# Isometries for unitarily invariant norms

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## Abstract

A survey of linear isometries for unitarily invariant norms on real or complex rectangular matrices is given which includes some latest development on the topic. A result on isometries for unitarily invariant norms without the linearity assumption is presented. Related results and problems are discussed.

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# 1 Introduction

Let $M_{m,n}(\mathbb{F})$ (respectively, $M_n(\mathbb{F})$) be the set of $m \times n$ (respectively, $n \times n$) matrices over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. We will use the notation $M_{m,n}$ and $M_n$ if the discussion is valid for both real or complex matrices. Furthermore, we will assume that $2 \le m \le n$ in our discussion.

Denote by

$$\mathcal{U}_k = \{U \in M_k(\mathbb{F}) : U^*U = I_k\}$$

the group of $k \times k$ orthogonal or unitary matrices according to $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. A norm $\|\cdot\|$ on $M_{m,n}$ is *unitarily invariant* if

$$\|UAV\| = \|A\|$$

for any $A \in M_{m,n}$ and unitary matrices $U \in \mathcal{U}_m$ and $V \in \mathcal{U}_n$. Common examples of unitarily invariant norms include:

1. The operator norm $M_{m,n}$ defined by

$$\|A\|_{op} = \sup\{\ell_2(Ax) : x \in \mathbb{F}^n, \ell_2(x) \le 1\}.$$

2. The trace norm on $M_n$ defined by

$$\|A\|_{\rm tr} = \operatorname{tr}|A|,$$

where $|A|$ is the unique positive semi-definite matrix satisfying $|A|^2 = A^*A$.

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3. The Frobenius norm on $M_{m,n}$ defined by

$$\|A\|_F = \{\mathrm{tr}\,(AA^*)\}^{1/2}.$$

Denote the singular values of $A \in M_{m,n}$ by

$$s_1(A) \geq \cdots \geq s_m(A),$$

which are the nonnegative square roots of eigenvalues of the matrix $AA^*$. By the singular value decomposition, for every $A \in M_{m,n}$ there are matrices $U \in \mathcal{U}_m$ and $V \in \mathcal{U}_n$ such that that $A = UDV$ where $D \in M_{m,n}$ with $(j,j)$ entry equal to $s_j(A)$ for $j = 1, \ldots, m$, and all other entries equal to zero. Thus, for any unitarily invariant norm $\|\cdot\|$ on $M_{m,n}$, $\|A\| = \|D\|$. A norm $f : \mathbb{R}^{1\times m} \to \mathbb{R}$ is a symmetric gauge function if $f(x) = f(\hat{x})$ for any $\hat{x}$ obtained from $x$ by permuting the entries and changing signs of the entries. Von Neumann [50] (see also [40]) showed that there is a one-one correspondence between a unitarily invariant norm $\|\cdot\|$ on $M_{m,n}$ and a symmetric gauge function $f : \mathbb{R}^{1\times m} \to \mathbb{R}$ such that

$$\|A\| = f(s_1(A), \ldots, s_m(A)).$$

For example, the symmetric gauge functions corresponding to the operator norm, the trace norm, and the Frobenius norm are $f(x) = \ell_\infty(x), \ell_1(x)$, and $\ell_2(x)$, respectively. More generally, one has the Schatten $p$-norm on $M_{m,n}$ defined by

$$S_p(A) = \ell_p(s_1(A), \ldots, s_m(A)).$$

Another important class of unitarily invariant norms are the Ky Fan $k$-norms on $M_{m,n}$ with $1 \leq k \leq m$ defined by

$$F_k(A) = \sum_{j=1}^{k} s_j(A).$$

The dominance theorem of Ky Fan asserts that two matrices $A, B \in M_{m,n}$ satisfy

$$\|A\| \leq \|B\| \qquad \text{for all unitarily invariant norm } \|\cdot\|$$

if and only if

$$F_k(A) \leq F_k(B) \qquad \text{for all } k = 1, \ldots, m;$$

see [15]. Suppose $\|\cdot\|$ is a norm on $M_{m,n}$. A linear map $\phi : M_{m,n} \to M_{m,n}$ is an (linear) isometry for $\|\cdot\|$ if

$$\|\phi(A)\| = \|A\| \qquad \text{for all } A \in M_{m,n}.$$

One readily checks that the collection of all isometries for $\|\cdot\|$ is a group of invertible operators acting on $M_{m,n}$. Such a group is called the isometry group of $\|\cdot\|$.

In this paper, we give a survey of results and proof techniques in the study of isometries for unitarily invariant norms; see Section 2. Then we characterize $\phi : M_{m,n} \to M_{m,n}$ such that

$$\|\phi(A) - \phi(B)\| = \|A - B\| \quad \text{for all } A, B \in M_{m,n}$$

without the linearity assumption in Section 3. In section 4, we discuss some related results and problems.

## 2 A brief survey of results and proof techniques

In [47], Schur proved that an analytic mapping $\phi$ on $M_{m,n}(\mathbb{C})$ satisfies

$$\|\phi(A)\|_{op} = \|A\|_{op} \qquad \text{for all } A \in M_{m,n}(\mathbb{C}) \tag{1}$$

if and only if there are unitary matrices $U$ and $V$ such that $\phi$ has the following standard form:

**(S1)** $A \mapsto UAV$, or

**(S2)** $m = n$ and $A \mapsto UA^tV$.

The proof was rather computational, and a consequence was that the analytic map $\phi$ satisfying (1) is actually linear. Russo [45] proved that a linear isometry $\phi$ for the trace norm on $M_n(\mathbb{C})$ must have the form (S1) or (S2). His proof used the fact that a trace norm isometry satisfies $\phi(\mathcal{E}) = \mathcal{E}$, where

$$\mathcal{E} = \{xy^* : x, y \in \mathbb{C}^n,\ \ell_2(x) = \ell_2(y) = 1\}$$

is the set of extreme points of the unit ball

$$\mathcal{B} = \{A \in M_n(\mathbb{C}) : \|A\|_{tr} \le 1\}.$$

In fact, one can use the result in [36] to deduce that a linear map satisfying $\phi(\mathcal{E}) = \mathcal{E}$ has the form (S1) or (S2).

This idea can be applied to characterize linear isometries for the operator norm on $M_n(\mathbb{C})$. Such isometries $\phi$ must satisfy $\phi(\mathcal{U}_n) = \mathcal{U}_n$, where $\mathcal{U}_n$ is the set of extreme points of the unit ball of the operator norm on $M_n(\mathbb{C})$. One can then use the result of [35] concerning linear maps mapping the set of unitary matrices into itself to deduce that an isometry for the operator norm has the form (S1) or (S2).

A similar idea can be found in an earlier paper of Kadison [25], in which he characterized surjective isometries from a $C^*$-algebra to another $C^*$-algebra. A key step is to show that the extreme points of the unit ball are maximal partial isometries satisfying a certain equation. In his subsequent paper [26], he used state preserving maps to study positive linear isomorphisms between $C^*$-algebras.

In case of matrix algebras and spaces, it is easy to describe the technique in terms of the dual transformation and dual norm as follows. Equip $M_{m,n}$ with the usual inner product $(A, B) = \operatorname{tr} AB^*$. For any norm $\|\cdot\|$ and linear map $\phi : M_{m,n} \to M_{m,n}$, the dual norm of $\|\cdot\|$ is defined by

$$\|A\|^* = \sup\{|(A, X)| : \|X\| \leq 1\},$$

and the dual transformation of $\phi$ is the unique linear map $\phi^* : M_{m,n} \to M_{m,n}$ such that

$$(\phi(A), B) = (A, \phi^*(B)) \qquad \text{for all } A, B \in M_{m,n}.$$

It is then easy to verify the following.

**Proposition 2.1** *Suppose $\|\cdot\|$ is a norm on $M_{m,n}$ and $\phi : M_{m,n} \to M_{m,n}$ is a linear map. Let $\mathcal{B} = \{A \in M_{m,n} : \|A\| \leq 1\}$. Then the following are equivalent.*

(a) *$\phi$ is an isometry for $\|\cdot\|$.*

(b) *$\phi(\mathcal{S}) = \mathcal{S}$, where $\mathcal{S} = \mathcal{B}$, the boundary of $\mathcal{B}$, or the set of extreme points of $\mathcal{B}$.*

(c) *$\phi^*$ is an isometry for $\|\cdot\|^*$.*

(d) *$\phi^*(\mathcal{S}^*) = \mathcal{S}^*$, where $\mathcal{S}^*$ is the unit ball, the unit sphere, or the set of the extreme points of the unit ball of $\|\cdot\|^*$ in $M_{m,n}$.*

The above proposition is actually valid for general finite dimensional normed vector spaces. Relating to our previous discussion, one can check that $\|\cdot\|_{op}$ and $\|\cdot\|_{tr}$ are dual to each other on $M_n$. Also, the dual transformation of a linear map in the form (S1) and (S2) will be of the form

$$B \mapsto U^*BV^* \quad \text{ and } \quad B \mapsto \overline{V}B^t\overline{U},$$

respectively. Thus, using duality and extreme point techniques together with the real analog of the results on unitary matrix preservers and rank one matrix preservers, we have the following result for both real and complex matrices.

**Proposition 2.2** *The following conditions are equivalent for a linear map $\phi : M_n \to M_n$.*

(a) *$\phi$ is an isometry for the operator norm or the trace norm.*

(b) *$\phi(\mathcal{S}) = \mathcal{S}$ for $\mathcal{S} = \mathcal{U}_n$ or $\{xy^* : x, y \in \mathbb{F}^n,\ x^*x = y^*y = 1\}$.*

(c) *$\phi$ has the form* (S1) *or* (S2).

While the focus of this paper is on matrix spaces, it is interesting to note that the ideas of studying extreme points, dual transformations, and linear maps leaving invariant rank one Hermitian idempotents, etc. have been used to treat infinite dimensional problems; see [11, 25, 46].

Suppose $1 < k < n$. Grone and Marcus [19] proved that linear isometries for the Ky-Fan $k$-norm on $M_n(\mathbb{C})$ have the form (S1) or (S2). In their proof, they showed that the extreme points of the unit ball of the Ky Fan $k$-norm in $M_n(\mathbb{C})$ consists of matrices of the form $k^{-1}V$ with $V \in \mathcal{U}_n$ and (rank one) matrices with singular values $1, 0, \ldots, 0$. Then they showed that each of the two types of extreme points are mapped to themselves under a linear isometry. The result will then follow (see Proposition 2.2). In [17], Grone extended the result in [35] to rectangular matrices by showing that a linear map mapping the set $\{X \in M_{m,n}(\mathbb{C}) : X^*X = I_m\}$ into itself has the form (S1) or (S2). Consequently, he could extend the result in [19] to rectangular matrices in [18]; see also [16].

Grone and Marcus in their paper [19] proposed the study of the isometries for the $(p, k)$ norm on $M_{m,n}(\mathbb{C})$ defined by

$$N_{p,k}(A) = \left\{ \sum_{j=1}^{k} (s_j(A))^p \right\}^{1/p}$$

for a given $k \in \{1, \ldots, m\}$ and $p \in [1, \infty]$. Evidently, $N_{1,k}$ is the Ky Fan $k$-norm, and $N_{m,p}$ is the Schatten $p$-norm. In particular, $N_{m,2}$ is the Frobenius norm, and the isometries are just unitary operators on $M_{m,n}(\mathbb{C})$. For Schatten $p$-norm with $p \neq 2$ on $M_n(\mathbb{C})$, it follows from a more general result of Arazy [2] (see also [3]) that the isometries are again of the standard form (S1) or (S2). A key idea of the proof in [2] was to show that if certain norm inequalities for Schatten $p$-norms become equalities for $A, B \in M_n$, then $A$ and $B$ are disjoint in the sense that $AB^* = 0_n = BA^*$; see McCarthy [38]. As a result, an isometry will preserve "disjoint" matrices, and one can then deduce that it is of the standard form. In [31], Li and Tsing showed that isometries for other $(p, k)$ norms on $M_{m,n}(\mathbb{C})$ are always of the standard form. Their proof used the idea in [2] and certain special features of the boundary of the unit ball of $M_{m,n}(\mathbb{C})$ with respect to the $(p, k)$ norm.

The results on $(p, k)$ norms described above were on complex matrices. One might think that the corresponding results for real matrices could be obtained in a similar way. It turns out that there was a surprise. In [24] (see also [8] and [13]) the authors showed that a linear isometry for the Ky Fan $k$-norm on $M_n(\mathbb{R})$ either has the standard form (S1), (S2), or the following special form:

**(S3)** $(\mathbb{F}, n, k) = (\mathbb{R}, 4, 2)$, and the isometry is a composition of a mapping of the standard form (S1) or (S2) with a mapping of form

$$A \mapsto (A + B_1AC_1 + B_2AC_2 + B_3AC_3)/2$$

or

$$A \mapsto (DA + B_1DAC_1 + B_2DAC_2 + B_3DAC_3)/2,$$

where $D = \operatorname{diag}(-1, 1, 1, 1)$ and

$$B_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad C_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$B_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$B_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad C_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Here $\otimes$ denotes the Kronecker product $(x_{ij}) \otimes Y = (x_{ij}Y)$. In [24], the authors used the extreme point techniques and showed that the extreme points of the unit ball of $M_n(\mathbb{R})$ consists of three connected components, namely, the set $\mathcal{E}_1$ of the set of rank one matrices with singular values $1, 0, \ldots, 0$, (ii) the set $\mathcal{E}_2$ of matrices of the form $k^{-1}X$ where $X \in \mathcal{U}_n$ with positive determinant, and (iii) the set $\mathcal{E}_2$ of matrices of the form $k^{-1}Y$ where $Y \in \mathcal{U}_n$ with negative determinant. If $(n, k) \neq (4, 2)$, one can show that an isometry will maps $\mathcal{E}_1$ to $\mathcal{E}_1$; so the isometry has the standard form (S1) or (S2). If $(n, k) = (4, 2)$, an isometry can indeed permute the connected components $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3$. The special maps in (S3) will help correct the situation. That is, a composition of the given non-standard isometry with one of the special maps in (S3) will give a standard isometry that maps $\mathcal{E}_1$ onto $\mathcal{E}_1$. Note that each of the sets $\{I_4, B_1, B_2, B_3\}$ and $\{I_4, C_1, C_2, C_3\}$ form a basis for the noncommutative algebra of real quaternions in $M_4(\mathbb{R})$. In fact, the proofs in both [24] and [13] depended on the theory of real quaternions; see [8] for an elementary computational proof using the symbolic computer package - Mathematica.

Let $c = (c_1, \ldots, c_m)$ be nonzero with $c_1 \geq \cdots \geq c_m \geq 0$. The $c$-spectral norm $N_c$ on $M_{m,n}(\mathbb{F})$ is defined by

$$N_c(A) = \sum_{j=1}^{m} c_j s_j(A).$$

When $c_1 = \cdots = c_k = 1 > 0 = c_{k+1} = \cdots = c_m$, $N_c$ reduces to the Ky Fan $k$-norm. Furthermore, the $c$-spectral norms can be viewed as the building blocks of unitarily invariant norms because for every unitarily invariant norm $\|\cdot\|$ on $M_{m,n}(\mathbb{F})$ there is a compact set $\mathcal{S} \subseteq \mathbb{R}^{1\times m}$ such that

$$\|A\| = \max\{N_c(A) : c \in \mathcal{S}\};$$

see [21, 28]. In [32], Li and Tsing studied linear maps $\phi$ such that

$$N_d(\phi(A)) = N_c(A) \qquad \text{for all } A \in M_{m,n}, \tag{2}$$

for two nonnegative nonzero vectors $c, d \in \mathbb{R}^{1\times m}$. It was shown that such a linear map $\phi$ exists if and only if

$$\phi^*(\mathcal{O}(d)) = \mathcal{O}(c), \tag{3}$$

where $\mathcal{O}(c)$ (respectively, $\mathcal{O}(d)$) consists of matrices $A \in M_{m,n}$ such that $(s_1(A), \ldots, s_m(A))$ equals $c$ (respectively, equals $d$). Furthermore, (3) holds if and only if $c$ and $d$ are multiple of each other. So, after normalization, one may assume that $c = d$. In such case, an linear map satisfying (2) has the form (S1), (S2), or (S3) provided $M_{m,n} = M_n(\mathbb{R})$ and $c_1 = c_2 + c_3 > 0 = c_4$. The study of $c$-spectral norms have been extended to the infinite dimensional space in [7].

One may wonder whether there are other special isometries for unitarily invariant norms on $M_{m,n}$. It turns out that the exceptional case can only happen in $M_4(\mathbb{R})$ with the form (S3). The following theorem was proved in [33].

**Theorem 2.3** *Suppose $\|\cdot\|$ is a unitarily invariant norm on $M_{m,n}$, which is not a multiple of the Frobenius norm. An isometry for $\|\cdot\|$ has the form* (S1), (S2), *or* (S3).

The proof was done by geometrical arguments. First, it was shown that an isometry $\phi$ for a unitarily invariant $\|\cdot\|$ on $M_{m,n}$ must also be an isometry for the Frobenius norm. As a result, if $\mathcal{B}$ and $\mathcal{B}_F$ are the unit balls of $\|\cdot\|$ and the Frobenius norm, then $\phi(a\mathcal{B} \cap b\mathcal{B}_F) = a\mathcal{B} \cap b\mathcal{B}_F$ for any $a, b > 0$. Furthermore, $\phi$ will be an isometry for the unitarily invariant norm

$$N(A) = \sup\{|(A, X)| : X \in a\mathcal{B} \cap b\mathcal{B}_F\}$$

in case $a\mathcal{B} \cap b\mathcal{B}_F \neq \emptyset$. Suppose $\|\cdot\|$ is not a multiple of the Frobenius norm. By a suitable choice of $a$ and $b$, the set $\mathcal{E}_1$ of matrices $A \in M_{m,n}$ with singular values $1, 0, \ldots, 0$, will be a maximal connected component of the set of extreme points of the unit ball of $N$, and one can show that $\phi(\mathcal{E}_1) = \mathcal{E}_1$ so that $\phi$ has the standard form (S1) or (S2) unless $M_{m,n} = M_4(\mathbb{R})$. In the exceptional case, $\phi(\mathcal{E}_1)$ can be $\mathcal{E}_1$, $\mathcal{E}_2$, or $\mathcal{E}_3$, where

$$\mathcal{E}_2 = \{X/2 : X \in \mathcal{U}_4,\ \det(X) = 1\} \quad \text{or} \quad \mathcal{E}_3 = \{Y/2 : Y \in \mathcal{U}_4,\ \det(Y) = -1\}$$

as defined above. If $\phi(\mathcal{E}_1) \neq \mathcal{E}_1$, then $\phi$ is of the form (S3).

In [14], an alternative proof of Theorem 2.3 was given using a group theoretic approach. First, note that the set of all isometries for a norm form a group $G$. For a unitarily invariant norm $\|\cdot\|$ on $M_{m,n}$, the isometry group $G$ is a subgroup of $\mathcal{U}_{mn}$: the group of unitary or orthogonal operators on $M_{m,n}$ depending on $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$. Moreover, $G$ contains the group $\mathcal{U}_m * \mathcal{U}_n$: the group of operators of the form (S1). Using some theory of Lie groups, one can determine all possible compact groups in $\mathcal{U}_{mn}$ that contains $\mathcal{U}_m * \mathcal{U}_n$. It turns out that the only possibilities are (a) $\mathcal{U}_{mn}$, (b) $\mathcal{U}_m * \mathcal{U}_n$, (c) the group generated by $\mathcal{U}_m * \mathcal{U}_n$ and the transposition operator $X \mapsto X^t$ in case $m = n$, or (d) the group generated by $\mathcal{U}_4 * \mathcal{U}_4$, the transposition operator, and the special operators in (S3) if $M_{m,n} = M_4(\mathbb{R})$. Consequently, these are the only possible isometry groups of a unitarily invariant norm on $M_{m,n}$.

In an earlier paper [49], Sourour characterized isometries for unitarily invariant norms on normed ideals of compact operators (with the suitable convergence conditions). He used the fact that an isometry for the norm has the form $\exp(iH)$ where $H$ is a norm Hermitian operator in the norm ideal. In [27], Li considered real linear maps on $M_n(\mathbb{C})$ leaving the Ky Fan $k$-norm invariant. This result will be further discussed and extended in the next section.

# 3 Isometries without the linearity assumption

In this section, we characterize mapping $\phi : M_{m,n} \to M_{m,n}$ satisfying

$$\|\phi(A) - \phi(B)\| = \|A - B\| \quad \text{for all} \quad A, B \in M_{m,n}. \tag{4}$$

Here we do not impose any linearity assumption on $\phi$. Nevertheless, by the result of Charzyński in [9], if $\phi$ satisfies (4), then the map $T(A) = \phi(A) - \phi(0)$ is real linear and satisfies

$$\|T(A)\| = \|A\| \quad \text{for all} \quad A \in M_{m,n}. \tag{5}$$

Thus, we can focus on real linear maps satisfying (5). In the real case, the result reduces to Theorem 2.3. We will prove the following theorem for the complex case.

**Theorem 3.1** *Let $\|\cdot\|$ be a unitarily invariant norm on $M_{m,n}(\mathbb{C})$, and let $T : M_{m,n}(\mathbb{C}) \to M_{m,n}(\mathbb{C})$ be an additive or real linear map. Then $T$ satisfies* (5) *if and only if there exist $U \in \mathcal{U}_m$ and $V \in \mathcal{U}_n$ such that one of the following holds.*

(a) *$T$ has the form*

$$A \mapsto UAV \quad \text{or} \quad A \mapsto U\overline{A}V.$$

(b) *$m = n$ and $T$ has the form*

$$A \mapsto UA^tV \quad \text{or} \quad A \mapsto UA^*V.$$

(c) *$\|\cdot\|$ is a multiple of the Frobenius norm, that is, $\|A\| = \gamma(\operatorname{tr} AA^*)^{\frac{1}{2}}$ for some $\gamma > 0$, and $T$ is a real orthogonal transformation on $M_{m,n}(\mathbb{C})$ with respect to the inner product $(A, B) = \operatorname{Re}(\operatorname{tr} AB^*)$.*

By Theorem 3.1 and the previous discussion, we immediately get the following.

**Theorem 3.2** *Let $\|\cdot\|$ be a unitarily invariant norm on $M_{m,n}(\mathbb{C})$. A map $\phi : M_{m,n}(\mathbb{C}) \to M_{m,n}(\mathbb{C})$ satisfies*

$$\|\phi(A) - \phi(B)\| = \|A - B\| \quad \text{for all} \quad A, B \in M_{m,n}(\mathbb{C}) \tag{6}$$

*if and only if there exist $S \in M_{m,n}(\mathbb{C})$, $U \in \mathcal{U}_m$ and $V \in \mathcal{U}_n$ such that one of the following holds.*

(a) *$\phi$ has the form*

$$A \mapsto UAV + S \quad \text{or} \quad A \mapsto U\overline{A}V + S.$$

(b) *$m = n$ and $\phi$ has the form*

$$A \mapsto UA^tV + S \quad \text{or} \quad A \mapsto UA^*V + S.$$

(c) *$\|\cdot\|$ is a multiple of the Frobenius norm, that is, $\|A\| = \gamma(\operatorname{tr} AA^*)^{\frac{1}{2}}$ for some $\gamma > 0$, and the map $A \mapsto \phi(A) - S$ is a real orthogonal transformation on $M_{m,n}(\mathbb{C})$ with respect to the inner product $(A, B) = \operatorname{Re}(\operatorname{tr} AB^*)$.*

To prove Theorem 3.1, we need several auxiliary results. We shall use $\{E_{11}, E_{12}, \ldots, E_{mn}\}$ to denote the standard basis for $M_{m,n}$, and let $\mathbf{i} = \sqrt{-1}$.

**Lemma 3.3** *Let $\|\cdot\|$ be a unitarily invariant norm on $M_{m,n}(\mathbb{C})$ such that $\|\cdot\|$ is not a multiple of the Frobenius norm. If $T$ satisfies (5), then*

(a) *$T$ preserves the real inner product $(X, Y) = \operatorname{Re}(\operatorname{tr} XY^*)$ and the Frobenius norm on $M_{m,n}(\mathbb{C})$, and*

(b) *$T$ maps the set of matrices with singular values $1, 0, \ldots, 0$ onto itself.*

*Proof.* Our proof is an adaptation and modification of that in [33]. Yet, there are some technical details required different treatment. We give the details for the sake of completeness.

(a) Let $G$ be the group of all real linear operators on $M_{m,n}(\mathbb{C})$ satisfying (5). For any $X \in M_m$ and $Y \in M_n$, let $T_{X,Y}$ and $T^c_{X,Y}$ be the operators of the form

$$A \mapsto XAY \quad \text{and} \quad A \mapsto X\overline{A}Y$$

respectively. Evidently,

$$G_0 = \{T_{U,V} : U \in \mathcal{U}_m, V \in \mathcal{U}_n\} \cup \{T^c_{U,V} : U \in \mathcal{U}_m, V \in \mathcal{U}_n\} \subseteq G.$$

One can find $2m^2$ (real) linearly independent elements $U$ in $\mathcal{U}_m$, and $n^2$ linearly independent elements $V$ in $\mathcal{U}_n$ to get $(2m^2)(n^2)$ linearly independent operators of the form $T_{U,V}$, and also $(2m^2)(n^2)$ linearly independent operators of the form $T^c_{U,V}$. Combining these two sets of operators, we get $4m^2n^2$ real linear operators on $M_{m,n}(\mathbb{C})$ which form a basis for the algebra of all real linear operators on $M_{m,n}(\mathbb{C})$. Thus, the real linear span of the set $G_0$ equals the set of all real linear operators on $M_{m,n}(\mathbb{C})$.

Because $G$ is a bounded group of operators, by the result of Auerbach [6], (see [12] for an elementary proof), there is an invertible real linear operator $S : M_{m,n}(\mathbb{C}) \to M_{m,n}(\mathbb{C})$ such that

$$SGS^{-1} = \{STS^{-1} : T \in G\}$$

is a subgroup of the real orthogonal group of operators on $M_{m,n}(\mathbb{C})$ with respect to the real inner product $(A, B) = \mathrm{Re}(\mathrm{tr}\,(AB^*))$. Then for any $T \in G_0$,

$$(STS^{-1})^*(STS^{-1}) = (S^{-1})^*T^*(S^*S)TS^{-1}$$

is the identity map on $M_{m,n}(\mathbb{C})$, where $T^*$ is the dual transformation of $T$. Since $T^* = T^{-1}$ for every $T \in G_0$, it follows that

$$(S^*S)T = T(S^*S) \quad \text{for all } T \in G.$$

Since the real span of $G_0$ equals the set of all real linear operators, $S^*S$ is a scalar map. As $S^*S$ is positive definite, there is a positive $k$ such that $k(S^*S)$ is the identity map. Then $\sqrt{k}S$ is real orthogonal. Hence,

$$SGS^{-1} = (\sqrt{k}S)G(\sqrt{k}S)^{-1} = G$$

is a subgroup of the set of all real orthogonal operators. That is, every operator in $G$ preserves the real inner product on $M_{m,n}(\mathbb{C})$ and the Frobenius norm.

(b) Since $\|\cdot\|$ is not a multiple of the Frobenius norm, there exists $R \in M_{m,n}(\mathbb{C})$ such that $\|R\|_F = 1 = \|E_{11}\|_F$ and $\|R\| \neq \|E_{11}\|$. Let $r = \|R\| \neq \|E_{11}\|$. We define another unitary invariant norm $N(\cdot)$ by

$$N(A) = \max\{|(A, X)| : X \in \mathcal{B}_F \cap r\mathcal{B}\} = \max\left\{\sum_{i=1}^{m} s_i(A)s_i(X) : X \in \mathcal{B}_F \cap r\mathcal{B}\right\},$$

where $\mathcal{B}$ and $\mathcal{B}_F$ are the unit balls of $\|\cdot\|$ and the Frobenius norm respectively. Note that $T$ is real orthogonal and $T(\mathcal{B}_F \cap r\mathcal{B}) = \mathcal{B}_F \cap r\mathcal{B}$, then

$$\begin{aligned} N(T(A)) &= \max\{|(T(A), X)| : X \in \mathcal{B}_F \cap r\mathcal{B}\} \\ &= \max\{|(T(A), T(X))| : X \in \mathcal{B}_F \cap r\mathcal{B}\} \\ &= \max\{|(A, X)| : X \in \mathcal{B}_F \cap r\mathcal{B}\} \\ &= N(A). \end{aligned}$$

Let $\mathcal{S}(A) = \{UAV : U \in \mathcal{U}_m, V \in \mathcal{U}_n\}$ and

$$\Omega = \{A \in M_{m,n}(\mathbb{C}) : \|A\|_F = \|E_{11}\|_F \quad \text{and} \quad N(A) = N(E_{11})\}.$$

Then $\mathcal{S}(E_{11}) \subseteq \Omega$ and $T(\Omega) = \Omega$. We claim that $\mathcal{S}(E_{11})$ is a maximum connected component of $\Omega$. Clearly, $\mathcal{S}(E_{11})$ is connected and is a closed subset of $\Omega$. It remains to show that $\mathcal{S}(E_{11})$ is open in $\Omega$. To this end, take any matrix $A \in \mathcal{S}(E_{11})$ and let $Z$ be a matrix in the boundary of $\mathcal{B}_F \cap r\mathcal{B}$ such that

$$N(A) = (A, Z) = \sum_{i=1}^{m} s_i(A)s_i(Z) = s_1(Z).$$

Note that $\|Z\|_F = 1$ and $\|Z\| = r \neq \|E_{11}\|$, $Z$ has at least two nonzero singular values. Hence, $s_2(Z) > 0$. Let $\epsilon = s_2(Z)[(m-1)s_1(Z)]^{-1} > 0$. Suppose $B \in \Omega$ and $\|B - A\|_F < \epsilon$. Then

$$s_2(B) = s_2(B) - s_2(A) \leq \left(\sum_{i=1}^{m}(s_i(B) - s_i(A))^2\right)^{1/2} \leq \|B - A\|_F < \epsilon.$$

Since $B \in \Omega$, we have $\sum_{i=1}^{m} s_i(B)^2 = \|B\|_F^2 = 1$. Clearly, $N(A) = N(B)$. As

$$\begin{aligned} N(A) &= s_1(Z) = s_1(Z)\sum_{i=1}^{m} s_i(B)^2 \leq s_1(Z)s_1(B)^2 + (m-1)s_1(Z)s_2(B)^2 \quad \text{and} \\ N(B) &\geq \sum_{i=1}^{m} s_i(B)s_i(Z) \geq s_1(Z)s_1(B) + s_2(Z)s_2(B) \geq s_1(Z)s_1(B)^2 + s_2(Z)s_2(B), \end{aligned}$$

we conclude that $(m-1)s_1(Z)s_2(B)^2 \geq s_2(Z)s_2(B)$. So $s_2(B)=0$; otherwise,

$$s_2(Z)s_2(B) = (m-1)s_1(Z)s_2(B)\epsilon > (m-1)s_1(Z)s_2(B)^2 \geq s_2(Z)s_2(B).$$

As a result, $B \in \mathcal{S}(E_{11})$ and hence $\mathcal{S}(E_{11})$ is open. Note that $\mathcal{S}(E_{11})$ is a real differentiable manifold in $M_{m,n}(\mathbb{C})$. The tangent space $\mathcal{T}_{E_{11}}(\mathcal{S}(E_{11}))$ of the manifold $\mathcal{S}(E_{11})$ at $E_{11}$ consists of $X \in M_{m,n}(\mathbb{C})$ such that there is a smooth curve $f : (-d,d) \to \mathcal{S}(E_{11})$ satisfying

$$f(t) = E_{11} + tX + O(t^2) \in \mathcal{S}(E_{11}).$$

For any $H = H^* \in M_m(\mathbb{C})$ and $K = K^* \in M_n(\mathbb{C})$,

$$f(t) = e^{itH}E_{11} + E_{11}e^{itK} = E_{11} + it(HE_{11} + E_{11}K) + O(t^2), \qquad t \in \mathbb{R}$$

is a curve in $\mathcal{S}(E_{11})$; conversely, $X \in \mathcal{T}_{E_{11}}(\mathcal{S}(E_{11}))$ implies that $E_{11} + tX + h(t) \in \mathcal{S}(E_{11})$ for some smooth function $h(t)$ of order $t^2$ and hence $X$ has imaginary $(1,1)$ entry, and zero $(i,j)$ entries if $i \geq 1$ or $j \geq 1$. Thus, we see that

$$\mathcal{T}_{E_{11}}(\mathcal{S}(E_{11})) = \{i(HE_{11} + E_{11}K) : H = H^* \in M_m(\mathbb{C}),\ K = K^* \in M_n(\mathbb{C})\}$$

is a real linear space of dimension $2(m+n)-3$.

Now, since $T(\Omega) = \Omega$ and $T$ is a homeomorphism, there is a maximal connected component $\mathcal{C}$ of $\Omega$ such that $T(\mathcal{C}) = \mathcal{S}(E_{11})$. On the other hand, let $A = T(E_{11})$. As $\mathcal{S}(E_{11})$ is a maximal connected component of $\Omega$ containing $T^{-1}(A)$ and

$$\mathcal{S}(A) = \{UAV : U \in \mathcal{U}_m, V \in \mathcal{U}_n\}$$

is a connected subset of $\Omega$, $T^{-1}(\mathcal{S}(A)) \subseteq \mathcal{S}(E_{11})$. By the singular value decomposition, there exist $U \in \mathcal{U}_m$ and $V \in \mathcal{U}_n$ such that $A = U(\sum_{j=1}^m s_j(A)E_{jj})V$. Let

$$A' = U\left(s_1(A)E_{12} + s_2(A)E_{21} + \sum_{j=3}^m s_j(A)E_{jj}\right)V.$$

Then $A' \in \mathcal{S}(A)$. Thus, both $T^{-1}(A)$ and $T^{-1}(A')$ are in $\mathcal{S}(E_{11})$, it follows that the rank of the matrix $T^{-1}(A-A')$ is at most 2. Let $C = T^{-1}((A-A')/\|A-A'\|_2)$. Then $C$ is at most rank 2. Because $(A-A')/\|A-A'\|_2$ is in $\mathcal{S}(E_{11})$, we see that $C \in T^{-1}(\mathcal{S}(E_{11})) = \mathcal{C}$.

Suppose $C$ has rank 2. By the singular value decomposition, there exist $U_1 \in \mathcal{U}_m$ and $V_1 \in \mathcal{U}_n$ such that $C = U_1(s_1(C)E_{11} + s_2(C)E_{22})V_1$. As $T(C) \in \mathcal{S}(E_{11})$, there exist $U_2 \in \mathcal{U}_m$ and $V_2 \in \mathcal{U}_n$ such that $T(C) = U_2E_{11}V_2$. We may assume that $U_1 = U_2 = I_m$ and $V_1 = V_2 =$

$I_n$, i.e., $C = s_1(C)E_{11} + s_2(C)E_{22}$ and $T(C) = E_{11}$; otherwise, replace $T$ by the mapping $X \mapsto U_2^* T(U_1 X V_1) V_2^*$. Since $T$ is a bijective linear map such that $T(\mathcal{C}) = \mathcal{S}(E_{11})$, $T$ will map the tangent space $\mathcal{T}_C(\mathcal{C})$ of $\mathcal{C}$ at $C$ onto $\mathcal{T}_{E_{11}}(\mathcal{S}(E_{11}))$. Because $\mathcal{C}$ is a maximal connected component of $\Omega$ containing $C$ and $\mathcal{S}(C)$ is a connected set of $\Omega$, we see that $\mathcal{S}(C) \subseteq \mathcal{C}$. Moreover, for any $t \in \mathbb{R}$, $H = H^* \in M_m(\mathbb{C})$ and $G = G^* \in M_n(\mathbb{C})$,

$$g(t) = e^{itH} C e^{itG} = C + it(HC + CG) + O(t^2) \in \mathcal{S}(C) \subseteq \mathcal{C}.$$

So, the tangent space of $\mathcal{T}_C(\mathcal{C})$ at $C$ contains the space

$$\mathcal{T}_C(\mathcal{S}(C)) = \{i(HC + CG) : H = H^* \in M_m(\mathbb{C}),\ G = G^* \in M_n(\mathbb{C})\}.$$

Since $T$ is bijective linear and $T(\mathcal{T}_C(\mathcal{S}(C))) \subseteq \mathcal{T}_{E_{11}}(\mathcal{S}(E_{11}))$,

$$\begin{aligned} 2(m+n) - 3 &= \dim \mathcal{T}_{E_{11}}(\mathcal{S}(E_{11})) \\ &\geq \dim \mathcal{T}_C(\mathcal{S}(C)) \\ &= \begin{cases} 4(m+n) - 12 & \text{if } s_1(C) = s_2(C), \\ 4(m+n) - 10 & \text{if } s_1(C) > s_2(C). \end{cases} \end{aligned}$$

We check that the above inequality holds if and only if $m = n = 2$ and $s_1(C) = s_2(C)$. In this case, it is impossible to have $\mathcal{C} = \mathcal{S}(C)$ as

$$\dim \mathcal{T}_C(\mathcal{S}(C)) = 4 < 5 = \dim \mathcal{T}_{E_{11}}(\mathcal{S}(E_{11})).$$

Therefore, $\mathcal{C}$ must contain some matrices $C'$ with $s_1(C') > s_2(C') > 0$. However, using the similar argument on $C'$ instead of $C$, we get

$$6 = \dim \mathcal{T}_{C'}(\mathcal{S}(C')) \leq \dim \mathcal{T}_{E_{11}}(\mathcal{S}(E_{11})) = 5,$$

which is impossible.

By the above arguments, $C$ must be a rank one matrix in $\Omega$, and hence $C \in \mathcal{S}(E_{11})$. Since both $\mathcal{C}$ and $\mathcal{S}(E_{11})$ are maximal connected components of $\Omega$ containing $C$, we conclude that $\mathcal{C} = \mathcal{S}(E_{11})$. ■

To complete the proof of Theorem 3.1, we need one more result, namely, Theorem 3.5, concerning real linear maps $L : M_{m,n}(\mathbb{C}) \to M_{m,n}(\mathbb{C})$ mapping rank one matrices to rank one matrices. Note that when $m = n$, a characterization of bijective additive maps preserving rank one matrices in both directions was given by Omladič and Šemrl [44].

We begin with the following observation, the proof of which is straightforward and will be omitted.

**Lemma 3.4** *Suppose $P \in M_m(\mathbb{C})$ and $Q \in M_n(\mathbb{C})$ are invertible. Then for any rank one matrix $E \in M_{m,n}(\mathbb{C})$, if there is a nonzero $\gamma \in \mathbb{C}$ such that $E + \gamma P E_{11} Q$ is also rank one, then either*

$$E = P\left(\sum_{i=1}^{m} \alpha_i E_{i1}\right) Q \quad \textit{or} \quad E = P\left(\sum_{j=1}^{n} \beta_j E_{1j}\right) Q \quad \textit{for some} \quad \alpha_i, \beta_j \in \mathbb{C}. \tag{7}$$

**Theorem 3.5** *Denote by $\mathcal{R}$ the set of rank one matrices in $M_{m,n}(\mathbb{C})$. If $L : M_{m,n}(\mathbb{C}) \to M_{m,n}(\mathbb{C})$ is real linear such that*

$$A \in M_{m,n}(\mathbb{C}) \;\; \textit{satisfies} \;\; A \in \mathcal{R} \quad \textit{if and only if} \quad L(A) \in \mathcal{R}, \tag{8}$$

*then $L$ is either linear or conjugate linear. Consequently, a real linear map $L : M_{m,n}(\mathbb{C}) \to M_{m,n}(\mathbb{C})$ satisfies (8) if and only if there are invertible matrices $P \in M_m(\mathbb{C})$ and $Q \in M_n(\mathbb{C})$ such that*

(a) *$L$ has the form*

$$A \mapsto PAQ \quad \textit{or} \quad A \mapsto P\overline{A}Q.$$

(b) *$m = n$ and $L$ has the form*

$$A \mapsto PA^tQ \quad \textit{or} \quad A \mapsto PA^*Q.$$

*Proof.* First, we show that for any matrix $R \in \mathcal{R}$, there is nonzero $\mu_R \in \mathbb{C}\backslash\mathbb{R}$ such that

$$L(\mathbf{i}R) = \mu_R L(R). \tag{9}$$

Since $L$ is real linear, it is sufficient to show the claim holds for all rank one matrices with operator norm one. Now take any matrix $R$ in $\mathcal{R}$ with operator norm one. Then we can write $R = XE_{11}Y$ for some $X \in \mathcal{U}_m$ and $Y \in \mathcal{U}_n$. Since $L$ satisfies (8), there are $U \in \mathcal{U}_m$ and $V \in \mathcal{U}_n$ such that $L(R) = U(\gamma E_{11})V$. Let

$$R_0 = X(\mathbf{i}E_{11})Y = \mathbf{i}R, \quad R_1 = XE_{12}Y \quad \text{and} \quad R_2 = XE_{21}Y.$$

Since $R_i, (R_i + R) \in \mathcal{R}$, we see that $L(R_i), L(R_i) + L(R) = L(R_i) + U(\gamma E_{11})V \in \mathcal{R}$ for $i = 0, 1, 2$. By Lemma 3.4, $L(R_i) = UE_iV$ where $E_i$ has the form (7). On the other hand, because $R_0 + R_1$ and $R_0 + R_2$ have rank one, we see that $E_0 + E_1$ and $E_0 + E_2$ also have rank one.

Suppose $E_0$ has the form $\sum_{i=1}^m \alpha_i E_{i1}$ with some $\alpha_2, \ldots, \alpha_m$ not equal to zero. Then $E_1$ and $E_2$ must also be of the same form, and $E_1 + E_2$ is a rank one matrix. But then

$L$ maps the rank two matrix $(R_1 + R_2)$ to the rank one matrix $U(E_1 + E_2)V$, which is impossible. Similarly, we can show that $E_0$ cannot be of the form $\sum_{j=1}^n \beta_j E_{1j}$ with some $\beta_2, \dots, \beta_n$ not equal to zero. Therefore, $E_0 = \mu_R E_{11}$ for some nonzero $\mu_R \in \mathbb{C}$. Hence, $L(\mathbf{i}R) = \mu_R U E_{11} V = \mu_R L(R)$.

Note that $\mu_R$ must not be in $\mathbb{R}$; otherwise

$$L(\mathbf{i}R - \mu_R R) = L(\mathbf{i}R) - L(\mu_R R) = \mu_R L(R) - \mu_R L(R) = 0,$$

contradicting the fact that $L$ maps rank one matrices to rank one matrices.

Next we show that $L(E_{ii}) \neq \alpha L(E_{ij})$ for all $\alpha \in \mathbb{C}$. Suppose there is $\alpha \in \mathbb{C}$ such that $L(E_{ii}) = \alpha L(E_{ij})$. As $\mu_{E_{ij}} \notin \mathbb{R}$, there exist $a, b \in \mathbb{R}$ such that $a + b\mu_{E_{ij}} = \alpha$. Then

$$\begin{aligned} & L(E_{ii} - (a + b\mathbf{i})E_{ij}) = L(E_{ii}) - aL(E_{ij}) - bL(\mathbf{i}E_{ij}) \\ = \; & L(E_{ii}) - (a + b\mu_{E_{ij}})L(E_{ij}) = \alpha L(E_{ij}) - (a + b\mu_{E_{ij}})L(E_{ij}) = 0. \end{aligned}$$

Again this is impossible as $E_{ii} - (a + b\mathbf{i})E_{ij}$ is a rank one matrix.

Finally we show that there is $\mu \in \{\mathbf{i}, -\mathbf{i}\}$ such that $T(\mathbf{i}E_{ij}) = \mu T(E_{ij})$ for all $i, j$. To see this, consider $R = E_{ii}, E_{ij}, (E_{ii} + E_{ij})$ and $(-\mathbf{i}E_{ii} + E_{ij})$ in (9). For simplicity, we write $\mu_{ii} = \mu_{E_{ii}}$, $\mu_{ij} = \mu_{E_{ij}}$, $\mu_1 = \mu_{(E_{ii}+E_{ij})}$ and $\mu_2 = \mu_{(-\mathbf{i}E_{ii}+E_{ij})}$. Then

$$\mu_{ii} L(E_{ii}) + \mu_{ij} L(E_{ij}) = L(\mathbf{i}(E_{ii} + E_{ij})) = \mu_1 L(E_{ii} + E_{ij}) = \mu_1 L(E_{ii}) + \mu_1 L(E_{ij})$$

and

$$L(E_{ii}) + \mu_{ij} L(E_{ij}) = L(\mathbf{i}(-\mathbf{i}E_{ii} + E_{ij})) = \mu_2 L(-\mathbf{i}E_{ii} + E_{ij}) = -\mu_2 \mu_{ii} L(E_{ii}) + \mu_2 L(E_{ij}).$$

Thus,

$$(\mu_{ii} - \mu_1)L(E_{ii}) = (\mu_1 - \mu_{ij})L(E_{ij}) \quad \text{and} \quad (1 + \mu_2\mu_{ii})L(E_{ii}) = (\mu_2 - \mu_{ij})L(E_{ij}).$$

Since $L(E_{ii}) \neq \alpha L(E_{ij})$ for all $\alpha \in \mathbb{C}$, we must have $\mu_{ii} = \mu_{ij} = \mu_1 = \mu_2$ and $\mu_{ii}^2 = -1$. Considering $E_{ii}$ and $E_{ji}$ in a similar way, we show that $\mu_{ii} = \mu_{ji}$. Take $\mu$ to be the common value of $\mu_{ij}$, then $\mu^2 = -1$ and $L(\mathbf{i}E_{ij}) = \mu L(E_{ij})$ for all $i, j$. Since $L$ is real linear, we conclude that either $L(\alpha E_{ij}) = \alpha L(E_{ij})$ for all $\alpha \in \mathbb{C}$ and $E_{ij}$, or $L(\alpha E_{ij}) = \overline{\alpha} L(E_{ij})$ for all $\alpha \in \mathbb{C}$ and $E_{ij}$. Hence, $L$ is either linear or conjugate linear.

Now, if $L$ is linear, then we can apply the result in [37] (see also [39]) to conclude that (a) or (b) holds; if $L$ is conjugate linear, we can apply the result in [37] to the map $A \mapsto \overline{L(A)}$ and conclude that (a) or (b) holds. So, we get the necessity of the second assertion in the theorem. The sufficiency of the assertion is clear. ■

*Proof of Theorem 3.1.* The sufficiency part is clear. We consider the necessity part. If $T$ is additive, then $T(qA) = qT(A)$ for all rational numbers $q$. For any real number $r$, there exists a sequence of rational numbers $\{r_k\}$ approaching $r$. Then

$$\begin{aligned}\|T(rA) - rT(A)\| &\leq \|T(rA) - T(r_kA)\| + \|T(r_kA) - rT(A)\| \\ &= \|rA - r_kA\| + \|r_kT(A) - rT(A)\| \\ &= |r - r_k|\|A\| + |r_k - r|\|T(A)\|\end{aligned}$$

will approach to 0 when $r_k$ tends to $r$, i.e., $T$ is real linear.

Suppose $\|\cdot\|$ is a multiple of the Frobenius norm, Then clearly $T$ preserves $\|\cdot\|$ if and only if $T$ is an orthogonal transformation on $M_{m,n}(\mathbb{C})$.

Suppose $\|\cdot\|$ is not a multiple of the Frobenius norm, and $T$ satisfies (5). By Lemma 3.3 (b), $T(\mathcal{S}(E_{11})) = \mathcal{S}(E_{11})$. So, $A \in M_{m,n}(\mathbb{C})$ has rank one if and only if $T(A)$ has rank one. By Theorem 3.5, $T$ is either linear or conjugate linear. Applying the result of [33] to $T$ if $T$ is linear, or to the map $A \mapsto \overline{T(A)}$ if $T$ is conjugate linear, we have the conclusion. ∎

# 4 Related results and problems

There are many interesting results and problems that deserve further research. We briefly mention some of them in the following.

First, there are study of unitarily invariant norms on other matrix and operator algebras or spaces. Moore and Trent [42] (see also [5]) characterized surjective isometries for the operator norm on nest algebras; Anoussis and Katavolos [1] characterized surjective isometries for the Schatten $p$-norms on nest algebras; Li, Šemrl, and Sourour [30] characterized surjective isometries for Ky Fan $k$-norms for block triangular matrix algebras, which can be viewed as all finite dimensional nest algebras. The problem of characterizing surjective isometries for general unitarily invariant norms on nest algebras (finite or infinite dimensional) is still open.

Related results on operator norm isometries on other algebras and spaces can be found in [4, 22, 23, 41, 43, 48]. It would be interesting to formulate and extend the results to other unitarily invariant norms.

Another direction is to study the isometry problems without the surjectivity assumption. In [10], the authors studied linear map $\phi : M_m(\mathbb{C}) \to M_n(\mathbb{C})$ such that

$$\|\phi(A)\|_{op} = \|A\|_{op} \quad \text{for all } A \in M_m(C). \tag{10}$$

If $\phi$ has the form

$$A \mapsto U[A \oplus g(A)]V \quad \text{or} \quad A \mapsto U[A^t \oplus g(A)]V \tag{11}$$

for some $U, V \in \mathcal{U}_n$ and contractive linear map $g : M_m(\mathbb{C}) \to M_{n-m}(\mathbb{C})$, then (10) holds. However, the converse may not hold in general. It was shown that if $n \leq 2m - 1$ and $\phi$ satisfies (10) then condition (11) holds. If $n \geq 2m$ there are linear maps satisfying (10) but not (11). The problem of characterizing linear maps $\phi : M_m(\mathbb{C}) \to M_n(\mathbb{C})$ satisfying (10) is still open. Also, not much is known about the real case.

While the structure of operator norm preserving maps between two matrix spaces is rather complicated, the corresponding problem for Ky Fan $k$-norms are more tractable for $k > 1$. Li, Poon, and Sze [29] showed that for positive integers $k$ and $k'$ with $k' > 1$. A linear map $\phi : M_{m,n}(\mathbb{C}) \to M_{r,s}(\mathbb{C})$ satisfies

$$F_k(\phi(A)) = F_{k'}(A) \quad \text{for all } A \in M_{m,n}(\mathbb{C}) \tag{12}$$

if and only if there are nonnegative integer $a$ and $b$ and partial isometries $P$ and $Q$ of suitable dimensions such that one of the following holds.

(a) $k' < m$, $k'(a + b) = k$, and $\phi$ has the form

$$A \mapsto (a + b)^{-1} P^*[(I_a \otimes A) \oplus (I_b \otimes A^t)]Q.$$

(b) $k' = m$, $k'(a + b) \leq k$, and $\phi$ has the form

$$A \mapsto P^*[(D_a \otimes A) \oplus (D_b \otimes A^t)]Q,$$

where $D_a$ and $D_b$ are positive diagonal matrices such that $\mathrm{tr}\,(D_a \oplus D_b) = 1$.

It would be interesting to see whether the same conclusion holds for other unitarily invariant norms, and for the real case.

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