Preservers of eigenvalue inclusion sets of matrix products

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Abstract

For a square matrix A, let $\mathcal{S}(A)$ be an eigenvalue inclusion set such as the Gershgorin region, the union of Cassini ovals, and the Ostrowski's set. Characterization is obtained for maps Φ on $n\times n$ matrices satisfying $\mathcal{S}(\Phi(A)\Phi(B)) = \mathcal{S}(AB)$ for all matrices A and B.

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1. Introduction

Motivated by pure and applied problems, researchers need to understand the eigenvalues of matrices. For example, in numerical analysis or population dynamics, a square matrix A satisfies $\lim_{m\to\infty} A^m = 0$ if and only if all eigenvalues have modulus less than 1; in stability theory of differential equations, the solution of the system of differential equations $x' = Ax$ is stable if and only if all the eigenvalues of A lie in the left half plane; in the study of quadratic forms a Hermitian matrix is positive definite if and only if all the eigenvalues lie on the positive real line; see [7]. However, sometimes there is no practical or efficient way to compute the eigenvalues exactly, say, because the dimension

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of the matrix is too high or numerical and measuring errors in the entries, etc. So, researchers consider eigenvalue inclusion sets; see [7, 14]. For instance, the well known Gershgorin theorem asserts that the eigenvalues of a matrix lie in the union of circular disks centered at the diagonal entries and radii determined by the off-diagonal entries (see the definition in Section 2). These allow one to estimate the location of the eigenvalues for a given matrix efficiently. To further improve the estimate, researchers apply simple transformations such as diagonal similarities to a matrix to get better or easier estimates of the eigenvalue location of the given matrix. In this connection, it is interesting to study maps on matrices that improve or leave invariant eigenvalue inclusion sets $\mathcal{S}(A)$ for matrices $A \in M_n$. In this paper, we consider such problems for several types of eigenvalue inclusion regions including the Gershgorin sets, the Ostrowski's sets, and the Brauer's sets, which are unions of Cassini ovals.

In fact, there is independent interest in studying maps on matrix spaces leaving invariant certain properties, functions or subsets. Such problems are known as *preserver problems*. Early study on the subject focused on linear preservers, i.e., linear maps having the preserving properties; see [10] and its references. Recently, researchers work on general preservers (also referred to as non-linear preservers); see [12] and its references.

To facilitate our discussion, we fixed some notations. Denote by M_n the set of $n \times n$ complex matrices, and $Sp(A)$ the set of eigenvalues of $A \in M_n$. Also $E_{11}, E_{12}, \dots, E_{nn}$ be the standard basis in M_n.

In [11], the authors showed that a linear map $\Phi : M_n \to M_n$ satisfies

$$Sp(\Phi(A)) = Sp(A) \qquad \text{for all } A \in M_n \tag{1.1}$$

if and only if there is an invertible $S \in M_n$ such that Φ has the form $A \mapsto S^{-1}AS$ or $A \mapsto S^{-1}A^tS$. In [5], it was shown that a multiplicative map $\Phi : M_n \to M_n$ satisfies (1.1) if and only if there is an invertible matrix $S \in M_n$ such that Φ has the form $A \mapsto S^{-1}AS$. By the result in [2, Theorem 1.1] (see also [8]), a map $\Phi : M_n \to M_n$ satisfies $Sp(\Phi(A) - \Phi(B)) = Sp(A - B)$ for all $A, B \in M_n$ if and only if there are $R, S \in M_n$, where S is invertible, such that Φ has the form $A \mapsto S^{-1}AS + R$ or $A \mapsto S^{-1}A^tS + R$. Also it was proven in [4] that a map $\Phi : M_n \to M_n$ satisfying $Sp(\Phi(A)\Phi(B)) = Sp(AB)$ for all $A, B \in M_n$ must have the form $A \mapsto \pm S^{-1}AS$ or $A \mapsto \pm S^{-1}A^tS$ with invertible $S \in M_n$. Notice that maps that preserve the spectral values of maximum modulus of products were also studied in [13]. In this connection, it is interesting to know what kinds of transformation on matrices will improve or leave invariant a certain eigenvalue inclusion set $\mathcal{S}(A)$ for $A \in M_n$. If one just assumes that $\mathcal{S}(A) = \mathcal{S}(\Phi(A))$ for every matrix A on Φ, the structure of Φ can be quite arbitrary. For instance, one can partition the set of matrices into equivalence classes so that two matrices A and B belong to the same class if $\mathcal{S}(A) = \mathcal{S}(B)$. If Φ sends each of these classes back to itself, then Φ satisfies $\mathcal{S}(A) = \mathcal{S}(\Phi(A))$ for every matrix A. So, it is reasonable to impose some condition on the map Φ relating the eigenvalue value containment sets of a pair of matrices. In [6], characterizations were obtained for maps Φ satisfying $\mathcal{S}(A - B) = \mathcal{S}(\Phi(A) - \Phi(B))$ for any $A, B \in M_n$. In applications, one often needs to consider the product or powers of matrices, and estimate their eigenvalues. For example, applications in wavelet analysis require the joint spectral radius, which is the maximum eigenvalue of matrix products over a set of matrices [9]. Therefore, we consider Φ satisfying $\mathcal{S}(\Phi(A)\Phi(B)) = \mathcal{S}(AB)$ for any two matrices A and B. It is shown that such maps have tractable structure. An important step in our study is to extract information of the eigenvalues of $\Phi(A)$ using $\mathcal{S}(A)$ and $\mathcal{S}(A^2) = \mathcal{S}(\Phi(A)^2)$. To achieve this, we use the following result in matrix theory; for example see [7, Theorem 3.2.4.2].

Proposition 1.1. *Suppose $A \in M_n$ has n distinct eigenvalues and $B \in M_n$ satisfies $AB = BA$. Then there is a complex polynomial $p(z)$ of degree at most $n-1$ such that $B = p(A)$.*

Though the general strategy we used to prove the results in Sections 2 and 3 are similar, the technical arguments in the proofs are quite different. Thus, instead of just saying that "by a similar argument as in the previous case", we present the proofs of the results for Ostrowski sets and Brauer's sets separately.

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2. Gershgorin and Ostrowski sets

Given a matrix $A = [a_{ij}] \in M_n$. Define

$$R_k = R_k(A) = \sum_{j \neq k} |a_{kj}| \quad \text{and} \quad C_k = C_k(A) = \sum_{j \neq k} |a_{jk}| \qquad k = 1, \ldots, n.$$

The Gershgorin set of A is defined by

$$G(A) = \bigcup_{k=1}^{n} G_k(A) \quad \text{with} \quad G_k(A) = \{\mu \in \mathbb{C} : |\mu - a_{kk}| \leq R_k\}.$$

The set $G_k(A)$ is called a Gershgorin disk of A. It is well known that the Gershgorin set contains all the eigenvalues of A, see, e.g. [7].

Let $\varepsilon \in [0, 1]$. The Ostrowski set of A is defined by

$$O_\varepsilon(A) = \bigcup_{k=1}^{n} O_{\varepsilon,j}(A) \quad \text{with} \quad O_{\varepsilon,k}(A) = \{\mu \in \mathbb{C} : |\mu - a_{kk}| \leq R_k^\varepsilon C_k^{1-\varepsilon}\}.$$

Clearly, the Ostrowski set is an extension of $G(A)$ as $O_1(A) = G(A)$ and $O_0(A) = G(A^t)$. It turns out that for any $\varepsilon \in [0,1]$, $O_\varepsilon(A)$ also contains all eigenvalues of A [7, Chapter 6]. We have the following result on preservers of the Gershgorin and Ostrowski sets.

Theorem 2.1. *Let $\varepsilon \in [0,1]$. A mapping $\Phi : M_n \to M_n$ satisfies*

$$O_\varepsilon(\Phi(A)\Phi(B)) = O_\varepsilon(AB) \qquad \text{for all } A, B \in M_n \tag{2.2}$$

if and only if there exist $c = \pm 1$, a permutation matrix P, and an invertible diagonal matrix D, where D is unitary unless $(n, \varepsilon) = (2, 1/2)$, such that

$$\Phi(A) = c(DP)A(DP)^{-1} \quad \text{for all} \quad A \in M_n.$$

We write $O(A)$ and $O_j(A)$ instead of $O_\varepsilon(A)$ and $O_{\varepsilon,j}(A)$, respectively, for notational simplicity if the meaning of ε is clear in the context. The following observations will be used in our proof.

Lemma 2.2. *Let $\varepsilon \in (0,1)$ and $A, B \in M_n$.*

(a) *If $O_\varepsilon(A^2)$ consists of n disjoint isolated points, then*

$$C_k(A)\, R_k(A) = 0 \quad \text{for all} \quad k = 1, \ldots, n. \tag{2.3}$$

(b) *If A and B each satisfy (2.3) and the set $O_\varepsilon(AB)$ consists of a collection of nonzero isolated points, then*

$$C_k(A)\, R_k(B) = 0 \quad \text{for all} \quad k = 1, \ldots, n.$$

Proof. (a) Suppose $O_\varepsilon(A^2)$ consists of n disjoint isolated points. Then for every k, $C_k(A^2)R_k(A^2) = 0$. So either $C_k(A^2) = 0$ or $R_k(A^2) = 0$. On the other hand, each disk of $O_\varepsilon(A^2)$ contains at least one eigenvalue of A^2 and therefore A^2 has n distinct eigenvalues. By Proposition 1.1, any matrix commuting with A^2 is a polynomial of A^2. In particular, A is a polynomial of A^2. Then $C_k(A^2) = 0$ implies $C_k(A) = 0$, or $R_k(A^2) = 0$ implies $R_k(A) = 0$. Thus, the result follows.

(b) Suppose $A = [a_{ij}]$ and $B = [b_{ij}]$ satisfy the hypothesis. Assume that $C_k(A)R_k(B) \neq 0$ for some k. Then $R_k(A) = 0$ and $C_k(B) = 0$. Write $A = P \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} P^t$ and $B = P \begin{bmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{bmatrix} P^t$ for some permutation matrix P, where $A_{11} = [a_{kk}] \in M_1$, $B_{11} = [b_{kk}] \in M_1$, and $A_{22}, B_{22} \in M_{n-1}$. Notice also that A_{21} and B_{12} are nonzero $(n-1) \times 1$ matrix and $1 \times (n-1)$ matrix respectively. Since

$$0 \notin O_\varepsilon(AB) = O_\varepsilon\left(P \begin{bmatrix} A_{11}B_{11} & A_{11}B_{12} \\ A_{21}B_{11} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix} P^t \right),$$

both A_{11} and B_{11} are nonzero and $O_\varepsilon(AB)$ contains a non-degenerate disk centered at $A_{11}B_{11}$. But this contradicts the hypothesis. Therefore, $C_k(A)R_k(B) = 0$. □

Proof of Theorem 2.1. For the sufficiency part, note that $O(X) = O(PXP^t)$ for any permutation matrix P, $O(X) = O(DXD^*)$ for any diagonal unitary matrix D, and $O(X) = O(DXD^{-1})$ for any invertible diagonal matrix D if $(n, \varepsilon) = (2, 1/2)$. If $\Phi(A) = \pm(DP)A(DP)^{-1}$, then $\Phi(A)\Phi(B) = DP(AB)P^tD^{-1}$ and $O(AB) = O(\Phi(A)\Phi(B))$.

For the necessity part, suppose $\Phi : M_n \to M_n$ satisfies (2.2). We first prove the case when $\varepsilon = 1$. Recall that $O_1(A) = G(A)$, the Gershgorin set of A. The proof is divided into Assertions 2.1–2.3.

Assertion 2.1. *Let $D = \mu\,\mathrm{diag}\,(1, \ldots, n)$ for $\mu > 1$. There exist a permutation matrix P and a diagonal matrix $R = \mathrm{diag}\,(r_1, \ldots, r_n)$ with $r_k \in \{1, -1\}$ such that $\Phi(D) = PRDP^t$.*

Proof. Since $G(\Phi(D)^2) = G(D^2) = \{\mu^2, (2\mu)^2, \ldots, (n\mu)^2\}$, we see that $\Phi(D)^2$ is a diagonal matrix with diagonal entries as in $G(D^2)$. Note that $\Phi(D)$ commutes with $\Phi(D)^2$, so by Proposition 1.1, $\Phi(D)$ is a polynomial of $\Phi(D)^2$. Thus, $\Phi(D)$ is a diagonal matrix with diagonal entries whose squares equal to $\mu^2, (2\mu)^2, \ldots, (n\mu)^2$. Then there are $r_1, \ldots, r_n \in \{-1, 1\}$ and a permutation matrix P such that $\Phi(D) = PRDP^t$ with $R = \mathrm{diag}\,(r_1, \ldots, r_n)$.

Assertion 2.2. *Following the notation in Assertion 2.1, there are $\nu_{11}, \ldots, \nu_{nn} \in \mathbb{C}$ with $|\nu_{ij}| = 1$ and $\nu_{ij}\nu_{ji} = 1$ such that*

$$\Phi(E_{ij}) = \nu_{ij} P E_{ij} P^t \quad \text{for all} \quad 1 \le i, j \le n.$$

Proof. Without loss of generality, we may assume that $P = I_n$ in Assertion 2.1. For every k, notice that $G((RD)\Phi(E_{kk})) = G(DE_{kk}) = \{k\mu, 0\}$. It follows that $\Phi(E_{kk})$ is a diagonal matrix and its j-th diagonal entry must be either 0 or $k/(r_j j)$. Observe that $G(\Phi(E_{kk})^2) = G(E_{kk}^2) = \{1, 0\}$. Thus, only the k-th diagonal entry of $\Phi(E_{kk})$ is nonzero. Hence, $\Phi(E_{kk}) = r_k E_{kk}$. Let $\nu_{kk} = r_k$. Clearly, $|\nu_{kk}| = \nu_{kk}\nu_{kk} = 1$. Then the assertion holds for $i = j$.

Assume $i \neq j$. Since $G(\Phi(E_{ij})(\nu_{kk}E_{kk})) = G(E_{ij}E_{kk}) = \{0\}$ for all $k \neq j$, only the j-th column of $\Phi(E_{ij})$ can contain nonzero entries. Similarly, $G((\nu_{kk}E_{kk})\Phi(E_{ij})) = G(E_{kk}E_{ij}) = \{0\}$ for all $k \neq i$, and so only the i-th row of $\Phi(E_{ij})$ can contain nonzero entries. Therefore, $\Phi(E_{ij}) = \nu_{ij}E_{ij}$ for some $\nu_{ij} \in \mathbb{C}$. Now as $G((\nu_{ii}E_{ii})(\nu_{ij}E_{ij})) = G(E_{ii}E_{ij})$ is a disk centered at 0 with radius 1, we have $|\nu_{ii}\nu_{ij}| = 1$ and hence $|\nu_{ij}| = 1$. Also $G(\nu_{ij}\nu_{ji}E_{ij}E_{ji}) = G(E_{ij}E_{ji}) = \{1, 0\}$ gives $\nu_{ij}\nu_{ji} = 1$.

Assertion 2.3. *For $\varepsilon = 1$, the map Φ has the asserted form as in Theorem 2.1.*

Proof. Assume that Assertions 2.1 and 2.2 hold with $P = I_n$. For any $A = [a_{ij}]$, let $B = [b_{ij}] = \Phi(A)$. By Assertion 2.2, $G((\nu_{ji}E_{ji})B) = G(E_{ji}A)$. Then $b_{ij} = \nu_{ji}^{-1}a_{ij} = \nu_{ij}a_{ij}$ and so

$$\Phi(A) = N \circ A = [\nu_{ij}a_{ij}] \quad \text{with} \quad N = [\nu_{ij}], \tag{2.4}$$

where $N \circ A$ is the Schur (entrywise) product of N and A. Without loss of generality, we may assume that $\nu_{11} = 1$; otherwise, replace Φ by $A \mapsto -\Phi(A)$. Let $X = E_{11} + E_{1k} + E_{k1} + E_{kk}$ with $k \neq 1$. Then $X^2 = 2X$. By (2.4), $\Phi(X) = E_{11} + \nu_{1k}E_{1k} + \nu_{k1}E_{k1} + \nu_{kk}E_{kk}$ and hence

$$\Phi(X)^2 = 2E_{11} + (1 + \nu_{kk})\nu_{1k}E_{1k} + (1 + \nu_{kk})\nu_{k1}E_{k1} + 2E_{kk}.$$

If $\nu_{kk} = -1$, then $\Phi(X)^2 = 2(E_{11} + E_{kk})$. But then $O(\Phi(X)^2) = \{2, 0\}$, which contradicts the fact that $O(X^2) = O(2X)$ is a non-degenerate disk centered at 2 with radius 2. Therefore, $\nu_{kk} = 1$ for all k. For $n = 2$, the map has the form $\Phi(A) = SAS^{-1}$ with $S = \operatorname{diag}(1, \nu_{21})$. Then the assertion holds when $n = 2$.

Suppose $n \geq 3$. Let $U = \operatorname{diag}(1, \nu_{21}, \ldots, \nu_{n1})$. Then U is a unitary matrix as $|\nu_{ij}| = 1$ for all i and j. By replacing Φ with the map $A \mapsto U^{-1}\Phi(A)U$, we may further assume that $\nu_{k1} = \nu_{1k} = 1$ for all k. Now the assertion holds if one can show that $\nu_{ij} = 1$ for all $2 \leq i, j \leq n$. To prove this, let $X = E_{11} + E_{1i} + E_{1j} + E_{i1} + E_{ii} + E_{ij}$ with $i \neq j$. Notice that $X^2 = 2X$. Then $G(X^2)$ is a disk centered at 2 with radius 4. By (2.4), $\Phi(X) = E_{11} + E_{1i} + E_{1j} + E_{i1} + E_{ii} + \nu_{ij}E_{ij}$ and hence

$$\Phi(X)^2 = 2(E_{11} + E_{1i} + E_{i1} + E_{ii}) + (1 + \nu_{ij})(E_{1j} + E_{ij}).$$

So $G(\Phi(X)^2)$ is the disk centered at 2 with radius $(2 + |1 + \nu_{ij}|)$. Then $O(\Phi(X)^2) = O(X^2)$ implies $\nu_{ij} = 1$. Thus, the assertion holds.

For $\varepsilon = 0$, the proof is similar. We now move to the case when $\varepsilon \in (0, 1)$. The proof of this case is more delicate, and we divide the proof into Assertions 2.4 – 2.6.

Assertion 2.4. *Let $D = \mu \operatorname{diag}(1, \ldots, n)$ with $\mu > 1$. Then there exist a permutation matrix P and a diagonal matrix $R = \operatorname{diag}(r_1, \ldots, r_n)$ with $r_j \in \{1, -1\}$ such that*

$$\Phi(D) = PRDP^t \quad \textit{and} \quad \Phi(D + E_{ij}) = P(RD + \nu_{ij}E_{ij})P^t \quad \textit{for all} \quad i \neq j,$$

where ν_{ij}'s are nonzero numbers such that $\nu_{ij}\nu_{ji} = 1$.

Proof. Since $O(\Phi(D)^2) = O(D^2) = \{\mu^2, \ldots, (n\mu)^2\}$, $\Phi(D)^2$ has diagonal entries $\mu^2, \ldots, (n\mu)^2$. Moreover, by Lemma 2.2 (a), $C_k(\Phi(D))\, R_k(\Phi(D)) = 0$ for all k. It follows that the diagonal entries of $\Phi(D)^2$ are squares of the diagonal entries of $\Phi(D)$. Thus, there are $r_1, \ldots, r_n \in \{-1, 1\}$ and a permutation matrix P such that the k-th diagonal entry of $P^t\Phi(D)P$ is $r_k k\mu$. Without loss of generality, we may assume that $P = I_n$; otherwise, we replace Φ by the map $A \mapsto P^t\Phi(A)P$.

Let $X_{ij} = D + E_{ij}$. We **claim** that for distinct i and j, $\Phi(D)$ and $\Phi(X_{ij})$ have the same diagonal entries. First, by a similar argument as in the first paragraph, one sees that $C_k(\Phi(X_{ij}))R_k(\Phi(X_{ij})) = 0$ for all k, and if $d_1, \ldots, d_n$ are the diagonal entries of $\Phi(X_{ij})$, $\{d_1^2, \ldots, d_n^2\} = \{\mu^2, \ldots, (n\mu)^2\}$. Notice also that $O(\Phi(X_{ij})\Phi(D)) = \{\mu^2, \ldots, (n\mu)^2\}$. By Lemma 2.2 (b),

$$C_k(\Phi(X_{ij}))\, R_k(\Phi(D)) = 0 \quad \text{for} \quad k = 1, \ldots, n. \tag{2.5}$$

Then the k-th diagonal entry of $\Phi(D)\Phi(X_{ij})$ is equal to the product of k-th diagonal entries of $\Phi(D)$ and $\Phi(X_{ij})$, i.e., $(r_k k\mu)d_k$, which is in the set $O(\Phi(D)\Phi(X_{ij}))$. Then we must have $d_k = r_k k\mu$. Hence, the claim holds.

Observe that $O(X_{ij}X_{kj}) = O(X_{ij}X_{ik}) = \{\mu^2, \ldots, (n\mu)^2\}$ for all $k \neq j$. Lemma 2.2 (b) and (2.2) yield

$$C_k(\Phi(X_{ij}))\, R_k(\Phi(X_{kj})) = C_k(\Phi(X_{ij}))\, R_k(\Phi(X_{ik})) = 0.$$

Suppose $C_k(\Phi(X_{ij})) \neq 0$. Then $R_k(\Phi(X_{kj})) = R_k(\Phi(X_{ik})) = 0$ and hence $R_k(\Phi(X_{kj})\Phi(X_{ik})) = 0$. Moreover, the k-th diagonal entry of $\Phi(X_{kj})\Phi(X_{ik})$ is equal to the product of the k-th diagonal entries of $\Phi(X_{kj})$ and $\Phi(X_{ik})$, which is $(k\mu)^2$. Then the set $O(\Phi(X_{kj})\Phi(X_{ik}))$ has a degenerate disk centered at $(k\mu)^2$. But this contradicts the fact that $O(X_{kj}X_{ik})$ has n disjoint disks and the disk centered at $(\mu k)^2$ is non-degenerate. Therefore, $C_k(\Phi(X_{ij})) = 0$ for all $k \neq j$. Similarly, one can show that $R_k(\Phi(X_{ij})) = 0$ for $k \neq i$ by the fact that $O(X_{ik}X_{ij}) = O(X_{kj}X_{ij}) = \{\mu^2, \ldots, (n\mu)^2\}$ for all $k \neq i$. Therefore, only the (i, j)-th off-diagonal entry of $\Phi(X_{ij})$ can be nonzero. With the above claim, $\Phi(X_{ij}) = RD + \nu_{ij}E_{ij}$ for some $\nu_{ij} \in \mathbb{C}$. Finally, since $O(\Phi(X_{ji})\Phi(X_{ij})) = O(X_{ji}X_{ij})$ has two non-degenerate disks centered at $(i\mu)^2$ and $(j\mu)^2 + 1$ with radius $|i\mu|$, one can conclude that $\nu_{ij}\nu_{ji} = 1$. Thus, the last part of the assertion holds.

Finally, we show that $\Phi(D)$ is a diagonal matrix. Once this is proved, we can conclude that $\Phi(D) = RD$ with $R = \operatorname{diag}(r_1, \ldots, r_n)$ and the assertion holds. Suppose $\Phi(D)$ is not diagonal. Then $C_j(\Phi(D)) \neq 0$ for some j. By (2.5), $C_j(\Phi(X_{ij})) = C_j(\Phi(X_{ji})) = 0$ for all $i \neq j$. Then the j-th diagonal entry of $\Phi(X_{ij})\Phi(X_{ji})$ is equal to the product of the j-th diagonal entries of $\Phi(X_{ij})$

and $\Phi(X_{ji})$, which is $(r_j j\mu)\cdot(r_j j\mu)=(j\mu)^2$. Moreover, $C_j(\Phi(X_{ij})\Phi(X_{ji}))=0$. Then the set $O(\Phi(X_{ij})\Phi(X_{ji}))$ has a degenerate disk centered at $(j\mu)^2$. But the set $O(X_{ij}X_{ji})$ contains n disjoint disks and the disk centered at $(j\mu)^2$ is non-degenerate. Thus, we have derived a contradiction.

Assertion 2.5. *Following the notation in Assertion 2.4 and define $\nu_{kk}=r_k$ for $1\le k\le n$, we have*

$$\Phi(E_{ij})=\nu_{ij}PE_{ij}P^t \quad \textit{for all} \quad 1\le i,j\le n.$$

Proof. Assume that Assertion 2.4 holds with $P=I_n$. Suppose first that $i\ne j$. By the fact that $O((RD)\Phi(E_{ij}))=O(DE_{ij})=\{0\}$, one sees that all diagonal entries of $\Phi(E_{ij})$ are zero. On the other hand, for any $s\ne t$, $O((RD+\nu_{st}E_{st})\Phi(E_{ij}))=O(X_{st}E_{ij})$, which is equal to $\{0\}$ or $\{1,0\}$ depending on $(s,t)\ne(j,i)$ or $(s,t)=(j,i)$. It follows that all entries of $\Phi(E_{ij})$ are zero, except the (i,j)-th entry which is equal to $\nu_{ji}^{-1}=\nu_{ij}$. Thus, $\Phi(E_{ij})=\nu_{ij}E_{ij}$ and the assertion holds for $i\ne j$.

Next, for $k=1,\dots,n$, $O((\nu_{ij}E_{ij})\Phi(E_{kk}))=O(E_{ij}E_{kk})=\{0\}$ for all $i\ne j$. Then all off-diagonal entries of $\Phi(E_{kk})$ have to be zero and hence $\Phi(E_{kk})$ is a diagonal matrix. Further, $O(\Phi(E_{kk})^2)=O(E_{kk}^2)=\{1,0\}$ implies that the diagonal entries are either -1, 0, or 1. Finally, $O((RD)\Phi(E_{kk}))=O(DE_{kk})=\{k,0\}$ implies that all diagonal entries have to be zero, except the k-th entry, which is equal to r_k, i.e., $\Phi(E_{kk})=r_kE_{kk}$.

Assertion 2.6. *For $\varepsilon\in(0,1)$, the map Φ has the asserted form as in Theorem 2.1.*

Proof. Assume that Assertions 2.4 and 2.5 hold with $P=I_n$. By a similar argument as in the first part of Assertion 2.3, one can assume that $\nu_{kk}=1$ for all k and for any $A=[a_{ij}]\in M_n$,

$$\Phi(A)=N\circ A=[\nu_{ij}a_{ij}] \quad \text{with} \quad N=[\nu_{ij}]. \tag{2.6}$$

For $(n,\varepsilon)=(2,1/2)$, the map has the form $\Phi(A)=SAS^{-1}$ with $S=\operatorname{diag}(1,\nu_{21})$. Therefore, the result follows in this case. In the following, we assume that $(n,\varepsilon)\ne(2,1/2)$.

We first claim that $|\nu_{ij}|=1$ for all $i\ne j$. Once the claim holds, then the assertion holds for $n=2$. For the case $\varepsilon\ne1/2$, by (2.6), $O(\nu_{ij}E_{ij}+\nu_{ji}E_{ji})=O(E_{ij}+E_{ji})$ is a disk centered at origin with radius 1. Then $\max\{|\nu_{ij}|^\epsilon|\nu_{ji}|^{1-\varepsilon},|\nu_{ji}|^\epsilon|\nu_{ij}|^{1-\varepsilon}\}=1$, and therefore $|\nu_{ij}|=1$. For the case $\varepsilon=1/2$ and $n\ge3$, let $X=E_{ii}+2E_{ij}+E_{ik}+E_{ji}$ and $Y=E_{ii}+E_{ij}+2E_{ik}+E_{ji}$ with $k\ne i,j$. Notice that $\Phi(I_n)=I_n$. Then

$$O(I_n\Phi(X))=O(I_nX)=O(I_nY)=O(I_n\Phi(Y))$$

yields $|\nu_{ji}|(2|\nu_{ij}|+|\nu_{ik}|) = |\nu_{ji}|(|\nu_{ij}|+2|\nu_{ik}|)$ and hence $|\nu_{ij}| = |\nu_{ik}|$. Similarly, one can show that $|\nu_{ji}| = |\nu_{jk}|$ and $|\nu_{ki}| = |\nu_{kj}|$. Then

$$|\nu_{ij}| = |\nu_{ik}| = |\nu_{ki}|^{-1} = |\nu_{kj}|^{-1} = |\nu_{jk}| = |\nu_{ji}| = |\nu_{ij}|^{-1},$$

and therefore $|\nu_{ij}| = 1$. Thus, the claim holds.

Now assume $n \geq 3$. Let $U = \operatorname{diag}(1, \nu_{21}, \ldots, \nu_{n1})$, which is a unitary matrix. By replacing Φ with the map $A \mapsto U^{-1}\Phi(A)U$, we may assume that $\nu_{1k} = \nu_{k1} = 1$ for all k. It remains to show that $\nu_{ij} = 1$ for all $2 \leq i, j \leq n$. To see this, consider $X = E_{11} + E_{1i} + E_{1j} + E_{i1} + E_{ii} + E_{ij}$ with $i \neq j$. Notice that $X^2 = 2X$. By (2.6), $\Phi(X) = E_{11} + E_{1i} + E_{1j} + E_{i1} + E_{ii} + \nu_{ij}E_{ij}$ and hence $\Phi(X)^2 = 2(E_{11}+E_{1i}+E_{i1}+E_{ii})+(1+\nu_{ij})(E_{1j}+E_{ij})$. Since $O(\Phi(X)^2) = O(X^2)$ is a disk centered at 2 with radius $2^{1+\epsilon}$. One can conclude that $\nu_{ij} = 1$. Thus, the result follows. □

3. Brauer's Set

In this section, we consider another eigenvalue inclusion set, the *Brauer's Set* [1] of a matrix $A = [a_{ij}] \in M_n$ which is defined by

$$C(A) = \bigcup_{1\leq i<j\leq n} C_{ij}(A),$$

where $C_{ij}(A)$, the (i,j)-th *Cassini oval* of A with $i \neq j$, is defined by

$$C_{ij}(A) = \{\mu \in \mathbb{C} : |(\mu - a_{ii})(\mu - a_{jj})| \leq R_i(A)R_j(A)\}.$$

There are discussions of the Cassini oval in the standard references. We have a similar result for a map that preserves the Brauer's set of product of matrices.

Theorem 3.1. *A mapping $\Phi : M_n \to M_n$ satisfies*

$$C(\Phi(A)\Phi(B)) = C(AB) \qquad \text{for all } A, B \in M_n \tag{3.7}$$

if and only if there exist $c = \pm 1$, a permutation matrix P and an invertible diagonal matrix D, where D is unitary if $n \geq 3$, such that

$$\phi(A) = c(DP)A(DP)^{-1} \quad \text{for all} \quad A \in M_n.$$

The following results about Cassini ovals will be used in the proof.

Lemma 3.2. *Let $A = [a_{ij}]$ and $B = [b_{ij}]$ in M_n.*

(a) *The set $C(A)$ consists of a collection of isolated points if and only if A has at most one row with nonzero off-diagonal entries.*

(b) *If $C(A^2)$ consists of n isolated points, then $C(A^2)$ contains squares of the diagonal entries of A and at most one row of A has nonzero off-diagonal entries.*

(c) *If each of the sets $C(A^2)$, $C(B^2)$, and $C(AB)$ consists of n nonzero isolated points, then A and B can have nonzero off-diagonal entries in one common row only.*

Proof. Part (a) is trivial by definition. To show part (b), suppose $C(A^2)$ consists of n isolated points. Then A^2 must have n distinct eigenvalues. By (a), at most one row of A^2 has nonzero off-diagonal entries. Moreover, A is a polynomial of A^2 by Proposition 1.1. Therefore, at most one row of A has nonzero off-diagonal entries. Thus, part (b) follows.

Now suppose A and B satisfy the hypothesis in (c). By (b), at most one row of each of A and B can have nonzero off-diagonal entires. Furthermore, all diagonal entries of A and B are nonzero. Assume A and B have nonzero off-diagonal entries in its i-th and j-th row, respectively. If $i \neq j$, then both the i-th and j-th rows of AB have nonzero off-diagonal entries. But then $C(AB)$ has a non-degenerate oval, which is a contradiction and part (c) holds. □

Proof of Theorem 3.1. Note that $C(X) = C(PXP^t)$ for any permutation matrix P, $C(X) = C(DXD^*)$ for any diagonal unitary matrix D, and $C(X) = C(DXD^{-1})$ for any invertible diagonal matrix D if $n = 2$. Hence, if $\Phi(A) = \pm(DP)A(DP)^{-1}$, then $\Phi(A)\Phi(B) = DP(AB)P^tD^{-1}$ and $C(AB) = C(\Phi(A)\Phi(B))$. We now prove the converse by dividing the proof into several assertions. In the following, we assume that $\Phi : M_n \to M_n$ satisfies (3.7).

Assertion 3.1. *Let $D = \mu\,\mathrm{diag}\,(1, \ldots, n)$ with $\mu > 1$. Then there exist a permutation matrix P and a diagonal matrix $R = \mathrm{diag}\,(r_1, \ldots, r_n)$ with $r_j \in \{1, -1\}$ such that*

$$\Phi(D) = PRDP^t \quad \textit{and} \quad \Phi(D + E_{ij}) = PR(D + \nu_{ij}E_{ij})P^t \quad \textit{for all} \quad i \neq j,$$

where ν_{ij}'s are nonzero numbers such that $\nu_{ij}\nu_{ji} = 1$.

Proof. Let $X_{ij} = D + E_{ij}$. Notice that for any $i \neq j$, $C(\Phi(D)^2) = C(\Phi(X_{ij})^2) = \{\mu^2, \ldots, (n\mu)^2\}$. By Lemma 3.2 (b), both $\Phi(D)$ and $\Phi(X_{ij})$ have at most one row with nonzero off-diagonal entries.

We claim that $\Phi(D)$ is a diagonal matrix. Suppose $\Phi(D)$ has a nonzero off-diagonal entry in the k-th row for some k. By the fact that $C(\Phi(D)\phi(X_{ij})) = \{\mu^2, \ldots, (n\mu)^2\}$ and Lemma 3.2 (c), only the k-th row of $\Phi(X_{ij})$ can contain nonzero off-diagonal entries. In particular, this observation holds for $\Phi(X_{12})$ and $\Phi(X_{21})$ and hence only the k-th row of $\Phi(X_{12})\Phi(X_{21})$ may have nonzero off-diagonal entries. Therefore, $C(\Phi(X_{12})\Phi(X_{21}))$ contains isolated points only, which contradicts that $C(X_{12}X_{21})$ has a non-degenerate oval. Therefore, the claim holds and $\Phi(D)$ is a diagonal matrix. Since $C(\Phi(D)^2) = \{\mu^2, \ldots, (n\mu)^2\}$, there exist a permutation matrix P and $R = \operatorname{diag}(r_1, \ldots, r_n)$ with $r_j \in \{1, -1\}$ such that $\Phi(D) = PRDP^t$.

Without loss of generality, we may assume that $P = I_n$. Now the fact that $C(\Phi(D)\Phi(X_{ij})) = \{\mu^2, \ldots, (n\mu)^2\}$ implies that $\Phi(X_{ij})$ has the same diagonal entries as $\Phi(D)$. Since $C(\Phi(X_{ij})\Phi(X_{ji})) = C(X_{ij}X_{ji})$ has two non-degenerate disks with centers $(i\mu)^2 + 1$ and $(j\mu)^2$, that implies that the (i, j)-th entry of $\Phi(X_{ij})$ and the (j, i)-th entry of $\Phi(X_{ji})$ must be nonzero. Suppose the (i, k)-th entry of $\Phi(X_{ij})$ is nonzero for some $k \neq i, j$ if $n \geq 3$. Recall that $\Phi(X_{ki})$ has nonzero (k, i)-th entry. Then one sees that $(i\mu)^2$ is not a center of any of the oval in $C(\Phi(X_{ij})\phi(X_{ki}))$. But this contradicts that $C(X_{ij}X_{ki})$ has two non-degenerate disks centered at $(i\mu)^2$ and $(k\mu)^2$. Therefore, $\Phi(X_{ij}) = RD + \nu_{ij}E_{ij}$ for some nonzero $\nu_{ij} \in \mathbb{C}$. Finally, since $C(\Phi(X_{ij})\Phi(X_{ji})) = C(X_{ij}X_{ji})$, one can conclude that $\nu_{ij}\nu_{ji} = 1$.

Assertion 3.2. *Following the notation in Assertion 3.1 and define $\nu_{kk} = r_k$, we have*

$$\Phi(E_{ij}) = \nu_{ij} P E_{ij} P^t \quad \textit{for all} \quad 1 \leq i, j \leq n.$$

Proof. Without loss of generality, we assume $P = I_n$ in Assertion 3.1. For any distinct i and j, since $C((RD)\Phi(E_{ij})) = C(DE_{ij}) = \{0\}$, all diagonal entries of $\Phi(E_{ij})$ are zero. Furthermore, as $C(\Phi(X_{st})\Phi(E_{ij}))$ is equal to $\{0\}$ or $\{1, 0\}$, depending on $(s, t) \neq (j, i)$ or $(s, t) = (j, i)$, only the (i, j)-th entry can be nonzero and equal to $\nu_{ji}^{-1} = \nu_{ij}$, i.e., $\Phi(E_{ij}) = \nu_{ij}E_{ij}$. Thus, the assertion holds for $i \neq j$. Now for any $k = 1, \ldots, n$, since $C((\nu_{ij}E_{ij})\Phi(E_{kk})) = C(E_{ij}E_{kk}) = \{0\}$ for all $i \neq j$, $\Phi(E_{kk})$ must be a diagonal matrix. Moreover, $C(\Phi(E_{kk}^2)) = \{1, 0\}$ and $C(\Phi(D)\Phi(E_{kk})) = \{k, 0\}$ implies that $\Phi(E_{kk}) = r_k E_{kk}$. Then the assertion follows.

Assertion 3.3. *The map Φ has the asserted form as in Theorem 3.1.*

Proof. Assume that $P = I_n$ in Assertions 3.1 and 3.2. By a similar argument as in the first part of Assertion 2.3, one can assume that $\nu_{kk} = 1$ for all k, and for any $A = [a_{ij}] \in M_n$,

$$\Phi(A) = N \circ A = [\nu_{ij}a_{ij}] \quad \text{with} \quad N = [\nu_{ij}]. \tag{3.8}$$

For the case when $n = 2$, the map has the form $\Phi(A) = SAS^{-1}$ with $S = \operatorname{diag}(1, \nu_{21})$. Therefore, the result follows if $n = 2$. In the following, we assume that $n \geq 3$.

Let $X = 2E_{ij} + E_{ik} + E_{ji}$ and $Y = E_{ij} + 2E_{ik} + E_{ji}$ with $k \neq i, j$. Notice that $\Phi(I_n) = I_n$. Then

$$C(I_n\Phi(X)) = C(I_nX) = C(I_nY) = C(I_n\Phi(Y)),$$

which is a disk centered at zero with radius $\sqrt{3}$. Thus, $|\nu_{ji}|(2|\nu_{ij}| + |\nu_{ik}|) = |\nu_{ji}|(|\nu_{ij}| + 2|\nu_{ik}|)$ and hence $|\nu_{ij}| = |\nu_{ik}|$. Similarly, one can show that $|\nu_{ji}| = |\nu_{jk}|$ and $|\nu_{ki}| = |\nu_{kj}|$. Then

$$|\nu_{ij}| = |\nu_{ik}| = |\nu_{ki}|^{-1} = |\nu_{kj}|^{-1} = |\nu_{jk}| = |\nu_{ji}| = |\nu_{ij}|^{-1}$$

and therefore $|\nu_{ij}| = 1$. Now let $U = \text{diag}\,(1, v_{21}, \ldots, v_{n1})$, which is a unitary matrix. By replacing Φ with the map $A \mapsto U^{-1}\Phi(A)U$, we may assume that $\nu_{1k} = \nu_{k1} = 1$ for all k. It remains to show that $\nu_{ij} = 1$ for all $2 \leq i, j \leq n$. To see this, consider $X = E_{11} + E_{1i} + E_{1j} + E_{i1} + E_{ii} + E_{ij}$. Notice that $X^2 = 2X$. By (3.8), $\Phi(X) = E_{11} + E_{1i} + E_{1j} + E_{i1} + E_{ii} + \nu_{ij}E_{ij}$ and hence $\Phi(X)^2 = 2(E_{11}+E_{1i}+E_{i1}+E_{ii})+(1+\nu_{ij})(E_{1j}+E_{ij})$. Since $C(\Phi(X)^2) = C(X^2)$ contains a disk centered at 2 with radius 8, one can conclude that $\nu_{ij} = 1$. Thus, the result follows. □

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