Additive rank-one preservers between spaces of rectangular matrices

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Abstract

Suppose $\mathbf{F}$ is a field and m, n, p, q are positive integers. Let $M_{mn}(\mathbf{F})$ be the set of all $m \times n$ matrices over $\mathbf{F}$, and let $M^1_{mn}(\mathbf{F})$ be its subset consisting of all rank-one matrices. A map $\phi : M_{mn}(\mathbf{F}) \to M_{pq}(\mathbf{F})$ is said to be an additive rank-one preserver if $\phi(M^1_{mn}(\mathbf{F})) \subseteq M^1_{pq}(\mathbf{F})$ and $\phi(A + B) = \phi(A) + \phi(B)$ for any $A, B \in M_{mn}(\mathbf{F})$. This paper describes the structure of all additive rank-one preservers from $M_{mn}(\mathbf{F})$ to $M_{pq}(\mathbf{F})$.

Keywords Field; Matrix space; Rank one; Additive preserver.

AMS Subject Classification 15A04, 15A03.

1 Introduction

In order to state precisely this article, we first introduce some concepts and fix the notation. Suppose $\mathbf{F}$ is a field and $\mathbf{F}^* = \mathbf{F}\setminus\{0\}$. For positive integers m and n, let $M_{mn}(\mathbf{F})$ be the set of all $m \times n$ matrices over $\mathbf{F}$, and let $M_n(\mathbf{F}) = M_{nn}(\mathbf{F})$ and $\mathbf{F}^n = M_{n1}(\mathbf{F})$. Denote by $M^r_{mn}(\mathbf{F})$ the subset of $M_{mn}(\mathbf{F})$ consisting of all rank-r matrices, and by $GL_n(\mathbf{F})$ the subset of $M_n(\mathbf{F})$ consisting of all nonsingular matrices. We use $\langle m\rangle$ to represent the set $\{1, 2, \ldots, m\}$. We denote by E_{ij} the matrix with 1 in the (i, j)th entry and 0 elsewhere (*Note: the dimension of E_{ij} will be dependent on the content*). For any matrix B, let $N(B)$, B^T and B^{-1} be the kernel space, transpose and inverse of B, respectively. For a map $\rho : \mathbf{F} \to \mathbf{F}$ and a matrix $A = [a_{ij}]$ over $\mathbf{F}$, we denote by A^ρ the matrix $[\rho(a_{ij})]$. Symbol $\oplus$ denotes the usual direct sum of matrices. An operator $\phi : M_{mn}(\mathbf{F}) \to M_{pq}(\mathbf{F})$ is said to be additive if $\phi(A + B) = \phi(A) + \phi(B)$ for any $A, B \in M_{mn}(\mathbf{F})$, linear if it is additive and satisfies $\phi(aA) = a\phi(A)$ for any $a \in \mathbf{F}$ and $A \in M_{mn}(\mathbf{F})$, and a rank-one preserver if $\phi(M^1_{mn}(\mathbf{F})) \subseteq M^1_{pq}(\mathbf{F})$.

In the recent several decades, characterizing linear/additive maps on spaces of matrices or operators that preserve certain properties has been an active area of research (see [5, 8, 11, 15] and the references therein). These are usually called *linear/additive preserver problems* in the literature. One of the most basic problems in linear/additive preserver problems is rank-one preserver problem, since some other questions about preservers have been solved with the help of rank-one preservers (see [1–4] and the references therein). Here we mention only partial results of rank-one preservers on spaces of rectangular matrices, which are closely related to this article. Marcus and Moyls [9] and Minc [10] described the structure of those linear rank-one preservers from $M_{mn}(\mathbf{F})$ into itself when $\mathbf{F}$ is any algebraically closed field of characteristic 0. Their difference lies in: Marcus and Moyls used multilinear algebra techniques, and Minc used only elementary matrix theory. Lim [7] characterized all invertible linear rank-one preservers from $M_{mn}(\mathbf{F})$ into itself when $\mathbf{F}$ is any field. Waterhouse [14] generalized the result of Lim to commutative rings with unit, but the invertibility assumption was still needed. Recently, Li *et al.* [6] obtained the following theorem (There is only a formal distinction to [6, Theorem 2.1]; they are in fact the same.) without the invertibility assumption.

Theorem 1 *Suppose $\mathbf{F}$ is any field and $f : M_{mn}(\mathbf{F}) \to M_{pq}(\mathbf{F})$ is a linear rank-one preserver. Then f has one of the following four forms:*

(I) *$p \geq m \geq 2$, $q \geq n \geq 2$ and $f(A) = X(A \oplus 0)Y$ for any $A \in M_{mn}(\mathbf{F})$, where $X \in GL_p(\mathbf{F})$ and $Y \in GL_q(\mathbf{F})$.*

(II) *$p \geq n \geq 2$, $q \geq m \geq 2$ and $f(A) = X(A^T \oplus 0)Y$ for any $A \in M_{mn}(\mathbf{F})$, where X and Y are defined as in (I).*

(III) *$f(A) = \xi(A)\gamma^T$ for any $A \in M_{mn}(\mathbf{F})$, where $\gamma \in \mathbf{F}^q \setminus \{0\}$ and $\xi : M_{mn}(\mathbf{F}) \to \mathbf{F}^p$ is a linear map such that $0 \notin \xi(M^1_{mn}(\mathbf{F}))$.*

(IV) *$f(A) = \beta\eta(A)^T$ for any $A \in M_{mn}(\mathbf{F})$, where $\beta \in \mathbf{F}^p \setminus \{0\}$ and $\eta : M_{mn}(\mathbf{F}) \to \mathbf{F}^q$ is a linear map such that $0 \notin \eta(M^1_{mn}(\mathbf{F}))$.*

Inspired by these works mentioned above, in this article we characterize the additive rank-one preservers from $M_{mn}(\mathbf{F})$ to $M_{pq}(\mathbf{F})$ over any field $\mathbf{F}$, i.e., investigating the following theorem.

Theorem 2 *For any field $\mathbf{F}$, a map $\phi : M_{mn}(\mathbf{F}) \to M_{pq}(\mathbf{F})$ is an additive rank-one preserver if and only if one of the following holds.*

(i) *There are $P \in M_{pm}(\mathbf{F})$, $Q \in M_{nq}(\mathbf{F})$ and an injective field endomorphism δ on $\mathbf{F}$ such that $N(P) \cap \left\{x^\delta \,|x \in \mathbf{F}^m\right\} = \{0\}$ and $N(Q^T) \cap \left\{y^\delta \,|y \in \mathbf{F}^n\right\} = \{0\}$, and ϕ has the form $A \mapsto PA^\delta Q$.*

(ii) *There are* $P \in M_{pn}(\mathbf{F})$, $Q \in M_{mq}(\mathbf{F})$ *and an injective field endomorphism* δ *on* $\mathbf{F}$ *such that* $N(P) \cap \{x^\delta \,|x \in \mathbf{F}^n\} = \{0\}$ *and* $N(Q^T) \cap \{y^\delta \,|y \in \mathbf{F}^m\} = \{0\}$, *and* ϕ *has the form* $A \mapsto P(A^\delta)^T Q$.

(iii) *There are nonzero* $v \in \mathbf{F}^q$ *and an additive map* $\mu : M_{mn}(\mathbf{F}) \to \mathbf{F}^p$ *such that* $0 \notin \mu(M^1_{mn}(\mathbf{F}))$ *and* ϕ *has the form* $A \mapsto \mu(A)v^T$.

(iv) *There are nonzero* $u \in \mathbf{F}^p$ *and an additive map* $\nu : M_{mn}(\mathbf{F}) \to \mathbf{F}^q$ *such that* $0 \notin \nu(M^1_{mn}(\mathbf{F}))$ *and* ϕ *has the form* $A \mapsto u\nu(A)^T$.

The proof of Theorem 2 will be shown in the next section. Now we remark Theorem 2 as follows:

1. When $p \geq m+n-1$, [6, Proposition 2.3] provided an example of linear rank-one preserver of the form (iv). Here, we also give another example of linear rank-one preserver of the form (iv) when $\mathbf{F}$ is a subfield of $\mathbf{R}$ of all real numbers. Let k be a positive integer, and let $f : M_{2,2k}(\mathbf{F}) \to M_{1,2k}(\mathbf{F})$ be defined by

$$f : \sum_{i\in\langle 2\rangle} \sum_{j\in\langle 2k\rangle} a_{ij}E_{ij} \mapsto \sum_{j\in\langle k\rangle} \left((a_{1,2j-1} - a_{2,2j})E_{1,2j-1} + (a_{1,2j} + a_{2,2j-1})E_{1,2j}\right).$$

   Clearly, f is a linear rank-one preserver of the form (iv). We can construct similar examples for the form described in (iii).

2. Note that, for any positive integers g and h, $\mathbf{R}$ is isomorphic to $M_{gh}(\mathbf{R})$ when they are viewed as additive groups. If $\xi_1 : M_{mn}(\mathbf{R}) \to \mathbf{R}$ and $\xi_2 : \mathbf{R} \to \mathbf{R}^g$ are additive group isomorphisms, then $\xi_2 \circ \xi_1$ is an additive group isomorphism from $M_{mn}(\mathbf{R})$ to $\mathbf{R}^g$. Hence there is a non-linear additive map μ (respectively, ν) satisfying (iii) (respectively, (iv)).

3. If $\mathbf{F}$ is some field that is not isomorphic to a proper subfield of itself (for example, $\mathbf{F}$ is a finite field), then any injective field endomorphism δ on $\mathbf{F}$ is a field automorphism.

Hence, $\left\{x^{\delta}\,|x \in \mathbf{F}^s\right\} = \mathbf{F}^s$. Then for any $A \in M_{rs}(\mathbf{F})$, $N(A) \cap \left\{x^{\delta}\,|x \in \mathbf{F}^s\right\} = \{0\}$ if and only if A has full column rank. Therefore, if the condition "**F** is not isomorphic to a proper subfield of itself" is added, the matrices P and Q defined in (i)/(ii) can be extended to some $p \times p$ invertible matrix X and $q \times q$ invertible matrix Y, respectively. Thus, the form described in (i) of Theorem 2 can be reduced to: $p \geq m \geq 2$, $q \geq n \geq 2$ and $A \mapsto X(A^{\delta} \oplus 0)Y$ with $X \in GL_m(\mathbf{F})$ and $Y \in GL_n(\mathbf{F})$. For the form described in (ii) of Theorem 2, we have the similar conclusion.

4. One also observe that Theorem 1 can be obtained by Theorem 2 by using similar argument. Indeed if we further assume that ϕ is linear, then the injective field endomorphism δ defined in (i)/(ii) must be linear too. Thus, δ must be the identity map, and hence the set $\{x^{\delta}\,|x \in \mathbb{F}^s\} = \mathbb{F}^s$.

5. If some appropriate restrictions on ϕ and $\mathbf{F}$ (for example, $\mathbf{F} = \mathbf{R}$ and ϕ preserves rank-one matrices in both directions) were added to Theorem 2, then Theorem 2 can be obtained directly from [12, 13]. However, the general case of Theorem 2 is not a direct corollary of [12, 13].

2 The Proof of Theorem 2

In this section, we investigate Theorem 2.

Proof of the sufficency part of Theorem 2. It is trivial that ϕ is an additive rank-one preserver if (iii) or (iv) holds. Suppose ϕ has the form described in (i). Clearly, ϕ is an additive map. It remains to show that ϕ is a rank-one preserver.

Note that for any $A \in M^1_{mn}(\mathbf{F})$,

$$\text{rank}(\phi(A)) = \text{rank}(PA^{\delta}Q) \leq \text{rank}A^{\delta} = 1.$$

Therefore, $\phi(M^1_{mn}(\mathbf{F})) \subseteq M^1_{pq}(\mathbf{F}) \cup \{0\}$. It suffices to show that $\phi(A) \neq 0$ for all $A \in M^1_{mn}(\mathbf{F})$. Since both the sets $N(P) \cap \left\{x^\delta \,| x \in \mathbf{F}^m \right\}$ and $N(Q^T) \cap \left\{y^\delta \,| y \in \mathbf{F}^n \right\}$ contain the zero vectors only, Px^δ and $Q^T y^\delta$ are nonzero for all nonzero $x \in \mathbf{F}^m$ and $y \in \mathbf{F}^n$. Then $\phi(xy^T) = P(xy^T)^\delta Q = Px^\delta(y^\delta)^T Q \neq 0$ for all $xy^T \in M^1_{mn}(\mathbf{F})$.

Similarly, we show that ϕ is an additive rank-one preserver if ϕ has the form described in (ii). ■

For the necessary part, we only need to prove the following two theorems.

Theorem 3 *Suppose* $\mathbf{F}$ *is any field, and* $\phi : M_{mn}(\mathbf{F}) \to M_{pq}(\mathbf{F})$ *is an additive rank-one preserver such that* $\mathrm{rank}(\phi(H) + \phi(G)) > 1$ *for some* $H, G \in M^1_{mn}(\mathbf{F})$*. Then* ϕ *has the form either (i) or (ii) of Theorem 2.*

Theorem 4 *Suppose* $\mathbf{F}$ *is any field, and* $\phi : M_{mn}(\mathbf{F}) \to M_{pq}(\mathbf{F})$ *is an additive rank-one preserver such that* $\mathrm{rank}(\phi(H) + \phi(G)) \leq 1$ *for any* $H, G \in M^1_{mn}(\mathbf{F})$*. Then* ϕ *has the form either (iii) or (iv) of Theorem 2.*

We first prove Theorem 4 as follows.

Proof of Theorem 4. For any $A \in M^1_{mn}(\mathbf{F})$, it follows from the definition of ϕ that $\phi(A) \in M^1_{pq}(\mathbf{F})$, and hence for each $A \in M^1_{mn}(\mathbb{F})$

$$\phi(A) = u_A v_A^T \tag{1}$$

for some nonzero $u_A \in \mathbf{F}^p$ and $v_A \in \mathbf{F}^q$. Let

$$u = u_{E_{11}} \quad \text{and} \quad v = v_{E_{11}}. \tag{2}$$

Case 1 Suppose the maximum number of linearly independent elements in $\phi(M^1_{mn}(\mathbf{F}))$ is 1. Then $\phi(A)$ and $\phi(E_{11})$ are linearly dependent for any $A \in M^1_{mn}(\mathbf{F})$. Thus, by (1) and (2), $\phi(A) = x_A uv^T$ for every $A \in M^1_{mn}(\mathbf{F})$, where $x_A \in \mathbf{F}^*$. Since every matrix in $M_{mn}(\mathbf{F})$ can be written as a sum of finitely many matrices in $M^1_{mn}(\mathbf{F})$, we derive from the additivity of ϕ that ϕ is of the form (iii)/(iv).

Case 2 Suppose the maximum number of linearly independent elements in $\phi(M^1_{mn}(\mathbf{F}))$ is greater than or equal to 2. Then there exists $B \in M^1_{mn}(\mathbf{F})$ such that $\phi(E_{11})$ and $\phi(B)$ are linearly independent. This, together with (1) and (2), implies that either

(a) u and u_B are linearly independent; or

(b) v and v_B are linearly independent.

When (a) holds, it follows from (1), (2) and $\text{rank}(\phi(E_{11}) + \phi(B)) \leq 1$ that v and v_B are linearly dependent, i.e., there is $c_B \in \mathbf{F}^*$ such that $v_B = c_B v$. This, together with (1), (2), $\text{rank}(\phi(E_{11}) + \phi(A)) \leq 1$ and $\text{rank}(\phi(B) + \phi(A)) \leq 1$ for any $A \in M^1_{mn}(\mathbf{F})$, implies that for any $A \in M^1_{mn}(\mathbf{F})$, v and v_A are linearly dependent. Thus, $v_A = c_A v$ for some $c_A \in \mathbf{F}^*$. Furthermore, $\phi(A) = c_A u_A v^T$. From the arbitrariness of $A \in M^1_{mn}(\mathbf{F})$, we can write (1) as $\phi(A) = \mu(A)v^T$ for any $A \in M^1_{mn}(\mathbf{F})$, where $\mu(A) = c_A u_A \in \mathbf{F}^p \setminus \{0\}$. Since every matrix in $M_{mn}(\mathbf{F})$ can be written as a sum of finitely many matrices in $M^1_{mn}(\mathbf{F})$, we derive from the additivity of ϕ that ϕ has the form (iii). Similarly, when (b) holds, one can conclude that ϕ has the form (iv). ■

Now we devote our attention to the proof of Theorem 3. This requires the following three lemmas.

Lemma 1 ([16, Lemma 1]) *Let* $\mathbf{F}$ *be any field and* $A, B \in M^1_{mn}(\mathbf{F})$ *satisfying* $A + B \in M^2_{mn}(\mathbf{F})$*. Then there are* $P \in GL_m(\mathbf{F})$ *and* $Q \in GL_n(\mathbf{F})$ *such that* $A = PE_{11}Q$ *and* $B = PE_{22}Q$*.*

Lemma 2 ([17, Lemma 2]) *Let* $C = [c_{gh}] \in M^1_{mn}(\mathbf{F})$*. If* $C + xE_{ij} \in M^1_{mn}(\mathbf{F})$ *for some* $x \in \mathbf{F}^*$*,* $i \in \langle m \rangle$*,* $j \in \langle n \rangle$*. Then* $C = \sum_{g=1}^{m} c_{gj} E_{gj}$ *or* $\sum_{h=1}^{n} c_{ih} E_{ih}$*.*

Lemma 3 *Suppose* $\mathbf{F}$ *is any field and* $\psi : M_2(\mathbf{F}) \to M_{pq}(\mathbf{F})$ *is an additive rank-one preserver. If* $\psi(I_2) \in M_{pq}^2(\mathbf{F})$*, then there are* $X \in GL_p(\mathbf{F})$*,* $Y \in GL_q(\mathbf{F})$ *and an injective field endomorphism* δ *on* $\mathbf{F}$ *such that* ψ *has the form*

$$A \mapsto X(A^\delta \oplus 0)Y \quad or \quad A \mapsto X((A^\delta)^T \oplus 0)Y.$$

Indeed, it is equivalent to say that there are linearly independent pairs $x_1, x_2 \in \mathbf{F}^p$ *and* $y_1, y_2 \in \mathbf{F}^q$ *such that either*

(i) $\psi(\lambda E_{ij}) = \delta(\lambda) x_i y_j^T$ *for all* $\lambda \in \mathbf{F}$*,* $i, j \in \langle 2 \rangle$*, or*

(ii) $\psi(\lambda E_{ij}) = \delta(\lambda) x_j y_i^T$ *for all* $\lambda \in \mathbf{F}$*,* $i, j \in \langle 2 \rangle$*.*

Proof. Since ψ is an additive rank one preserver, we have $\psi(E_{11}), \psi(E_{22}) \in M_{pq}^1(\mathbf{F})$. This, together with $\psi(E_{11}) + \psi(E_{22}) = \psi(I_2) \in M_{pq}^2(\mathbf{F})$ and Lemma 1, implies that

$$\psi(E_{11}) = PE_{11}Q, \ \psi(E_{22}) = PE_{22}Q \tag{3}$$

for some $P \in GL_p(\mathbf{F})$ and $Q \in GL_q(\mathbf{F})$.

Define a map $\psi_0 : M_2(\mathbf{F}) \to M_{pq}(\mathbf{F})$ by

$$\psi_0(A) = P^{-1}\psi(A)Q^{-1}, \ \forall A \in M_2(\mathbf{F}). \tag{4}$$

Then ψ_0 is an additive rank-one preserver, and further, we can derive from (3) that

$$\psi_0(E_{11}) = E_{11}, \quad \psi_0(E_{22}) = E_{22}. \tag{5}$$

For any $x \in \mathbf{F}^*$, since $xE_{12}, \ E_{11} + xE_{12}, \ E_{22} + xE_{12} \in M_2^1(\mathbf{F})$, it follows from (5) that $\psi_0(xE_{12}), \ E_{11} + \psi_0(xE_{12}), \ E_{22} + \psi_0(xE_{12}) \in M_{pq}^1(\mathbf{F})$. By Lemma 2, there is $a_x \in \mathbf{F}^*$ such that $\psi_0(xE_{12}) = a_x E_{12}$ or $a_x E_{21}$. Since ψ_0 is an additive rank-one preserver and $xE_{12} - yE_{12} \in M_2^1(\mathbf{F})$ for any distinct $x, y \in \mathbf{F}^*$, it is seen that either

$$\psi_0(\lambda E_{12}) = \pi(\lambda) E_{12}, \ \forall \lambda \in \mathbf{F} \tag{6}$$

or

$$\psi_0(\lambda E_{12}) = \pi(\lambda)E_{21},\ \forall \lambda \in \mathbf{F}, \tag{7}$$

where $\pi : \mathbf{F} \to \mathbf{F}$ is an injective additive map. Similarly, either

$$\psi_0(\lambda E_{21}) = \mu(\lambda)E_{12},\ \forall \lambda \in \mathbf{F} \tag{8}$$

or

$$\psi_0(\lambda E_{21}) = \mu(\lambda)E_{21},\ \forall \lambda \in \mathbf{F}, \tag{9}$$

where $\mu : \mathbf{F} \to \mathbf{F}$ is an injective additive map.

Case 1 Suppose (6) and (8) hold simultaneously. Then it follows from (5) that $\psi_0(E_{11}+E_{12}+E_{21}+E_{22}) = (\pi(1)+\mu(1))E_{12}+E_{11}+E_{22} \in M_{pq}^2(\mathbf{F})$, which contradicts that $E_{11}+E_{12}+E_{21}+E_{22} \in M_2^1(\mathbf{F})$ and ψ_0 is an additive rank-one preserver.

Case 2 Suppose (7) and (9) hold simultaneously. By an argument similar to Case 1, one can derive a contradiction.

Case 3 Suppose (6) and (9) hold simultaneously. For any $\lambda \in \mathbf{F}$, it follows from rank$(\lambda E_{11}+E_{11}) \leq 1$ and $E_{12} + \lambda E_{11},\ E_{21} + \lambda E_{11} \in M_2^1(\mathbf{F})$ that rank$(\psi_0(\lambda E_{11}) + \psi_0(E_{11})) \leq 1$ and $\psi_0(E_{12}) + \psi_0(\lambda E_{11}),\ \psi_0(E_{21}) + \psi_0(\lambda E_{11}) \in M_{pq}^1(\mathbf{F})$. Using Lemma 2, (5), (6) and (9), we have

$$\psi_0(\lambda E_{11}) = \delta(\lambda)E_{11},\ \forall \lambda \in \mathbf{F}, \tag{10}$$

where $\delta : \mathbf{F} \to \mathbf{F}$ is an injective additive map with $\delta(1) = 1$. Similarly,

$$\psi_0(\lambda E_{22}) = \varkappa(\lambda)E_{22},\ \forall \lambda \in \mathbf{F}, \tag{11}$$

where $\varkappa : \mathbf{F} \to \mathbf{F}$ is an injective additive map with $\varkappa(1) = 1$. Because of $\lambda E_{11}+\lambda E_{12}+E_{21}+E_{22}$, $E_{11} + \lambda E_{12} + E_{21} + \lambda E_{22} \in M_2^1(\mathbf{F})$, it follows from (6) and (9)—(11) that $\delta(\lambda)E_{11} + \pi(\lambda)E_{12} + \mu(1)E_{21} + E_{22}$, $E_{11} + \pi(\lambda)E_{12} + \mu(1)E_{21} + \varkappa(\lambda)E_{22} \in M_{pq}^1(\mathbf{F})$, and hence

$$\delta(\lambda) = \varkappa(\lambda) = \pi(\lambda)\mu(1),\ \forall \lambda \in \mathbf{F}. \tag{12}$$

Similarly,

$$\delta(\lambda) = \varkappa(\lambda) = \pi(1)\mu(\lambda),\ \forall \lambda \in \mathbf{F}. \tag{13}$$

If we denote $U = \mu(1)^{-1} \oplus I_{p-1}$ and $V = \mu(1) \oplus I_{q-1}$, then it is easy to verify from $\delta(1) = 1$, (6) and (9)—(13) that

$$\psi_0(\lambda E_{ij}) = \delta(\lambda) U E_{ij} V,\ \forall \lambda \in \mathbf{F},\ i, j \in \langle 2 \rangle. \tag{14}$$

For any $a, b \in \mathbf{F}$, because of $abE_{11} + aE_{12} + bE_{21} + E_{22} \in M_2^1(\mathbf{F})$, it follows from (14) that $\delta(ab)E_{11} + \delta(a)E_{12} + \delta(b)E_{21} + E_{22} \in M_{pq}^1(\mathbf{F})$, and hence $\delta(ab) = \delta(a)\delta(b)$. Since $\delta : \mathbf{F} \to \mathbf{F}$ is an injective additive map with $\delta(1) = 1$, it can be concluded that δ is an injective field endomorphism on $\mathbf{F}$. This, together with (14) and the additivity of ψ_0, ψ_0 has the form $A \mapsto U(A^\delta \oplus 0)V$. Thus, with (4), ψ has the same form too.

Case 4 Suppose (7) and (8) hold simultaneously. Then by an argument similar to Case 3, ψ has the form $A \mapsto U((A^\delta)^T \oplus 0)V$. ■

Based on the above preparations, one can prove Theorem 3 as follows.

Proof of Theorem 3. Suppose there are $H, G \in M_{mn}^1(\mathbf{F})$ such that $\mathrm{rank}(\phi(H) + \phi(G)) > 1$. Then $\mathrm{rank}(H + G) > 1$. Since $\mathrm{rank}(H + G) \leq \mathrm{rank}H + \mathrm{rank}G = 2$, we conclude that $H + G \in M_{mn}^2(\mathbf{F})$. By singular decomposition, there are $U \in GL_m(\mathbf{F})$ and $V \in GL_n(\mathbf{F})$ such that $H + G = U(I_2 \oplus 0)V$.

Define $\psi_1 : M_2(\mathbf{F}) \to M_{pq}(\mathbf{F})$ by $\psi_1(B) = \phi(U(B \oplus 0)V)$ for all $B \in M_2(\mathbf{F})$. Then ψ_1 is an additive rank-one preserver. Furthermore,

$$\psi_1(I_2) = \phi(U(I_2 \oplus 0)V) = \phi(H + G) = \phi(H) + \phi(G),$$

and hence $\mathrm{rank}\psi_1(I_2) \leq \mathrm{rank}\phi(H) + \mathrm{rank}\phi(G)$. This, together with $\phi(M_{mn}^1(\mathbf{F})) \subseteq M_{pq}^1(\mathbf{F})$ and $\mathrm{rank}\psi_1(I_2) = \mathrm{rank}(\phi(H) + \phi(G)) > 1$, implies that $\psi_1(I_2)$ has rank two. Then by Lemma 3, there are $P \in GL_p(\mathbf{F})$, $Q \in GL_q(\mathbf{F})$ and an injective field endomorphism δ on $\mathbf{F}$ such that

$$\phi(U(B \oplus 0)V) = \psi_1(B) = P(B^\delta \oplus 0)Q \quad \text{for all} \quad B \in M_2(\mathbf{F})$$

or

$$\phi(U(B \oplus 0)V) = \psi_1(B) = P((B^\delta)^T \oplus 0)Q \quad \text{for all} \quad B \in M_2(\mathbf{F}).$$

Replacing ϕ by the maps $A \mapsto P^{-1}\phi(UAV)Q^{-1}$ or $A \mapsto (P^{-1}\phi(UAV)Q^{-1})^T$, we may assume that

$$\phi(B \oplus 0) = B^\delta \oplus 0 \quad \text{for all } B \in M_2(\mathbf{F}). \tag{15}$$

Now for any $j \in \langle n \rangle \setminus \{1, 2\}$, since

$$E_{21} + \phi(E_{2j}) = \phi(E_{21} + E_{2j}) \quad \text{and} \quad E_{22} + \phi(E_{2j}) = \phi(E_{22} + E_{2j})$$

are rank one, we check that at least one of

$$E_{11} + \phi(E_{2j}) = \phi(E_{11} + E_{2j}) \quad \text{and} \quad E_{12} + \phi(E_{2j}) = \phi(E_{12} + E_{2j})$$

has rank two. Otherwise, we have $\phi(E_{2j}) = 0$, but this contradicts to $\phi(M^1_{mn}(\mathbf{F})) \subseteq M^1_{pq}(\mathbf{F})$. Let E_{1k}, where $k = 1$ or 2, be the matrix for which $\phi(E_{1k} + E_{2j})$ has rank two. We define $\psi_2 : M_2(\mathbf{F}) \to M_{pq}(\mathbf{F})$ by

$$\psi_2 \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \phi\left(b_{11}E_{1k} + b_{12}E_{1j} + b_{21}E_{2k} + b_{22}E_{2j}\right) \quad \text{for all} \quad \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \in M_2(\mathbf{F}).$$

Then ψ_2 is an additive rank-one preserver and $\psi_2(I_2) = \phi(E_{1k} + E_{2j})$ has rank two. From (15), we have

$$\psi_2(\lambda E_{11}) = \phi(\lambda E_{1k}) = \delta(\lambda) e_1 e_k^T \quad \text{and} \quad \psi_2(\lambda E_{21}) = \phi(\lambda E_{2k}) = \delta(\lambda) e_2 e_k^T \quad \text{for all} \quad \lambda \in \mathbf{F}.$$

Then by Lemma 3, there is $y_j \in \mathbf{F}^q$ such that y_j and e_k are linearly independent and

$$\phi(\lambda E_{1j}) = \psi_2(\lambda E_{12}) = \delta(\lambda) e_1 y_j^T \quad \text{and} \quad \phi(\lambda E_{2j}) = \psi_2(\lambda E_{22}) = \delta(\lambda) e_2 y_j^T \quad \text{for all} \quad \lambda \in \mathbf{F}.$$

Let $Y^T = \left[\begin{array}{ccc} y_1 & \cdots & y_n \end{array}\right]$ with $y_1 = e_1$ and $y_2 = e_2$. Then with (15), we have

$$\phi\left(\lambda E_{ij}\right) = \delta(\lambda) e_i e_j^T Y \quad \text{for all} \;\; \lambda \in \mathbf{F},\; i \in \langle 2\rangle \text{ and } j \in \langle n\rangle. \tag{16}$$

Now for any $i \in \langle m\rangle \setminus \{1,2\}$ and $j \in \langle n\rangle \setminus \{k\}$, we check that at least one of $\phi(E_{1k} + E_{ij})$ and $\phi(E_{2k} + E_{ij})$ has rank two. Let E_{lk}, where $l = 1$ or 2, be the matrix for which $\phi(E_{lk} + E_{ij})$ has rank two. Similarly, with Lemma 3, (16) and by considering the map $\psi_3 : M_2(\mathbf{F}) \to M_{pq}(\mathbf{F})$ defined by

$$\psi_3 \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \phi\left(b_{11}E_{lk} + b_{12}E_{lj} + b_{21}E_{ik} + b_{22}E_{ij}\right) \quad \text{for all} \quad \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \in M_2(\mathbf{F}),$$

we conclude that there is $x_i \in \mathbf{F}^p$ such that x_i and e_l are linearly independent and

$$\phi\left(\lambda E_{ik}\right) = \psi_3\left(\lambda E_{21}\right) = \delta(\lambda) x_i e_k^T Y \quad \text{and} \quad \phi\left(\lambda E_{ij}\right) = \psi_3\left(\lambda E_{22}\right) = \delta(\lambda) x_i e_j^T Y \quad \text{for all} \;\; \lambda \in \mathbf{F}.$$

Let $X = \left[\begin{array}{ccc} x_1 & \cdots & x_m \end{array}\right]$ with $x_1 = e_1$ and $x_2 = e_2$. Then with (16), we have

$$\phi\left(\lambda E_{ij}\right) = \delta(\lambda) X e_i e_j^T Y = \delta(\lambda) X E_{ij} Y \quad \text{for all} \;\; \lambda \in \mathbf{F},\; i \in \langle m\rangle \text{ and } j \in \langle n\rangle.$$

As ϕ is additive, we deduce that $\phi\left(A\right) = X A^\delta Y$ for all $A \in M_{mn}(\mathbf{F})$.

Finally, for any nonzero $x \in \mathbf{F}^m$ and $y \in \mathbf{F}^n$, $\phi\left(xy^T\right) = X x^\delta (y^\delta)^T Y \neq 0$ as ϕ is a rank-one preserver. Therefore, $X x^\delta$ and $Y^T y^\delta$ are nonzero for all nonzero $x \in \mathbf{F}^m$ and $y \in \mathbf{F}^n$, i.e., $N(X) \cap \left\{x^\delta \,|x \in \mathbf{F}^m\right\} = \{0\}$ and $N(Y^T) \cap \left\{y^\delta \,|y \in \mathbf{F}^n\right\} = \{0\}$. ■

3 Concluding remarks

This article characterized the additive rank-one preservers from $M_{mn}(\mathbf{F})$ to $M_{pq}(\mathbf{F})$ over any field $\mathbf{F}$ without the surjectivity assumption. As shown in [1–4], some preserver problems on matrix spaces can be reduced to rank-one preserver problems. This provides the possibility for removing the surjectivity assumption of some results on additive preserver problems.

Further work is to solve some preserver problems between spaces of rectangular matrices by reducing them to the results obtained in this article.

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