A Note On The Perturbation Of Positive Matrices By Normal And Unitary Matrices

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Abstract

In a recent paper, Neumann and Sze considered for an $n \times n$ nonnegative and irreducible matrix A, the minimization and maximization of $\rho(A+S)$, the spectral radius of $A+S$, as S ranges over all the doubly stochastic matrices. They showed that both extremal values are always attained at an $n \times n$ permutation matrix. As a permutation matrix is a particular case of a normal matrix whose spectral radius is 1, we consider here, for positive matrices A such that $A+N$ is a nonnegative matrix, for all normal matrices N whose spectral radius is 1, the minimization and maximization problems of $\rho(A+N)$ as N ranges over all such matrices. We show that the extremal values always occur at an $n \times n$ real unitary matrix. We compare our results with a less recent work of Han, Neumann, and Tastsomeros in which the maximum value of $\rho(A+X)$ over all $n \times n$ real matrices X of Frobenius norm $\sqrt{n}$ was sought.

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1 Introduction

In the recent paper [10], one of the questions considered by the authors was the following **additive spectral perturbation problem**: Given a nonnegative matrix $A \in \mathbb{R}^{n,n}$, then find

$$\min_{S \in \Omega_n} \rho(A+S) \quad \text{and} \quad \max_{S \in \Omega_n} \rho(A+S). \tag{1.1}$$

Here $\rho(\cdot)$ is the spectral radius of a matrix and Ω_n is the **set of all $n \times n$ doubly stochastic matrices**. It was shown that both problems attain their solution in $\mathcal{P}_n$, the **set of all $n \times n$ permutation matrices**. Recall that the spectral radius of a nonnegative matrix is also known by the name of the Perron root and that the spectral theory for nonnegative matrices

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is also known as the Perron–Frobenius theory, see Berman and Plemmons [2].

Let $\mathcal{N}_n$ be **the set of $n \times n$ normal matrices with spectral radius 1 in $\mathbb{R}^{n,n}$**. Since permutation matrices are, in particular, normal matrices with spectral radius 1, we can immediately conclude that

$$\min_{N\in\mathcal{N}_n} \rho(A+N) \ \le\ \min_{S\in\Omega_n} \rho(A+S) \ \ \textbf{and}\ \ \max_{S\in\Omega_n} \rho(A+S) \ \le\ \max_{N\in\mathcal{N}_n} \rho(A+N). \tag{1.2}$$

Let $\mathcal{U}_n$ be the **set of $n \times n$ real unitary (orthogonal) matrices**. Clearly, $\mathcal{U}_n \subset \mathcal{N}_n$. In this note, for nonnegative matrices $A = (a_{i,j}) \in \mathbb{R}^{n,n}$ with $a_{i,j} \ge 1$, for all $i, j = 1, \ldots, n$, we investigate the minimum and maximum spectral perturbation expressions, over the normal matrices of spectral radius 1, which appear in (1.2). Note that for such matrices A, the matrix sums $A + N$ and $A + U$ are nonnegative, for all $N \in \mathcal{N}_n$ and for all $U \in \mathcal{U}_n$, respectively. In our main result, Theorem 2.2, we shall show that for such positive matrices A, whose entries are necessarily bounded below by 1, the solution to both problems is attained in the set of the real unitary matrices. Namely, we establish that:

$$\min_{N\in\mathcal{N}_n} \rho(A+N) \ =\ \min_{U\in\mathcal{U}_n} \rho(A+U) \ \ \text{and}\ \ \max_{N\in\mathcal{N}_n} \rho(A+N) \ =\ \max_{U\in\mathcal{U}_n} \rho(A+U) \tag{1.3}$$

Recalling that the Frobenius norm of a matrix in $\mathcal{U}_n$ is $\sqrt{n}$, we can further write that:

$$\min_{X\in\mathcal{X}_n} \rho(A+X) \ \le\ \min_{U\in\mathcal{U}_n} \rho(A+U) \quad \textbf{and} \quad \max_{U\in\mathcal{U}_n} \rho(A+U) \ \le\ \max_{X\in\mathcal{X}_n} \rho(A+X), \tag{1.4}$$

where $\mathcal{X}_n$ is the **set of all $n \times n$ real matrices whose Frobenius norm is $\sqrt{n}$**. In [8], Han, Neumann, and Tsatsomeros investigated the maximization problem of $\rho(A+X)$, where A is an $n \times n$ nonnegative matrix and $X \in \mathbb{R}^{n,n}$ varies over $\mathcal{X}_n$. It was shown that when A is (also) irreducible, the maximizing element is an rank one matrix in $\mathcal{X}_n$.

We develop our main results of this note in the next section.

We comment that much background material on nonnegative matrices and the Perron Frobenius theory can be found in the book by Berman and Plemmons [2]. Furthermore, a partial list of works which consider perturbation problems for nonnegative matrices can be found in the papers: Cohen [3], Deutsch and Neumann [4], Elsner [5], Friedland [6], Golub and Meyer [7], Han, Neumann, and Tsatsomeros [8], and Jonson, Loewy, Olesky, and van den Driessche [9].

2 Main Results

In this section we develop the main results of this paper which were as displayed in (1.2) and (1.3). An auxiliary lemma which will be essential to prove our results is Lemma 2.1 in [10]. The lemma is a special case of the more general result, namely, Lemma 2.2 in [1].

Lemma 2.1 ([10, Lemma 2.1]) *Suppose that* T_1 *and* T_2 *are irreducible nonnegative matrices in* $\mathbb{R}^{n,n}$ *such that* $\operatorname{rank}(T_1 - T_2) = 1$. *Then the map* f_{T_1,T_2} *defined by*

$$f_{T_1,T_2}(\alpha) \;:=\; \rho(\alpha T_1 + (1-\alpha)T_2), \quad \alpha \in [0,1],$$

is either a strictly monotone function or a constant function on $[0,1]$.

Recall from Section 1 that $\mathcal{N}_n$ denotes the set of $n \times n$ normal matrices with spectral radius one in $\mathbb{R}^{n,n}$, while $\mathcal{U}_n$ denotes the set of $n \times n$ orthogonal matrices. The main result of our paper is as follows:

Theorem 2.2 *Let* $A \in \mathbb{R}^{n,n}$ *be a positive matrix such that* $A + N \geq 0$, *for all* $N \in \mathcal{N}_n$. *Then*

$$\min_{U \in \mathcal{U}_n} \rho(A+U) \;=\; \min_{N \in \mathcal{N}_n} \rho(A+N) \tag{2.1}$$

and

$$\max_{U \in \mathcal{U}_n} \rho(A+U) \;=\; \max_{N \in \mathcal{N}_n} \rho(A+N). \tag{2.2}$$

Proof. We shall prove here only the claim in (2.1) as the claim in (2.2) follows similarly. Furthermore, continuity arguments allow us to assume that $A+N$ is nonnegative irreducible for all N in $\mathcal{N}_n$.

Suppose that (2.1) is false. Then there exists a normal matrix $N_0 \in \mathcal{N}_n$ such that

$$\rho(A+N_0) \;=\; \min_{N \in \mathcal{N}_n} \rho(A+N) \;<\; \min_{U \in \mathcal{U}_n} \rho(A+U).$$

In fact, if there is more than one normal matrix yielding the above minimum, let us assume that we have chosen N_0 which has the maximum number of eigenvalues, say p, with modulus one. Clearly, $p < n$. As N_0 is not only normal, but also real, there is an orthogonal matrix W such that

$$N_0 \;=\; W(A_1 \oplus \cdots \oplus A_k)W^*,$$

where each A_i is either a 1×1 real matrix or 2×2 real matrix of the form $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$, with the modulus of eigenvalue(s) of A_1 is (are) strictly less than one.

Suppose, first, that A_1 is a 1×1 real matrix with a real entry a. Then $-1 < a < 1$. Set

$$N_1 \;:=\; W([1] \oplus A_2 \oplus \cdots \oplus A_k)W^* \quad \textbf{and} \quad N_2 \;:=\; W([-1] \oplus A_2 \oplus \cdots \oplus A_k)W^*.$$

Then both N_1 and N_2 are normal and, together, have $p+1$ eigenvalues of modulus one. Furthermore, because of the choice of N_0, we have that:

$$\rho(A+N_0) \;=\; \min_{N \in \mathcal{N}_n} \rho(A+N) \;<\; \min\{\rho(A+N_1), \rho(A+N_2)\}. \tag{2.3}$$

Observe that $N_0 = \alpha N_1 + (1-\alpha)N_2$, where $\alpha = \frac{a+1}{2}$. Let $T_i = A + N_i$, for $i = 0, 1, 2$, in which case

$$T_0 \;=\; \alpha T_1 + (1-\alpha)T_2$$

and

$$T_1 - T_2 \;=\; N_1 - N_2 \;=\; W([2] \oplus 0_{n-1})W^*$$

is a rank one matrix. As, by our assumptions T_1 and T_2 are nonnegative matrices, Lemma 2.1 is applicable and hence the map f_{T_1,T_2} is either strictly monotone or a constant function on $[0,1]$. Thus,

$$\min\{\rho(A+N_1), \rho(A+N_2)\} \;=\; \min\{f_{T_1,T_2}(1), f_{T_1,T_2}(0)\} \;\leq\; f_{T_1,T_2}(\alpha) \;=\; \rho(A+N_0).$$

But this contradicts to (2.3).

Suppose next that A_1 is a 2×2 matrix of the form $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$. Note that A_1 has complex eigenvalues $a \pm ib$. Let $r = \sqrt{a^2+b^2}$. Then $0 < r < 1$. Define

$$B_1 \;:=\; \begin{bmatrix} c & d \\ -b & a \end{bmatrix}, \quad B_2 \;:=\; \begin{bmatrix} -c & -d \\ -b & a \end{bmatrix}$$

and

$$B_3 \;:=\; \begin{bmatrix} c & d \\ -d & c \end{bmatrix}, \quad B_4 \;:=\; \begin{bmatrix} c & d \\ d & -c \end{bmatrix}, \quad B_5 \;:=\; \begin{bmatrix} -c & -d \\ -d & c \end{bmatrix}, \quad B_6 \;:=\; \begin{bmatrix} -c & -d \\ d & -c \end{bmatrix}$$

with $c = a/r$ and $d = b/r$. Note that B_3, B_4, B_5, and B_6 are real normal with eigenvalues of modulus one. Furthermore,

$$A_1 \;=\; \alpha B_1 + (1-\alpha)B_2, \quad B_1 \;=\; \alpha B_3 + (1-\alpha)B_4 \quad \textbf{and} \quad B_2 \;=\; \alpha B_5 + (1-\alpha)B_6$$

for $\alpha = \frac{1+r}{2}$. Now, for $i = 1, \ldots, 6$, let $N_i := W(B_i \oplus A_2 \oplus \cdots \oplus A_k)W^*$. Note that the matrices N_3, N_4, N_5, and N_6 are real normal with $p+2$ eigenvalues of modulus one. Thus, due to the choice of N_0, we can write that

$$\rho(A+N_0) \;=\; \min_{N\in\mathcal{N}_n} \rho(A+N) \;<\; \min\{\rho(A+N_i) : i = 3,4,5,6\}. \tag{2.4}$$

Now let $T_i = A + N_i$, for $i = 0, 1, \ldots, 6$. Note that the T_i's are irreducible nonnegative matrices and that

$$T_0 \;=\; \alpha T_1 + (1-\alpha)T_2, \quad T_1 \;=\; \alpha T_3 + (1-\alpha)T_4 \quad \textbf{and} \quad T_2 \;=\; \alpha T_5 + (1-\alpha)T_6,$$

for $\alpha = \frac{1+r}{2}$ with $0 < \alpha < 1$. Also, for $(i,j) \in \{(1,2), (3,4), (5,6)\}$,

$$T_i - T_j \;=\; W\left((B_i - B_j) \oplus 0_{n-2}\right)W^*$$

are rank one matrices. But then, by Lemma 2.1, the maps f_{T_1,T_2}, f_{T_3,T_4}, and f_{T_5,T_6} are monotone functions. Thus,

$$\begin{array}{rcl} \min\{\rho(T_1),\rho(T_2)\} &=& \min\{f_{T_1,T_2}(1), f_{T_1,T_2}(0)\} \ \le \ f_{T_1,T_2}(\alpha) \ = \ \rho(T_0), \\ \min\{\rho(T_3),\rho(T_4)\} &=& \min\{f_{T_3,T_4}(1), f_{T_3,T_4}(0)\} \ \le \ f_{T_3,T_4}(\alpha) \ = \ \rho(T_1), \\ \min\{\rho(T_5),\rho(T_6)\} &=& \min\{f_{T_5,T_6}(1), f_{T_5,T_6}(0)\} \ \le \ f_{T_5,T_6}(\alpha) \ = \ \rho(T_2). \end{array}$$

Then from (2.4),

$$\rho(T_0) \ < \ \min\{\rho(T_3),\rho(T_4),\rho(T_5),\rho(T_6)\} \ \le \ \min\{\rho(T_1),\rho(T_2)\} \le \rho(T_0),$$

a contradiction. Our proof is now complete. □

As a consequence of Theorem 2.2, the authors' previous work in [10], and the work in [8], we can state the following chain of perturbation inequalities:

Theorem 2.3 *Let $A = (a_{i,j})$ be an $n \times n$ positive matrix whose entries are bounded below by 1. Then:*

$$\min_{X\in\mathcal{X}_n} \rho(A+X) \ \le \ \min_{U\in\mathcal{U}_n} \rho(A+U) \ = \ \min_{N\in\mathcal{N}_n} \rho(A+N) \ \le \ \min_{P\in\mathcal{P}_n} \rho(A+P) \ = \ \min_{S\in\Omega_n} \rho(A+S) \quad (2.5)$$

and

$$\max_{S\in\Omega_n} \rho(A+S) \ = \ \max_{P\in\mathcal{P}_n} \rho(A+P) \ \le \ \max_{N\in\mathcal{N}_n} \rho(A+N) \ = \ \max_{U\in\mathcal{U}_n} \rho(A+U) \ \le \ \max_{X\in\mathcal{X}_n} \rho(A+X). \quad (2.6)$$

Furthermore, the maximum in the rightmost expression in (2.6) is achieved at a rank 1 positive matrix (whose Frobenius norm is $\sqrt{n}$).

We close the paper with an example illustrating our results. Let

$$A \ = \ \begin{bmatrix} 4 & 4 \\ 1 & 1 \end{bmatrix}.$$

As A has constant column sums equal to 5, it easily follows that $\rho(A) = 5$. Furthermore adding any 2×2 doubly stochastic matrix to A, will result in a matrix whose column sums are a constant 6 and so

$$\max_{P\in\mathcal{P}_2} \rho(A+P) = \max_{S\in\Omega_2} \rho(A+S) \ = \ 6.$$

Next, numerically we can find the $\max_{U\in\mathcal{U}_n} \rho(A+U)$ occurs at

$$U \ \approx \ \begin{bmatrix} 0.9239 & -0.3827 \\ 0.3827 & 0.9239 \end{bmatrix}$$

so that

$$6.1168 \ = \ \rho(A+U) \ \approx \ \max_{U\in\mathcal{U}_n} \rho(A+U).$$

Thus we see that in general the extremum of $\rho(A+U)$ as U ranges over the $n \times n$ real unitary matrices can differ from the extremum that $\rho(A+P)$ attains over the $n \times n$ permutation matrices. Finally, consider the rank 1 matrix

$$X \;=\; \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \in \mathcal{X}_2.$$

As $A+X$ has constant column sums equal to $5+\sqrt{2}$,

$$\rho(A+X) \;=\; 5+\sqrt{2} \;\approx\; 6.4142 \;>\; 6.1168 \;=\; \rho(A+U).$$

Furthermore, it can be ascertained from [8] that $\rho(A)+\sqrt{2}$ is the maximum value that $\rho(A+X)$ can attain over $\mathcal{X}_n$. This example shows that the inequalities in (2.5) and (2.6) can be strict.

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