Mappings on Matrices:
Invariance of Functional Values of Matrix Products [1]

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Abstract

Let $\mathcal{M}_n$ be the algebra of all $n \times n$ matrices over a field $\mathbb{F}$, where $n \geq 2$. Let $\mathcal{S}$ be a subset of $\mathcal{M}_n$ containing all rank one idempotents. We study mappings $\phi : \mathcal{S} \to \mathcal{M}_n$ such that $F(\phi(A)\phi(B)) = F(AB)$ for various families of functions F including all the unitary similarity invariant functions on real or complex matrices. Very often, these mappings have the form

$$A \mapsto \mu(A)S(\sigma(a_{ij}))S^{-1} \qquad \text{for all } A = (a_{ij}) \in \mathcal{S}$$

for some invertible $S \in \mathcal{M}_n$, field monomorphism σ of $\mathbb{F}$, and an $\mathbb{F}^*$-valued mapping μ defined on $\mathcal{S}$. For real matrices, σ is often the identity map; for complex matrices, σ often the identity map or the conjugation map: $z \mapsto \bar{z}$. A key idea in our study is reducing the problem to the special case when $F : \mathcal{M}_n \to \{0, 1\}$ is defined by

$$F(X) = \begin{cases} 0 & \text{if } X = 0, \\ 1 & \text{otherwise.} \end{cases}$$

In such a case, one needs to characterize $\phi : \mathcal{S} \to \mathcal{M}_n$ such that $\phi(A)\phi(B) = 0$ if and only if $AB = 0$. We show that such a map has the standard form described above on rank one matrices in $\mathcal{S}$.

AMS Classifications: 15A04, 15A60, 15A18.

Keywords: Zero product preservers, unitary similarity invariant functions.

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1 Introduction

Let $\mathcal{M}_n$ be the algebra of all $n \times n$ matrices over a field $\mathbb{F}$, where $n \geq 2$. There has been considerable interest in studying preserver problems on $\mathcal{M}_n$, which concern the characterization of mapping $\phi : \mathcal{M}_n \rightarrow \mathcal{M}_n$ leaving invariant a set, a function, or a relation. In early study, the mappings were often assumed to be linear, and the quest to describe these mappings is collectively called linear preserver problems; see [11, 15]. Recently, researchers have considered additive preservers, multiplicative preservers, or other milder assumptions, see [18]. For example, given a function $F : \mathcal{M}_n \rightarrow \mathbb{F}$, $\mathcal{S} \subseteq \mathcal{M}_n$, and $\mathcal{T} \subseteq \mathbb{F}$, researchers characterize $\phi : \mathcal{S} \rightarrow \mathcal{M}_n$ such that

$$F(\phi(A) + \mu\phi(B)) = F(A + \mu B) \qquad \text{for all } A, B \in \mathcal{S}, \ \mu \in \mathcal{T},$$

and particular attention is on the cases when $\mathcal{T} = \{1\}$ or $\{-1\}$; see [2, 4, 5, 8, 21]. Another problem is to characterize $\phi : \mathcal{S} \rightarrow \mathcal{M}_n$ such that

$$F(\phi(A)\phi(B)) = F(AB) \qquad \text{for all } A, B \in \mathcal{S}; \tag{1.1}$$

see [6, 13, 14, 19]. In this paper, we consider this problem for various families of functions F including all the unitary similarity invariant norms on real or complex matrices. Very often, these mappings have the form

$$A \mapsto \mu(A)S(\sigma(a_{ij}))S^{-1} \qquad \text{for all } A = (a_{ij}) \in \mathcal{S}$$

for some invertible $S \in \mathcal{M}_n$, field monomorphism σ of $\mathbb{F}$, and an $\mathbb{F}^*$-valued mapping μ defined on $\mathcal{S}$. For real matrices, σ is often the identity map; for complex matrices, σ often the identity map or the conjugation map: $z \mapsto \bar{z}$. Note that we do not require $AB \in \mathcal{S}$ even if $A, B \in \mathcal{S}$ in our setting. However, we will require that $\mathcal{S}$ contains all rank one idempotents.

In studying preservers, one may develop special techniques to deal with a specific problem; one may develop a general technique to treat a class of problems; one may also obtain a basic preserver result so that other preserver problems can be reduced to it. For instance, many linear preserver problems can be reduced to rank one preservers or nilpotent preservers. In our study, many general problems are reduced to the special case when $F : \mathcal{M}_n \rightarrow \{0, 1\}$ is defined by

$$F(X) = \begin{cases} 0 & \text{if } X = 0, \\ 1 & \text{otherwise.} \end{cases}$$

In such a case, one needs to characterize mappings $\phi : \mathcal{S} \rightarrow \mathcal{M}_n$ such that

$$\phi(A)\phi(B) = 0 \quad \text{if and only if} \quad AB = 0.$$

We will call such mappings zero product preservers. Many researchers have study bijective zero product preservers; see Section 2 for more details. In our study, the bijectivity assumption is not required. We show that if $n \geq 3$, then for every zero product preserver ϕ, there exist an invertible $S \in \mathcal{M}_n$, a field monomorphism σ on $\mathbb{F}$, and an $\mathbb{F}^*$-valued mapping μ defined on the rank one matrices in $\mathcal{S}$ such that

$$\phi(A) = \mu(A)S(\sigma(a_{ij}))S^{-1} \text{ for all rank one } A = (a_{ij}) \in \mathcal{S}.$$

Our paper is organized as follows. In Section 2, we used a result in [17] to characterize zero product preservers. In Section 3, we discuss some immediate applications of the theorem on zero product preservers to rank preservers and mappings on $\mathbb{F}^n$ that preserve orthogonal pairs. In Section 4, we study mappings on $\mathcal{S}$ satisfying (1.1) for unitary similarity invariant functions F including all unitarily invariant norms on real and complex matrices. A self-contained elementary proof of Theorem 2.1 (without invoking the result in [17]) is given in Section 5. It will be apparent from the proof there that when $n = 2$, we do not always have a field monomorphism σ as in the case of $n \geq 3$. Also, the idea of the proof may be useful in further extending the result on zero product preservers to matrices over more general rings such as division rings.

In our discussion, $\mathbb{F}^*$ denotes the multiplicative group of all nonzero elements in $\mathbb{F}$, $\{e_1, \ldots, e_n\}$ denotes the standard basis for $\mathbb{F}^n$, and $e = e_1 + \cdots + e_n$. The standard basis for $\mathcal{M}_n$ is denoted by $\{E_{11}, E_{12}, \ldots, E_{nn}\}$, and $\mathcal{M}_n^m$ denotes the semigroup of matrices in $\mathcal{M}_n$ having rank at most m, where $m \in \{1, \ldots, n\}$.

For any matrix $A = (a_{ij}) \in \mathcal{M}_n$ and field homomorphism σ of $\mathbb{F}$, denote by A_σ the matrix whose (i, j)-th entry is $\sigma(a_{ij})$, i.e., $A_\sigma = (\sigma(a_{ij}))$. Note that

(a) $A_\sigma B_\sigma = (AB)_\sigma$ for all $A, B \in \mathcal{M}_n$.

If σ is a monomorphism, we also have

(b) $B_\sigma^{-1} = (B^{-1})_\sigma$ for all invertible $B \in \mathcal{M}_n$; and

(c) $\operatorname{rank} A_\sigma = \operatorname{rank} A$ for all $A \in \mathcal{M}_n$.

2   Zero product preserving mappings

Motivated by theory and applications, researchers have studied the basic preserver result on linear, additive, or bijective mappings ϕ preserving zero products in both directions, i.e.,

$$\phi(A)\phi(B) = 0 \quad \Longleftrightarrow \quad AB = 0 \quad \text{for all} \quad A, B \in \mathcal{S}$$

on various subsets $\mathcal{S}$ of an algebra; see [3, 7, 12, 16, 17].

In [12], Molnár studied zero product preservers on the set of all bounded linear rank one idempotent operators acting on the Banach space $\mathcal{X}$. If $\mathcal{X}$ is complex and has finite dimension n, then operators on $\mathcal{X}$ can be identified as $n \times n$ complex matrices. Molnár showed, among others, that if $n \geq 3$, bijective zero product preservers on the $n \times n$ rank one idempotent matrices have the form

$$(a_{ij}) \mapsto S(\sigma(a_{ij}))S^{-1} \tag{2.1}$$

for some invertible matrix S and field automorphism σ.

In [17], Šemrl used the Fundamental Theorem of Projective Geometry to give a short proof of an improved version of the above result. Specifically, he considered matrices over

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any field, and mappings preserving zero products in one direction only, which may not be bijective. These zero product preservers have the same form as above, except that σ can now be a field monomorphism.

Remark Theorem 1.2 in [17] only asserts that the mapping ϕ has the form (2.1) for some field endomorphism σ. Nevertheless, one readily verifies that a nonzero field endomorphism is automatically injective. It is clear from the proof of Theorem 1.2 in [17] that σ is nonzero.

Compared with the result of Šemrl in [17], ours has to assume that ϕ preserves zero products in both directions as a trade off for not requiring rank one idempotents be mapped to rank one idempotents. Indeed the two conditions are more or less equivalent, as we will see that under the stronger assumption, ϕ maps rank one idempotents to scalar multiples of rank one idempotents.

We are not able to exhaust all those ϕ preserving zero products in one direction only. There are other examples than that described above. For instance, the ϕ that maps all matrices to scalar multiples of a fixed square-zero matrix. It is also clear that such ϕ may not preserve the set of rank one matrices.

The main theorem of this section is the following. An infinite dimensional version of the result under a different setting and the bijective assumption is prove in [19, Theorem 4.1].

Theorem 2.1 *Let $\mathcal{S}$ be a subset of $\mathcal{M}_n$ containing all rank one idempotents. Suppose $\phi : \mathcal{S} \to \mathcal{M}_n$ is zero product preserving, i.e.,*

$$\phi(A)\phi(B) = 0 \quad \Longleftrightarrow \quad AB = 0 \quad \text{for all} \quad A, B \in \mathcal{S}. \tag{2.2}$$

Then there are functions f, $g : \mathbb{F}^n \to \mathbb{F}^n$ such that $y^t x = 0$ if and only if $g(y)^t f(x) = 0$, and there is a $\mathbb{F}^*$-valued mapping μ on the rank one matrices in $\mathcal{S}$ such that

$$\phi(xy^t) = \mu(xy^t)f(x)g(y)^t$$

for any rank one matrix $xy^t \in \mathcal{S}$. Suppose $n \geq 3$. Then there exist an invertible $S \in \mathcal{M}_n$ and a field monomorphism σ on $\mathbb{F}$ such that $f(x) = Sx_\sigma$ and $g(y) = (S^{-1})^t y_\sigma$, i.e.,

$$\phi(A) = \mu(A)SA_\sigma S^{-1} \quad \text{for all rank one } A \in \mathcal{S}.$$

Furthermore, if $A = M(I_k \oplus 0_{n-k})N \in \mathcal{S}$ for some invertible $M, N \in \mathcal{M}_n$ and $k > 1$, then

$$\phi(A) = SM_\sigma(A_k \oplus 0_{n-k})N_\sigma S^{-1}$$

for some matrix $A_k \in \mathcal{M}_k$. Moreover, if $\mathbb{F}$ has the property that all its nonzero (field) endomorphisms are automorphisms, then A and $\phi(A)$ have the same rank for each $A \in \mathcal{S}$.

It follows from Theorem 2.1 that a zero product preserving map is always rank one preserving and rank k non-increasing for $k > 1$. We are indebted to Peter Šemrl who showed us the last assertion of the theorem and the following example showing that ϕ may indeed decrease rank if $\mathbb{F}$ is the complex field.

Suppose $n \geq 3$, $\mathcal{M}_n$ is the set of $n \times n$ matrices over $\mathbb{C}$, and $\mathcal{S} \subseteq \mathcal{M}_n$ consists of all rank one idempotents and the matrix $E_{11} + E_{22}$. By the result in [9], there exists $a, b, c \in \mathbb{C}$ and a field monomorphism σ of $\mathbb{C}$ such that a, b, c are algebraically independent of $\sigma(\mathbb{C})$, i.e., if $p(z_1, z_2, z_3)$ is a polynomial with coefficients in $\sigma(\mathbb{C})$ such that $p(a, b, c) = 0$ then p is the zero polynomial. Define $\phi : \mathcal{S} \to \mathcal{M}_n$ such that $\phi(P) = P_\sigma$ for all rank one idempotent P and

$$\phi(E_{11} + E_{22}) = \begin{pmatrix} a & b \\ ca & cb \end{pmatrix} \oplus 0_{n-2}.$$

Then ϕ is a zero product preserving map and rank $(E_{11} + E_{22}) = 2 > 1 = \text{rank}(\phi(E_{11} + E_{22}))$.

We need several lemmas to prove Theorem 2.1. The first one is a characterization of (multiples of) rank one idempotents in terms of product zero. For any $A \in \mathcal{M}_n$, let $N(A) = \{x \in \mathbb{F}^n : Ax = 0\}$ and $R(A) = \{Ax : x \in \mathbb{F}^n\}$ be the null space and the column space of A, respectively.

Lemma 2.2 *Let $A_1, \ldots, A_n \in \mathcal{M}_n$. Then*

$$A_j A_i = A_i A_j = 0 \neq A_j^2 \quad \text{for all } i \neq j \tag{2.3}$$

if and only if there exist an invertible S, and nonzero numbers $r_1, \ldots, r_n$ such that $S^{-1} A_j S = r_j E_{jj}$ for $j = 1, \ldots, n$.

Proof. Suppose $n \geq 3$ and that $A_1, \ldots, A_n$ satisfy (2.3) with one of the A_i's having rank greater than one. Say, rank $A_1 > 1$. Then $\dim N(A_1) = n - \text{rank } A_1 < n - 1$. As $A_1 A_j = 0$, $R(A_j) \subseteq N(A_1)$ for all $2 \leq j \leq n$, and hence $R(A_2) + \cdots + R(A_n) \subseteq N(A_1)$. There must be some j, $2 < j \leq n$, such that

$$R(A_j) \subseteq R(A_2) + \cdots + R(A_{j-1}).$$

Otherwise,

$$\dim(R(A_2) + \cdots + R(A_n)) \geq n - 1 > \dim N(A_1).$$

We also have $R(A_2), \ldots, R(A_{j-1}) \subseteq N(A_j)$. Hence

$$R(A_j) \subseteq R(A_2) + \cdots + R(A_{j-1}) \subseteq N(A_j),$$

and $A_j^2 = 0$, which contradicts (2.3). So each A_j has rank one.

It is clear that the same conclusion holds when $n = 2$.

Since $A_j^2 \neq 0$, A_j is not a nilpotent. So, each A_j has one nonzero eigenvalue and is similar to an upper triangular matrix with one nonzero row.

Now $A_1, \ldots, A_n$ are mutually commuting. They are simultaneously triangularizable. Take an invertible $S \in \mathcal{M}_n$ so that $S^{-1} A_1 S = r_1 E_{11}$. Then $A_1 A_j = A_j A_1 = 0$ implies that $S^{-1} A_j S = [0] \oplus B_j$ for $j = 2, \ldots, n$. Since $B_i B_j = B_j B_i = 0$ for all $i \neq j$, we can use an inductive argument to show that there is an invertible T of the form $[1] \oplus T_0$ such that

$T^{-1}([0] \oplus B_j)T = r_j E_{jj}$ for $j = 2, \ldots, n$. Replacing S by ST, we have $S^{-1}A_j S = r_j E_{jj}$ for $j = 1, \ldots, n$.

The converse is clear. ∎

Now let $\mathcal{S}$ be a subset of $\mathcal{M}_n$ containing all rank one idempotents, and $\phi : \mathcal{S} \to \mathcal{M}_n$ be a mapping satisfying (2.2), i.e.,

$$\phi(A)\phi(B) = 0 \quad \Longleftrightarrow \quad AB = 0 \quad \text{for all} \quad A, B \in \mathcal{S}.$$

By Lemma 2.2, ϕ maps rank one idempotents to scalar multiples of rank one idempotents. It is clear that the scalars can be arbitrary. It is also clear that for every $A \in \mathcal{S}$, $\phi(A)$ can only be determined up to a scalar multiple. For any $A, B \in \mathcal{M}_n$, we write $A \equiv B$ if $A = \lambda B$ for some nonzero $\lambda \in \mathbb{F}$. Note that

$$A_1 \equiv A_2 \ \text{ and } \ B_1 \equiv B_2 \quad \Longrightarrow \quad A_1 B_1 \equiv A_2 B_2.$$

Moreover, $A \equiv 0$ if and only if $A = 0$.

Lemma 2.3 *The mapping ϕ maps rank one matrices to rank one matrices. Moreover, for any rank one matrices A and B in $\mathcal{S}$,*

(a) *$R(A) = R(B)$ implies $R(\phi(A)) = R(\phi(B))$; and*

(b) *$N(A) = N(B)$ implies $N(\phi(A)) = N(\phi(B))$.*

Consequently,
$$A \equiv B \quad \Longrightarrow \quad \phi(A) \equiv \phi(B).$$

Proof. Suppose $R(A) = R(B)$ for rank one matrices A and B. Take a nonzero $x \in R(A)$ and form a basis $\{x_1, \ldots, x_n\}$ for $\mathbb{F}^n$ with $x_1 = x$. Let $A_i = x_i y_i^t$, where $[y_1 | \cdots | y_n]^t$ is the inverse of the matrix $[x_1 | \cdots | x_n]$. Then $A_1, \ldots, A_n$ satisfy (2.3), and so do $\phi(A_1), \ldots, \phi(A_n)$. By Lemma 2.2, there is an invertible $S \in \mathcal{M}_n$ such that $\phi(A_i) = r_i S E_{ii} S^{-1}$ for some nonzero r_i. Since $A_i x = 0$, $A_i A = A_i B = 0$ for all $i = 2, \ldots, n$. By (2.2),

$$\phi(A_i)\phi(A) = \phi(A_i)\phi(B) = 0 \quad \text{for all} \quad i = 2, \ldots, n.$$

Hence $R(\phi(A)), R(\phi(B)) \subseteq N(\phi(A_2)) \cap \cdots \cap N(\phi(A_n))$. As $\dim N(\phi(A_2)) \cap \cdots \cap N(\phi(A_n)) = 1$, we see that $\phi(A)$ has rank one, and $R(\phi(A)) = R(\phi(B))$. So, the first assertion and condition (a) hold. Part (b) can be obtained by a similar argument.

The last assertion follows from the fact that for rank one matrices A and B, $A \equiv B$ if and only if $R(A) = R(B)$ and $N(A) = N(B)$. ∎

Now, we can present **the proof of Theorem 2.1.**

By Lemma 2.3, we see that $\phi(xy^t) = \mu(xy^t)f(x)g(y)^t$ for some functions f, g defined on $\mathbb{F}^n$ and $\mathbb{F}^*$-valued mapping μ on the rank one matrices in $\mathcal{S}$. Also, for any rank one matrix

$xy^t \in \mathcal{S}$, xy^t is nilpotent if and only if $\phi(xy^t)$ is. Hence $y^t x = 0$ if and only if $g(y)^t f(x) = 0$, as asserted.

Suppose $n \geq 3$. Since ϕ maps rank one idempotents to multiples of rank one idempotents, by [17, Theorem 1.2] (and the remark before the theorem), there exist an invertible $S \in \mathcal{M}_n$ and a field monomorphism σ on $\mathbb{F}$ such that

$$\phi(A) \equiv S A_\sigma S^{-1}$$

for all rank one idempotents $A \in \mathcal{S}$. In other words,

$$\phi(xy^t) \equiv S(xy^t)_\sigma S^{-1} = (Sx_\sigma)((S^{-1})^t y_\sigma)^t$$

for all rank one matrices xy^t such that $y^t x = 1$. Comparing the formula to the preceding representation, we may choose $f(x) = Sx_\sigma$ and $g(y) = (S^{-1})^t y_\sigma$.

Now, suppose $A \in \mathcal{S}$ has the form $A = M(I_k \oplus 0_{n-k})N$ for some invertible $M, N \in \mathcal{M}_n$ and $k > 1$. The assertion is trivial if $k = n$. Suppose $k < n$. For any $j = k+1, \ldots, n$, let $B = M E_{jj} M^{-1}$ and $C = N^{-1} E_{jj} N$. Then

$$\phi(B) = \mu(B) S B_\sigma S^{-1} = \mu(B) S (M E_{jj} M^{-1})_\sigma S^{-1} = \mu(B) S M_\sigma E_{jj} M_\sigma^{-1} S^{-1}.$$

Similarly,

$$\phi(C) = \mu(C) S N_\sigma^{-1} E_{jj} N_\sigma S^{-1}.$$

Clearly, $BA = 0 = AC$. Then $\phi(B)\phi(A) = 0 = \phi(A)\phi(C)$. It follows that

$$\phi(A) = S M_\sigma (A_k \oplus 0_{n-k}) N_\sigma S^{-1}$$

for some $A_k \in \mathcal{M}_k$.

Finally, suppose $\mathbb{F}$ has the additional property that all of its nonzero endomorphisms are automorphisms. Then we may replace ϕ by $A \mapsto \mu(A)^{-1} S^{-1} \phi(A_{\sigma^{-1}}) S$ and assume that $\phi(P) = P$ for all rank one idempotent $P \in \mathcal{S}$. Then for every $A \in \mathcal{S}$, we have $AP = 0$ if and only if $\phi(A)P = 0$, and $PA = 0$ if and only if $P\phi(A) = 0$, for all rank one idempotent P. Thus, $\phi(A)$ and A have the same image and kernel. So, A and $\phi(A)$ have the same rank. The proof of the theorem is complete. ∎

3   Rank preservers on $\mathcal{M}_n^m$ and orthogonality preservers on $\mathbb{F}^n$

As an immediate application of Theorem 2.1, we characterize rank preserving mappings on the semigroup $\mathcal{M}_n^m$ of $\mathcal{M}_n$, i.e., $\phi : \mathcal{M}_n^m \to \mathcal{M}_n$ such that

$$\text{rank}\,(\phi(A)\phi(B)) = \text{rank}\,(AB) \quad \text{for all} \quad A, B \in \mathcal{M}_n^m. \tag{3.1}$$

Clearly such a ϕ satisfies (2.2). When $m = n = 2$, one can apply Theorem 2.1 to get some information about ϕ on rank one matrices. However, ϕ can map the set of invertible matrices into itself in any way we like. So, we consider $n \geq 3$ in the following.

Theorem 3.1 *Suppose $n \geq 3$. A mapping $\phi : \mathcal{M}_n^m \to \mathcal{M}_n$ satisfying (3.1):*

$$\operatorname{rank}(\phi(A)\phi(B)) = \operatorname{rank}(AB) \quad \text{for all} \quad A, B \in \mathcal{M}_n^m$$

if and only if there exist an invertible $S \in \mathcal{M}_n$, and a field monomorphism σ on $\mathbb{F}$ such that for any $A = M(I_k \oplus 0_{n-k})N \in \mathcal{S}$ with invertible M, N,

$$\phi(A) = SM_\sigma(A_k \oplus 0_{n-k})N_\sigma S^{-1}.$$

for some invertible matrix $A_k \in \mathcal{M}_k$.

Proof. Suppose ϕ has the desired form. For any $A = M(I_k \oplus 0_{n-k})N \in \mathcal{M}_n^m$ with M, N invertible,

$$\phi(A) = SM_\sigma(A_k \oplus 0_{n-k})N_\sigma S^{-1} = SP_A A_\sigma S^{-1} = SA_\sigma Q_A S^{-1}$$

for $P_A = M_\sigma(A_k \oplus I_{n-k})M_\sigma^{-1}$ and $Q_A = N_\sigma^{-1}(A_k \oplus I_{n-k})N_\sigma$. Then for any $A, B \in \mathcal{M}_n^m$,

$$\phi(A)\phi(B) = SP_A A_\sigma B_\sigma Q_B S^{-1} = SP_A(AB)_\sigma Q_B S^{-1}.$$

Since S, P_A and Q_B are invertible,

$$\operatorname{rank}(\phi(A)\phi(B)) = \operatorname{rank}(AB)_\sigma = \operatorname{rank}(AB).$$

The sufficiency part holds.

For the necessity part, note that ϕ must also satisfy (2.2). By Theorem 2.1, there exist an invertible $S \in \mathcal{M}_n$ and a field monomorphism σ on $\mathbb{F}$ such that for any $A = M(I_k \oplus 0_{n-k})N \in \mathcal{M}_n^m$ with invertible M, N,

$$\phi(A) = SM_\sigma(A_k \oplus 0_{n-k})N_\sigma S^{-1}$$

for some $A_k \in \mathcal{M}_k$. We need to show A_k is invertible. Let $B = N^{-1}(I_k \oplus 0_{n-k}) \in \mathcal{M}_n^m$. Then $AB = M(I_k \oplus 0_{n-k})$ has rank k. Hence,

$$\phi(A)\phi(B) = SM_\sigma(A_k \oplus 0_{n-k})N_\sigma S^{-1}SN_\sigma^{-1}(B_k \oplus 0_{n-k})S^{-1} = SM_\sigma(A_kB_k \oplus 0_{n-k})S^{-1}$$

has rank k. It follows that A_k and B_k are invertible. ∎

Next, we show that our main theorem can be used to study mappings on $\mathbb{F}^n$ preserving orthogonality; see [1]. We write $u \equiv v$ if u is a scalar multiple of v.

Proposition 3.2 *Let $n \geq 3$ and let $f : \mathbb{F}^n \to \mathbb{F}^n$ be a mapping such that*

$$x^t y = 0 \quad \Longleftrightarrow \quad f(x)^t f(y) = 0$$

for every $x, y \in \mathbb{F}^n$. Then there exist $S \in \mathcal{M}_n$ with $S^t S = I_n$, a field monomorphism σ on $\mathbb{F}$ such that

$$f(x) \equiv Sx_\sigma \quad \text{for all } x \in \mathbb{F}^n.$$

Instead of proving this proposition, we present the result and proof for the slightly more involved version for the inner product $(x, y) = y^* x$ on $\mathbb{C}^n$.

Proposition 3.3 *Let $n \geq 3$ and let $f : \mathbb{C}^n \to \mathbb{C}^n$ be a mapping such that*

$$x^* y = 0 \quad \Longleftrightarrow \quad f(x)^* f(y) = 0$$

for every $x, y \in \mathbb{C}^n$. Then there exist a unitary $S \in \mathcal{M}_n$ and $\sigma : \mathbb{C} \to \mathbb{C}$ of the form $z \mapsto z$ or $z \mapsto \bar{z}$ such that

$$f(x) \equiv S x_\sigma \quad \text{for all } x \in \mathbb{C}^n.$$

Proof. For every nonzero rank one matrix $A \in \mathcal{M}_n$, write $A = xy^*$ and define

$$\phi(A) = f(x) f(y)^*.$$

Also let $\phi(0) = 0$. Then it is easy to check that $\phi : \mathcal{M}_n^1 \to \mathcal{M}_n$ is zero product preserving.

By Theorem 2.1, there exist an invertible $S \in \mathcal{M}_n$, a field monomorphism σ on $\mathbb{C}$ and a mapping $\mu : \mathcal{M}_n^1 \setminus \{0\} \to \mathbb{C}^*$ such that for all nonzero $x, y \in \mathbb{C}^n$,

$$f(x) f(y)^* = \phi(xy^*) = \mu(xy^*) S(xy^*)_\sigma S^{-1} = \mu(xy^*) (S x_\sigma)((y^*)_\sigma S^{-1}).$$

We conclude that

$$f(x) \equiv S x_\sigma \quad \text{and} \quad f(y) \equiv (S^{-1})^* ((y^*)_\sigma)^*.$$

Putting $x = y$, we get $S x_\sigma \equiv (S^{-1})^* ((x^*)_\sigma)^*$, and hence

$$(S^* S) x_\sigma \equiv ((x^*)_\sigma)^*.$$

For $x = e_j$, we have $(S^* S) e_j \equiv e_j$. All e_j are eigenvectors of $S^* S$ and hence $S^* S$ is a diagonal matrix. But we also have $(S^* S) e \equiv e$. The diagonal matrix $S^* S$ is indeed a scalar matrix. Absorbing the scalar into the function μ, we may assume that $S^* S$ is the identity matrix, or equivalently, that S is unitary.

Now $x_\sigma \equiv ((x^*)_\sigma)^*$ for every $x \in \mathbb{C}^n$. For every $\lambda \in \mathbb{C}$, let $x = (\lambda, 1, 0, \ldots, 0)^t$. Since

$$(\sigma(\lambda), 1, 0, \ldots, 0)^t = x_\sigma \equiv ((x^*)_\sigma)^* = (\overline{\sigma(\overline{\lambda})}, 1, 0, \ldots, 0)^t,$$

$\overline{\sigma(\overline{\lambda})} = \sigma(\lambda)$. But then $\overline{\sigma(\lambda)} = \sigma(\lambda)$ for every $\lambda \in \mathbb{R}$ so that σ maps $\mathbb{R}$ into $\mathbb{R}$. It then follows that σ has the form $z \mapsto z$ or $z \mapsto \bar{z}$; see [20]. ∎

Peter Šemrl pointed out that the above proposition is the non-bijective finite-dimensional version of Uhlhorn's theorem in quantum mechanics; see [22] and also [12].

4 Unitarily Invariant and Unitary Similarity Invariant Functions

In this section, we focus on the case when $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Let $\mathcal{U}_n = \{U \in \mathcal{M}_n : U^*U = I_n\}$. We study $\phi : \mathcal{S} \to \mathcal{M}_n$ such that

$$F(\phi(A)\phi(B)) = F(AB) \qquad \text{for all } A, B \in \mathcal{S}, \tag{4.1}$$

where $F : \mathcal{M}_n \to [0, \infty)$ satisfies the following conditions:

(F1) $F(A) = 0$ if and only if $A = 0$;

(F2) There is $p \in \mathbb{R}^*$ such that $F(\lambda A) = |\lambda|^p F(A)$ for all $\lambda \in \mathbb{F}^*$;

(F3) $F(UAV) = F(A)$ for all $U, V \in \mathcal{U}_n$.

This class of functions includes all common matrix norms on $\mathcal{M}_n$ such as the spectral norm $\|A\| = \max\{\|Ax\| : x \in \mathbb{F}^n, \ \|x\| \leq 1\}$, the Frobenius norm $\|A\|_F = \operatorname{tr}(A^*A)^{1/2}$. However the triangle inequality is not assumed.

Note also that we may always assume that $p = 1$ in (F2). Otherwise, we can replace F by the mapping $A \mapsto |F(A)|^{1/p}$. A function F satisfying (F3) is known as a unitarily invariant function. Evidently, condition (4.1) will still hold after this replacement. *We will always assume this in our discussion.*

We have the following result.

Theorem 4.1 *Let $F : \mathcal{M}_n \to [0, \infty)$ satisfy (F1) – (F3). Suppose $n \geq 3$, and $\phi : \mathcal{S} \to \mathcal{M}_n$ is a mapping satisfying*

$$F(\phi(A)\phi(B)) = F(AB) \quad \text{for all} \quad A, B \in \mathcal{S}. \tag{4.2}$$

Then there exist a matrix $W \in \mathcal{U}_n$, and mappings $\psi_L, \psi_R : \mathcal{S} \to \mathcal{U}_n$ satisfying

$$\psi_L(A)A = A\psi_R(A) \quad \text{for all } A \in \mathcal{S}$$

such that one of the following holds.

(a.1) $F(\psi_L(A)AB\psi_R(B)) = F(AB)$ for all $A, B \in \mathcal{S}$, and ϕ has the form

$$A \mapsto W\psi_L(A)AW^* = WA\psi_R(A)W^*.$$

(a.2) $\mathbb{F} = \mathbb{C}$, $F(\overline{\psi_L(A)AB\psi_R(B)}) = F(AB)$ for all $A, B \in \mathcal{S}$, and ϕ has the form

$$A \mapsto W\overline{\psi_L(A)A}W^* = W\overline{A\psi_R(A)}W^*.$$

Suppose $A = XDY$, where $X, Y \in \mathcal{U}_n$ and $D = \mathrm{diag}\,(s_1(A), \ldots, s_n(A))$. Then $\psi_L(A)A = A\psi_R(A)$ if and only if $X^*\psi_L(A)XD = DY\psi_R(A)^*Y^*$. So, each of the matrices $X^*\psi_L(A)X$ and $Y\psi_R(A)^*Y^*$ is a direct sum of square blocks according to the multiplicities of the singular values of A. Moreover, the blocks corresponding to the nonzero singular values are the same in the two matrices.

It turns out that we can further strengthen Theorem 4.1 by replacing the assumption (F3) by the following weaker condition on $F : \mathcal{M}_n \to \mathbb{F}$.

(F3') $F(U^*AU) = F(A)$ for all $U \in \mathcal{U}_n$.

A function F satisfying (F3') is called a unitarily similarity invariant function. Clearly, a unitarily invariant function is also unitary similarity invariant. The following result shows that preservers of unitary similarity invariant functions have the same structure of preservers of unitarily invariant functions, and more can be said if F is not unitarily invariant on rank one matrices.

Theorem 4.2 *Let* $F : \mathcal{M}_n \to [0, \infty)$ *satisfy* (F1), (F2) *and* (F3'). *Suppose* $n \geq 3$, *and* $\phi : \mathcal{S} \to \mathcal{M}_n$ *is a mapping satisfying*

$$F(\phi(A)\phi(B)) = F(AB) \quad \text{for all} \quad A, B \in \mathcal{S}.$$

Then condition (a.1) *or* (a.2) *of Theorem 4.1 holds. Moreover, if* $\{|F(X)/s_1(X)| : X \in \mathcal{M}_n^1\}$ *is not a singleton, then there is a mapping* $\mu : \mathcal{S} \to \Pi = \{z \in \mathbb{F} : |z| = 1\}$ *such that* $\psi_L(A) = \psi_R(A) = \mu(A)I$, *and thus one of the following holds.*

(b.1) ϕ *has the form*

$$A \mapsto \mu(A)WAW^*.$$

(b.2) $\mathbb{F} = \mathbb{C}$, $F(\overline{AB}) = F(AB)$ *for all* $A, B \in \mathcal{S}$, *and* ϕ *has the form*

$$A \mapsto \mu(A)W\overline{A}W^*.$$

We need only prove Theorem 4.2, and Theorem 4.1 will then follow. We begin with a condition under which F is essentially the largest singular value on rank one matrices. Denote by $s_1(A) \geq \ldots \geq s_n(A)$ the singular values of A.

Lemma 4.3 *Suppose* $n \geq 3$. *If there is a vector* $b \in \mathbb{F}^n$, *which is not a multiple of* e_1, *such that*

$$F(e_1 y^*) = F(by^*) \quad \text{for all } y \in \mathbb{F}^n.$$

Then

$$F(X) = s_1(X)F(E_{11}) \quad \text{for all } X \in \mathcal{M}_n^1. \tag{4.3}$$

Proof. Since b is not a multiple of e_1, there exists $U = [\lambda] \oplus U_1 \in \mathcal{U}_n$ with $|\lambda| = 1$ such that $Ub = c = (c_1, c_2, 0, \ldots, 0)$ with $c_1 \geq 0$ and $c_2 > 0$. As

$$F(cy^*) = F(U^* cy^* U) = F(by^* U) = F(e_1 y^* U) = F(Ue_1 y^*) = F(\lambda e_1 y^*) = F(e_1 y^*),$$

we may assume that $b = (b_1, b_2, 0, \ldots, 0)$ with $b_1 \geq 0$ and $b_2 > 0$.

Let $t \in [0, \pi/2]$. There exists a $\mu_t > 0$ such that for any $\hat{t} \in (t - \mu_t, t + \mu_t) \cap [0, \pi/2]$, there are vectors $y = (\sin t, y_2, \ldots, y_n)$ and $z = (\sin \hat{t}, z_2, \ldots, z_n)$ with $\|y\| = \|z\| = 1$ such that $|y^* b| = |z^* b|$. This is possible as $n \geq 3$.

Now the rank one matrix $e_1 y^*$ is unitarily similar to $\sin t E_{11} + \cos t E_{12}$, and that by^* is unitarily similar to a unit multiple of

$$|y^* b| E_{11} + \sqrt{\|by^*\|^2 - |y^* b|^2} E_{12} = |y^* b| E_{11} + \sqrt{\|b\|^2 - |y^* b|^2} E_{12}.$$

Hence

$$F(\sin t E_{11} + \cos t E_{12}) = F(e_1 y^*) = F(by^*) = F(|y^* b| E_{11} + \sqrt{\|b\|^2 - |y^* b|^2} E_{12}).$$

Similarly,

$$F(\sin \hat{t} E_{11} + \cos \hat{t} E_{12}) = F(|z^* b| E_{11} + \sqrt{\|b\|^2 - |z^* b|^2} E_{12}).$$

As $|y^* b| = |z^* b|$,

$$F(\sin t E_{11} + \cos t E_{12}) = F(\sin \hat{t} E_{11} + \cos \hat{t} E_{12}).$$

It follows from the compactness of $[0, \pi/2]$ that $F(\sin t E_{11} + \cos t E_{12}) = F(E_{11})$ for all $t \in [0, \pi/2]$. Thus, $F(X) = s_1(X) F(E_{11})$ for all $X \in \mathcal{M}_n^1$. ∎

Lemma 4.4 *Suppose $n \geq 3$. If there are matrices $A, B \in \mathcal{M}$ with B not a multiple of A such that*

$$F(AX) = F(BX) \quad \text{for all } X \in \mathcal{M}_n^1,$$

then F satisfies (4.3).

Proof. Using Lemma 4.3, it is not difficult to see that (4.3) holds if there are vectors b, c in $\mathbb{F}^n$, one is not a multiple of the other, such that

$$F(by^*) = F(cy^*) \quad \text{for all } y \in \mathbb{F}^n.$$

Now if B is not a multiple of A, there is an x in $\mathbb{F}^n$ such that Bx is not a multiple of Ax. But we have

$$F(Axy^*) = F(Bxy^*) \quad \text{for all } y \in \mathbb{F}^n.$$

The conclusion follows. ∎

Lemma 4.5 *Suppose F satisfies (4.3). Then for any matrices $A, B \in \mathcal{M}_n$,*

$$F(AX) = F(BX) \quad \text{for all } X \in \mathcal{M}_n^1 \tag{4.4}$$

if and only if there is a $U \in \mathcal{U}_n$ such that $B = UA$. Similarly,

$$F(YA) = F(YB) \quad \text{for all } Y \in \mathcal{M}_n^1 \tag{4.5}$$

if and only if there is a $V \in \mathcal{U}_n$ such that $B = AV$.

Proof. Since F satisfies (4.3),

$$s_1(AX) = s_1(BX) \quad \text{for all } X \in \mathcal{M}_n^1.$$

For every nonzero x in $\mathbb{F}^n$,

$$\|Ax\| = s_1(Axe_1^*) = s_1(Bxe_1^*) = \|Bx\|.$$

Hence there is a $U \in \mathcal{U}_n$ such that $B = UA$.

The other assertion can be obtained similarly. ∎

Now we are ready to give **the proof of Theorem 4.2.**

In view of the comment before Theorem 4.1, we may assume that $p = 1$ in (F2). Clearly, if ϕ satisfies (4.2), then it also satisfies (2.2). By Theorem 2.1, there exist an invertible $S \in \mathcal{M}_n$, a field monomorphism σ on $\mathbb{F}$, and a $\mathbb{F}^*$-valued mapping μ on the rank one matrices in $\mathcal{S}$ such that

$$\phi(A) = \mu(A) S A_\sigma S^{-1} \quad \text{for any rank one matrix } A.$$

We claim that $|\sigma(\alpha)| = |\alpha|$ for every $\alpha \in \mathbb{F}$. For any $\alpha \in \mathbb{F}$, take $A = E_{11} + \alpha E_{1n}$. Then $\phi(A) = \mu(A) S(E_{11} + \sigma(\alpha) E_{1n}) S^{-1}$. For any $1 \le k \le n$,

$$|\mu(A)\mu(E_{1k})| \, F(SE_{1k}S^{-1}) = F(\phi(A)\phi(E_{1k})) = F(AE_{1k}) = F(E_{1k})$$

and

$$|\sigma(\alpha)\mu(A)\mu(E_{nk})| \, F(SE_{1k}S^{-1}) = F(\phi(A)\phi(E_{nk})) = F(AE_{nk}) = |\alpha|F(E_{1k}).$$

It follows that $|\sigma(\alpha)| = |\mu(E_{1k})/\mu(E_{nk})| \, |\alpha|$. Since $\sigma(1) = 1$, $|\mu(E_{1k})| = |\mu(E_{nk})|$, and hence $|\sigma(\alpha)| = |\alpha|$ for all $\alpha \in \mathbb{F}$. It is well-known that σ must either be the identity or the complex conjugation if $\mathbb{F} = \mathbb{C}$; see [20] for example. Replacing ϕ by $A \mapsto \phi(\overline{A})$, if necessary, we may assume that σ is the identity on $\mathbb{F}$.

By the singular value decomposition, $S = UDV$ for some $U, V \in \mathcal{U}_n$ and diagonal matrix $D = \operatorname{diag}(d_1, \ldots, d_n)$, where $d_1 \ge \cdots \ge d_n > 0$. Replacing ϕ by $A \mapsto U^*\phi(V^*AV)U$, we may assume that S is the diagonal matrix D.

Actually D is a scalar matrix. Note that $\phi(E_{kk}) = \mu(E_{kk})E_{kk}$ for all $k = 1, \ldots, n$. Hence

$$|\mu(E_{kk})|^2 F(E_{kk}) = F(\phi(E_{kk})^2) = F(E_{kk}^2) = F(E_{kk}),$$

and $|\mu(E_{kk})| = 1$. Since $\phi(E_{11} + E_{1n}) = \mu(E_{11} + E_{1n})(E_{11} + d_1 d_n^{-1} E_{1n})$,

$$|\mu(E_{11} + E_{1n})\mu(E_{11})| \, F(E_{11}) = F(\phi(E_{11} + E_{1n})\phi(E_{11})) = F((E_{11} + E_{1n})E_{11}) = F(E_{11})$$

and

$$d_1 d_n^{-1}|\mu(E_{11}+E_{1n})\mu(E_{nn})| \, F(E_{1n}) = F(\phi(E_{11}+E_{1n})\phi(E_{nn})) = F((E_{11}+E_{1n})E_{nn}) = F(E_{1n}).$$

We have $|\mu(E_{11} + E_{1n})\mu(E_{11})| = 1$, and $d_1 d_n^{-1}|\mu(E_{11} + E_{1n})\mu(E_{nn})| = 1$. It follows that $d_1 = d_n$, and that D is a scalar matrix. In conclusion, we have

$$\phi(X) = \mu(X)X \quad \text{for any rank one matrix } X.$$

We now show that $|\mu(X)| = 1$ for any rank one matrix X. Take any rank one matrix X. If $X^2 \neq 0$, then

$$|\mu(X)|^2 F(X^2) = F(\phi(X)^2) = F(X^2),$$

and hence $|\mu(X)| = 1$. If $X^2 = 0$, there exists a rank one matrix Y such that both XY and Y^2 are nonzero. We have

$$|\mu(X)|F(XY) = |\mu(X)\mu(Y)|F(XY) = F(\phi(X)\phi(Y)) = F(XY).$$

Hence $|\mu(X)| = 1$ too.

Finally, for any matrix $A \in \mathcal{S}$,

$$F(\phi(A)X) = F(\phi(A)\phi(X)) = F(AX) \quad \text{and} \quad F(X\phi(A)) = F(\phi(X)\phi(A)) = F(XA)$$

for any rank one matrix X, as $|\mu(X)| = 1$. By Lemma 4.4, either

(i) $\phi(A)$ is some unit multiple of A for all $A \in \mathcal{S}$; or

(ii) $F(X) = s_1(X)F(E_{11})$ for any rank one matrix X.

If (i) holds, μ can be extended to a $\mu : \mathcal{S} \to \Pi$ such that $\phi(A) = \mu(A)A$ for all $A \in \mathcal{S}$. If (ii) holds, then by Lemma 4.5, there exist unitary matrices $U_A, V_A \in \mathcal{U}_n$ such that $\phi(A) = U_A A = A V_A$. Define mappings $\psi_L, \psi_R : \mathcal{S} \to \mathcal{U}_n$ by $\psi_L(A) = U_A$ and $\psi_R(A) = V_A$. Then $\phi(A) = \psi_L(A)A = A\psi_R(A)$, and the result follows. ∎

As mentioned before Theorem 4.1 covers many functions such as all unitarily invariant norms on $\mathcal{M}_n$ including the spectral norm $\|A\|$ and the Frobenius norm $\|A\|_F = \operatorname{tr}(A^*A)^{1/2}$. Next, we consider unitary similarity invariant functions F so that condition (b.1) or/and (b.2) of Theorem 4.2 hold. For $\mathbb{F} = \mathbb{C}$, define the numerical range of A by

$$W(A) = \{x^*Ax : x \in \mathbb{C}^n, \ x^*x = 1\}$$

and the numerical radius of A by

$$r(A) = \{|\mu| : \mu \in W(A)\}.$$

It is known and easy to check that $r(A) = 0$ if and only if $A = 0$, and that $r(U^*AU) = r(A) = r(\overline{A})$ for all $A \in \mathcal{M}_n$. Note that $r(E_{11}) = 1 \neq 1/2 = r(E_{12})$. So, if $n \geq 3$, and $\phi : \mathcal{S} \to \mathcal{M}_n$ is a mapping satisfying (4.2) with $F(A) = r(A)$, then condition (b.1) or (b.2) of Theorem 4.2 holds.

Now, if $n \geq 3$, and $\phi : \mathcal{S} \to \mathcal{M}_n$ is a mapping satisfying (4.2) with $F(A) = W(A)$, then $r(\phi(A)\phi(B)) = r(AB)$ for all $A, B \in \mathcal{M}_n$. Furthermore, if $A = iE_{11}$ then

$$W(A) = \{it : t \in [0,1]\} \neq \{-it : t \in [0,1]\} = W(\overline{A}).$$

We see that ϕ satisfies (b.1) of Theorem 4.2 only.

More generally, let $C \in \mathcal{M}_n$. The C-numerical range of $A \in \mathcal{M}_n$ is defined by

$$W_C(A) = \{\operatorname{tr}(CU^*AU) : U \in \mathcal{U}_n\}$$

and the C-numerical radius of A is defined by

$$r_C(A) = \max\{|\mu| : \mu \in W_C(A)\}.$$

When $C = E_{11}$, these reduce to $W(A)$ and $r(A)$. One may see [10] for some general background on the C-numerical range and C-numerical radius. If C is a non-scalar matrix with nonzero trace, then (F1) - (F3) hold for $F(A) = r_C(A)$. If C is positive semi-definite, then we can extend the analysis on $r(A)$ and $W(A)$ in the preceding paragraphs to $r_C(A)$ and $W_C(A)$ and get the same conclusion on ϕ. We have the following.

Corollary 4.6 *Let $\mathbb{F} = \mathbb{C}$ and $C \in \mathcal{M}_n$ be a non-scalar positive semi-definite matrix, where $n \geq 3$. Suppose $\phi : \mathcal{S} \to \mathcal{M}_n$ is a mapping satisfying (4.2) for $F(A) = r_C(A)$ or $W_C(A)$. Then condition (b.1) or (b.2) of Theorem 4.2 holds if $F(A) = r_C(A)$; and condition (b.1) of Theorem 4.2 holds if $F(A) = W_C(A)$.*

Continue to assume $\mathbb{F} = \mathbb{C}$. A norm ν on $\mathcal{M}_n$ is unitary similarity invariant or weakly unitarily invariant if $\nu(U^*AU) = \nu(A)$ for all $U \in \mathcal{U}_n$ and $A \in \mathcal{M}_n$. It is known (e.g., see [10]) that for every unitary similarity invariant norm ν, there is a compact subset $\mathcal{K}$ of $\mathcal{M}_n$ such that

$$\nu(A) = \max\{r_C(A) : C \in \mathcal{K}\}.$$

So, the C-numerical radius can be viewed as the building blocks for usi norms. It would be interesting to extend Corollary 4.6 to general unitary similarity invariant norms that are not unitarily invariant on $\mathcal{S}$.

5   A self-contained elementary proof for Theorem 2.1

In this section, we give a self-contained elementary proof of the second part of Theorem 2.1. The proof continues from Lemma 2.3. Replacing ϕ by $A \mapsto S^{-1}\phi(A)S$ for some suitable $S \in \mathcal{M}_n$, we may assume that

$$\phi(E_{jj}) = r_j E_{jj} \quad \text{for all} \quad 1 \leq j \leq n. \tag{5.1}$$

Lemma 5.1 *For any rank one matrix A, the (i,j)-th entry of $\phi(A)$ is zero if and only if that of A is.*

Proof. Suppose a_{ij}, the (i,j)-th entry of the matrix A, is 0. Then since A has rank one, either the i-th row, or the j-th column, of A is zero. Respectively, $E_{ii}A = 0$ or $AE_{jj} = 0$. Because of (5.1), either $r_{ii}E_{ii}\phi(A) = 0$, or $\phi(A)r_jE_{jj} = 0$. In both cases, the (i,j)-th entry of $\phi(A)$ is zero.

The converse is similar. ∎

Lemma 5.2 *Replacing ϕ by the mapping $A \mapsto D^{-1}\phi(A)D$ for some invertible diagonal $D \in \mathcal{M}_n$, we may further assume that*

$$\phi(ee^t) = ey^t \tag{5.2}$$

for some $y = (y_1, \ldots, y_n)^t$ with all y_i nonzero.

Proof. Since ϕ maps rank one matrices to rank one matrices, $\phi(ee^t) = xz^t$ for some $x = (x_1, \ldots, x_n)^t, z = (z_1, \ldots, z_n)^t \in \mathbb{F}^n$. By Lemma 5.1, all entries of $\phi(ee^t)$ are nonzero, and hence all x_i and z_i are nonzero. Let $D = \mathrm{diag}\,(x_1, \ldots, x_n)$. Then $x = De$, and

$$D^{-1}\phi(ee^t)D = D^{-1}xz^tD = ez^tD.$$

Replacing ϕ by $A \mapsto D^{-1}\phi(A)D$, the new mapping satisfies (5.2) with $y = D^t z$. ∎

Lemma 5.3 *Suppose $\phi : \mathcal{S} \to \mathcal{M}_n$ satisfies (2.2), (5.1) and (5.2). When $n \geq 3$, there exists a field monomorphism σ on $\mathbb{F}$ such that*

$$\phi(A) \equiv A_\sigma$$

for all rank one matrix A in $\mathcal{S}$ with nonzero trace.

Proof. We claim that for each $j \in \{1, \ldots, n\}$, there exists a monomorphism σ_j on $\mathbb{F}$ such that

$$\phi(A) \equiv A_{\sigma_j} \quad \text{for all rank one } A \in \mathcal{S} \text{ with } a_{jj} \neq 0. \tag{5.3}$$

Suppose (5.3) is proved. Then for any $i, j \in \{1, \ldots, n\}$ with $i \neq j$, and any nonzero $\alpha \in \mathbb{F}$,

$$
\begin{aligned}
& E_{ii} + E_{jj} + \sigma_i(\alpha)E_{ij} + \sigma_i(\alpha^{-1})E_{ji} \\
\equiv\ & \sigma_i(1)E_{ii} + \sigma_i(1)E_{jj} + \sigma_i(\alpha)E_{ij} + \sigma_i(\alpha^{-1})E_{ji} \\
\equiv\ & \phi(E_{ii} + E_{jj} + \alpha E_{ij} + \alpha^{-1}E_{ji}) \\
\equiv\ & \sigma_j(1)E_{ii} + \sigma_j(1)E_{jj} + \sigma_j(\alpha)E_{ij} + \sigma_j(\alpha^{-1})E_{ji} \\
\equiv\ & E_{ii} + E_{jj} + \sigma_j(\alpha)E_{ij} + \sigma_j(\alpha^{-1})E_{ji}.
\end{aligned}
$$

Thus $\sigma_i(\alpha) = \sigma_j(\alpha)$. Choosing σ to be the common monomorphism, the conclusion follows.

We prove (5.3) by a sequence of assertions. Assume that $j = 1$ for simplicity.

Assertion 1 *There exist injective mappings $f_2, \ldots, f_n, g_2, \ldots, g_n$ on $\mathbb{F}$ such that*

$$\phi\left(\begin{pmatrix} 1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}(1 \ y_2 \ \cdots \ y_n)\right) \equiv \begin{pmatrix} 1 \\ f_2(x_2) \\ \vdots \\ f_n(x_n) \end{pmatrix}(1 \ g_2(y_2) \ \cdots \ g_n(y_n))$$

for all $x_2, \ldots, x_n, y_2, \ldots, y_n \in \mathbb{F}$. Furthermore, $f_j(1) = 1$ and $g_j(\alpha) = -f_j(-\alpha^{-1})^{-1}$ for all nonzero $\alpha \in \mathbb{F}$.

Proof. For each $j > 1$ and $\alpha \neq 0$, we have by Lemma 5.1 nonzero β_α and γ_α such that

$$\phi(E_{11} - \alpha^{-1}E_{1j}) \equiv E_{11} - \beta_\alpha^{-1}E_{1j} \quad \text{and} \quad \phi(E_{11} - \alpha^{-1}E_{j1}) \equiv E_{11} - \gamma_\alpha^{-1}E_{j1}.$$

Define f_j, $g_j : \mathbb{F} \to \mathbb{F}$ by $f_j(0) = g_j(0) = 0$,

$$f_j(\alpha) = \beta_\alpha \quad \text{and} \quad g_j(\alpha) = \gamma_\alpha$$

for all nonzero α.

Let $\alpha \neq 0$. Then $(E_{11} + \alpha E_{1j})(E_{11} - \alpha^{-1}E_{j1}) = 0$. Hence

$$\begin{aligned}(1 + f_j(-\alpha^{-1})^{-1}g_j(\alpha)^{-1})E_{11} &= (E_{11} - f_j(-\alpha^{-1})^{-1}E_{1j})(E_{11} - g_j(\alpha)^{-1}E_{j1}) \\ &= \phi(E_{11} + \alpha E_{1j})\phi(E_{11} - \alpha^{-1}E_{j1}) = 0.\end{aligned}$$

It follows that $g_j(\alpha) = -f_j(-\alpha^{-1})^{-1}$.

Next we show that f_j is injective. Suppose $f_j(\alpha) = f_j(\beta)$ for some nonzero $\alpha \neq \beta$. Then

$$\begin{aligned}\phi(E_{11} - \beta E_{1j})\phi(E_{11} + \alpha E_{j1}) &= (E_{11} - f_j(\beta)^{-1}E_{1j})(E_{11} - g_j(-\alpha^{-1})^{-1}E_{j1}) \\ &= (E_{11} - f_j(\alpha)^{-1}E_{1j})(E_{11} + f_j(\alpha)E_{j1}) = 0.\end{aligned}$$

It is impossible as $(E_{11} - \beta^{-1}E_{1j})(E_{11} + \alpha E_{j1}) = (1 - \alpha\beta^{-1})E_{11} \neq 0$. Thus f_j is injective.

The mapping g_j is also injective, as $g_j(\alpha) = -f_j(-\alpha^{-1})^{-1}$ for all nonzero α.

Now let $A = (1 \ x_2 \ \cdots \ x_n)^t(1 \ y_2 \ \cdots \ y_n)$. Then $\phi(A)$ is rank one with nonzero $(1,1)$-th entry so that

$$\phi(A) \equiv W = (1 \ u_2 \ \cdots \ u_n)^t(1 \ v_2 \ \cdots \ v_n)$$

for some $u_2, \ldots, u_n, v_2, \ldots, v_n$ in $\mathbb{F}$. By Lemma 5.1, $u_i = 0$ if and only if $x_i = 0$, and $v_i = 0$ if and only if $y_i = 0$. Now if $x_j \neq 0$, then $(E_{11} - x_j^{-1}E_{1j})A = 0$. We have

$$(E_{11} - f_j(x_j)^{-1}E_{1j})W \equiv (E_{11} - f_j(x_j)^{-1}E_{1j})\phi(A) = 0,$$

and hence $1 - f_j(x_j)^{-1}u_j = 0$ or $u_j = f_j(x_j)$. Similarly, we get that $v_j = g_j(y_j)$.

Finally since ϕ satisfies (5.2),

$$(1 \ f_2(1) \ \cdots \ f_n(1))^t(1 \ g_2(1) \ \cdots \ g_n(1)) \equiv \phi(ee^t) = (1 \ \cdots \ 1)^t(y_1 \ \cdots \ y_n)$$

for some nonzero $y_1, \ldots, y_n \in \mathbb{F}$. It follows that $1 = f_2(1) = \cdots = f_n(1)$. ∎

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Assertion 2 *For any distinct $i, j > 1$, and any $x_i, x_j, y_i, y_j \in \mathbb{F}$,*

$$1 + x_i y_i + x_j y_j = 0 \quad \Longleftrightarrow \quad 1 + f_i(x_i)g_i(y_i) + f_j(x_j)g_j(y_j) = 0. \tag{5.4}$$

Proof. Take $A = E_{11} + x_i E_{1i} + x_j E_{1j}$ and $B = E_{11} + y_i E_{i1} + y_j E_{i1}$. The conclusion follows from (2.2). ∎

Assertion 3 *For each i, $f_i = g_i$, and they are multiplicative.*

Proof. Take $(x_i, x_j) = (\alpha, 1)$ and $(y_i, y_j) = (\beta, -\alpha\beta - 1)$. By (5.4), $1 + f_i(\alpha)g_i(\beta) + g_j(-\alpha\beta - 1) = 0$. Now take $(x_i, x_j) = (\alpha', 1)$ and $(y_i, y_j) = (\beta', -\alpha'\beta' - 1)$ with $\alpha'\beta' = \alpha\beta$, then $1 + f_i(\alpha')g_i(\beta') + g_j(-\alpha'\beta' - 1) = 0$. It follows that

$$1 + f_i(\alpha)g_i(\beta) + g_j(-\alpha\beta - 1) = 0 = 1 + f_i(\alpha')g_i(\beta') + g_j(-\alpha'\beta' - 1),$$

i.e., $f_i(\alpha)g_i(\beta) = f_i(\alpha')g_i(\beta')$ for any $\alpha\beta = \alpha'\beta'$. Then for any α, β,

$$f_i(\alpha)f_i(\beta)g_i(1) = f_i(\alpha)f_i(1)g_i(\beta) = f_i(\alpha)g_i(\beta) = f_i(\alpha\beta)g_i(1).$$

Since $g_i(1) \neq 0$, f_i is multiplicative.

Since f_i is injective and multiplicative, $f_i(-1) = -1$, and so $g_i(1) = f_i(-1^{-1})^{-1} = 1$. Thus,

$$g_i(\alpha) = g_i(\alpha)f_i(1) = g_i(1)f_i(\alpha) = f_i(\alpha) \quad \text{for all } \alpha \in \mathbb{F}.$$

∎

Assertion 4 $f_2 = \cdots = f_n = g_2 = \cdots = g_n$.

Proof. It suffices to prove that $f_2 = f_j$ for any $j \geq 2$. Take $(x_2, x_j) = (-\alpha - 1, \alpha)$ and $(y_2, y_j) = (1, 1)$ in (5.4). Then we have

$$1 + f_2(-\alpha - 1) + f_j(\alpha) = 1 + f_2(-\alpha - 1)g_2(1) + f_j(\alpha)g_j(1) = 0.$$

Hence, $f_j(\alpha) = -f_2(-\alpha - 1) - 1 = f_2(\alpha + 1) - 1$. In particular, $1 = f_j(1) = f_2(2) - 1$, i.e., $f_2(2) = 2$. Interchanging the roles of x_2 and x_j, we get $f_2(\alpha) = f_j(\alpha + 1) - 1$ and $f_j(2) = 2$. Then

$$\begin{aligned}
f_j(\alpha) - 1 &= f_2(\alpha - 1) = f_2(2)f_2((\alpha - 1)/2) = 2[f_j((\alpha - 1)/2 + 1) - 1] \\
&= f_j(2)f_j((\alpha - 1)/2 + 1) - 2 = f_j(\alpha + 1) - 2,
\end{aligned}$$

i.e., $f_j(\alpha + 1) = f_j(\alpha) + 1$. Thus, $f_2(\alpha) = f_j(\alpha + 1) - 1 = f_j(\alpha)$. ∎

Assertion 5 *The mappings $f_2 = \cdots = f_n = g_2 = \cdots = g_n$ are additive.*

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Proof. For any nonzero $\alpha, \beta \in \mathbb{F}$, take $(x_2, x_3) = (-\alpha/(\alpha+\beta), -\beta/(\alpha+\beta))$ with $(y_2, y_3) = (1, 1)$, we have

$$1 - f_2(\alpha+\beta)^{-1} f_2(\alpha) - f_2(\alpha+\beta)^{-1} f_2(\beta) = 1 + f_2(-\alpha/(\alpha+\beta)) + f_3(-\beta/(\alpha+\beta)) = 0.$$

Thus, $f_2(\alpha+\beta) = f_2(\alpha) + f_2(\beta)$, i.e., f_2 is additive. ∎

Assertion 6 *There exists a σ_1 such that (5.3) holds.*

Proof. Let $\sigma_1 = f_2 = \cdots = f_n = g_2 = \cdots = g_n$. Then by above assertions, σ_1 is a field monomorphism on $\mathbb{F}$ and $\phi(A) = (\sigma(a_{ij}))$ for all $A = (a_{ij}) \in \mathcal{M}_n^1$ with $a_{11} = 1$. Now for any $B = (b_{ij}) \in \mathcal{M}_n^1$ with $b_{11} \neq 0$, $B \equiv (1/b_{11})B$. By Lemma 2.3,

$$\phi(B) \equiv \phi((1/b_{11})B) \equiv (\sigma(b_{ij}/b_{11})) \equiv \sigma(b_{11}^{-1})(\sigma(b_{ij})) \equiv (\sigma(b_{ij})).$$

The assertion follows. ∎

Suppose $n \geq 3$. By Lemma 5.3, there exist an invertible $S \in \mathcal{M}_n$ and a field monomorphism σ on $\mathbb{F}$ such that

$$\phi(A) \equiv SA_\sigma S^{-1} \tag{5.5}$$

for all rank one matrix A in $\mathcal{S}$ with nonzero trace. Now, suppose $B = xy^t \in \mathcal{S} \setminus \{0\}$ has zero trace. Let $u, v \in \mathbb{F}^n$ be such that $u^t x = y^t v = 1$. Then (5.5) holds for $A = xu^t$ and $A = vy^t$. By Lemma 2.3, $R(\phi(B)) = R(xu^t)$ and $N(\phi(B)) = N(\phi(vy^t))$. Hence, $\phi(B) \equiv SB_\sigma S^{-1}$. Combining the above arguments, we see that for each rank one A, $\phi(A) = \lambda_A SA_\sigma S^{-1}$ for some nonzero λ_A. Define $\mu : \mathcal{S} \cap \mathcal{M}_n^1 \to \mathbb{F}^*$ by $\mu(A) = \lambda_A$. The desired conclusion follows. ∎

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