \\ 0 & 1 \end{bmatrix},$$

where f and g are polynomials in λ with degree $d_1 + \cdots + d_p$ and $d_2 + \cdots + d_p$, respectively. Thus, there is a nonzero λ such that $f(\lambda) \neq g(\lambda)$. Let $s = d_1 + \cdots + d_p$ and

$$W = \begin{bmatrix} W_1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{with} \quad W_1 = \begin{bmatrix} -f(\lambda) & 0 \\ \lambda^s & 0 \end{bmatrix}.$$

Suppose

$$P = A_{i_1} \cdots A_{i_{t-1}} = A_{i_{m-1}} \cdots A_{i_{m-t+1}} \quad \text{and} \quad Q = A_{i_{m-t+1}} \cdots A_{i_{m-1}} = A_{i_{t-1}} \cdots A_{i_1}$$

and let $A_{i_m} = Q^{-1} W P^{-1}$. Then $A_{i_1} \cdots A_{i_m}$ and $A_{i_m} \cdots A_{i_1}$ equal

$$P \begin{bmatrix} U_1^{d_1} V_1^{e_1} \cdots U_1^{d_p} V_1^{e_p} W_1 & 0 \\ 0 & 0 \end{bmatrix} P^{-1} \quad \text{and} \quad Q^{-1} \begin{bmatrix} W_1 V_1^{e_p} U_1^{d_p} \cdots V_1^{e_1} U_1^{d_1} & 0 \\ 0 & 0 \end{bmatrix} Q,$$

where

$$U_1^{d_1} V_1^{e_1} \cdots U_1^{d_p} V_1^{e_p} W_1 = \begin{bmatrix} 0 & 0 \\ \lambda^s & 0 \end{bmatrix} \quad \text{and} \quad W_1 V_1^{e_p} U_1^{e_p} \cdots V_1^{e_1} U_1^{d_1} = \begin{bmatrix} -\lambda^s f(\lambda) & -f(\lambda) g(\lambda) \\ \lambda^{2s} & \lambda^s g(\lambda) \end{bmatrix},$$

respectively. Note that $A_{i_1} \cdots A_{i_m}$ is nilpotent while $A_{i_m} \cdots A_{i_1}$ is not, which contradicts our assumption. Hence, we must have $(i_1, \ldots, i_{m-1}) = (i_{m-1}, \ldots, i_1)$.

One can easily adapt the proof of the finite dimensional case to the infinite dimensional case to get the desired conclusion. $\square$

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