Characterizations of inverse M-matrices with special zero patterns

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Abstract

In this paper, we provide some characterizations of inverse M-matrices with special zero patterns. In particular, we give necessary and sufficient conditions for k-diagonal matrices and symmetric k-diagonal matrices to be inverse M-matrices. In addition, results for triadic matrices, tridiagonal matrices and symmetric 5-diagonal matrices are presented as corollaries.

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1. Introduction

A matrix A is called an *M-matrix* if A has non-positive off-diagonal entries and the eigenvalues of A have positive real part. There are many equivalent characterizations of M-matrices, see [3], for instance, A is an M-matrix if A is nonsingular and A^{-1} is a nonnegative matrix. However, in general a nonnegative matrix is not necessarily the inverse of an M-matrix. A nonsingular matrix A is called an *inverse M-matrix* if A^{-1} is an M-matrix. A first study in finding sufficient conditions for a nonnegative symmetric matrix to be an inverse M-matrix was conducted in [11] by T.L. Markham, and it was also shown in [11] that the inverse of a type-D matrix A with positive $(1,1)$th entry is a tridiagonal M-matrix. Since then, many efforts have been devoted to characterize nonnegative matrices whose inverses are M-matrices [1, 6, 7, 13], and certain special inverse M-matrices such as ultrametric matrices have been investigated

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in [8, 9, 10, 12]. Researchers call this problem the *inverse M-matrix problem* [13]. However, until now only few sufficient conditions were developed.

The aim of this paper is to provide some characterizations for nonnegative matrices with special zero patterns to be inverse M-matrices. A necessary and sufficient condition for a matrix to be an inverse M-matrix will be given in Section 2, and this main result will be used in Section 3 to study certain special matrices, namely, k-diagonal matrices and triadic matrices.

We first fix some notation. Denote by $\langle n\rangle$ the index set $\{1,\ldots,n\}$ for positive integer n. For notation convenience, we set $\langle n\rangle=\emptyset$ if $n\leq 0$. Let α and β be nonempty ordered subsets of $\langle n\rangle$, both of strictly increasing integers. Then $A[\alpha,\beta]$ is the submatrix of A with rows indexed by α and columns indexed by β. For simplicity, we write $A[\alpha]=A[\alpha,\alpha]$. It is not surprising that inverse M-matrices inherit certain considerable properties from M-matrices. Here, we list some properties that will be frequently used in this paper.

Suppose A is an inverse M-matrix.

(P1) A is a nonnegative matrix with positive diagonal entires.

(P2) All principal submatrices of A are inverse M-matrices.

(P3) For any permutation matrix P, P^TAP is an inverse M-matrix.

(P4) For any $\alpha\subseteq\langle n\rangle$, the Schur complement of $A/A[\alpha]$ is an inverse M-matrix.

To present the next property, we require the following definition. A nonnegative matrix $B=[b_{ij}]$ is called *zero-pattern invariant* if for any i,j, the (i,j)th entry of B equals zero if and only if

$$b_{ij}=0 \quad\Longleftrightarrow\quad b_{ik}b_{kj}=0 \quad \text{for all } k.$$

Indeed, if B is zero-pattern invariant, then every power B^n of B has the same zero pattern as B. Let $A=[a_{ij}]$ be an inverse M-matrix. Then (P1) implies that A has positive diagonal entires and (P4) implies that the Schur complement $A/[a_{kk}]$ is an inverse M-matrix for all k and hence $A/[a_{kk}]$ is nonnegative. Then for any distinct i, j and k,

$$a_{ij}-\frac{a_{ik}a_{kj}}{a_{kk}}\geq 0.$$

It follows that $a_{ij}=0$ implies $a_{ik}a_{kj}=0$ for all k. Then

$$a_{ij}=0 \quad\Longrightarrow\quad \sum_k a_{ik}a_{kj}=0 \quad\Longrightarrow\quad a_{ij}a_{jj}=0 \quad\Longrightarrow\quad a_{ij}=0.$$

Thus, we have the following property.

(P5) Every inverse M-matrix is zero-pattern invariant.

It has to be noted that (P5) is equivalent to a well known fact that the directed graph of every inverse M-matrix is transitively closed. That is, in the directed graph of an inverse M-matrix, there exists a path form i to j if and only if there is an edge from i to j (see e.g., [7] and [10]). For a more detailed description of inverse M-matrices, we refer readers to [3, 5].

2. Main result

We now present the main theorem of the paper.

Theorem 1. *Suppose $A = [a_{ij}]$ is an $n \times n$ nonnegative matrix with positive diagonal entries. Define the ordered index sets*

$$\gamma_i = \{k \in \langle n \rangle : a_{ik} > 0\} \quad \text{and} \quad \rho_j = \{k \in \langle n \rangle : a_{kj} > 0\} \quad \text{for all } i, j \in \langle n \rangle.$$

Then the following are equivalent.

(a) *A is an inverse M-matrix;*
(b) *A is zero-pattern invariant and the principal submatrix $A[\gamma_i]$ is an inverse M-matrix for all $i \in \langle n \rangle$;*
(c) *A is zero-pattern invariant and the principal submatrix $A[\rho_j]$ is an inverse M-matrix for all $j \in \langle n \rangle$.*

Proof. The implications (a) $\Rightarrow$ (b) and (a) $\Rightarrow$ (c) clearly follow from (P2) and (P5). We now prove (b) $\Rightarrow$ (a). The proof for (c) $\Rightarrow$ (a) is similar.

Assume (b) holds. Fixed any arbitrary $i \in \langle n \rangle$. We choose a sequence $i_1, \ldots, i_m \in \langle n \rangle$ with $i_1 = i$ such that

$$\gamma_{i_{k+1}} \backslash \gamma_{i_k} \neq \emptyset \quad \text{for all } k = 1, \ldots, m-1 \quad \text{and} \quad \bigcup_{k=1}^{m} \gamma_{i_k} = \langle n \rangle.$$

Define $\alpha_1 = \gamma_{i_1}$ and $\alpha_k = \gamma_{i_k} \backslash \left(\gamma_{i_1} \cup \cdots \cup \gamma_{i_{k-1}}\right)$ for $k = 2, \ldots, m$. Then for any $k < \ell$,

$$\alpha_k \cap \alpha_\ell = \emptyset \quad \text{and} \quad \bigcup_{k=1}^{m} \alpha_k = \langle n \rangle.$$

Suppose $k < \ell$ and take any arbitrary $(r, s) \in \alpha_k \times \alpha_\ell$. Notice that $r \in \gamma_{i_k}$ while $s \notin \gamma_{i_k}$. Hence, $a_{i_k r} \neq 0$ and $a_{i_k s} = 0$. Then zero-pattern invariant property ensures that $a_{i_k r} a_{rs} = 0$ and thus $a_{rs} = 0$. In short,

$$a_{rs} = 0 \quad \text{for all} \quad (r, s) \in \alpha_k \times \alpha_\ell \quad \text{with} \quad k < \ell.$$

From this, there exists a permutation matrix P such that

$$P^T A P = \begin{bmatrix} A[\alpha_1] & 0 & \cdots & 0 \\ * & A[\alpha_2] & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ * & \cdots & * & A[\alpha_m] \end{bmatrix}.$$

Furthermore, since $\gamma_{i_k} \subseteq \alpha_1 \cup \cdots \cup \alpha_k$, $A[\gamma_{i_k}]$ is permutationally similar to

$$\begin{bmatrix} A[\gamma_{i_k} \backslash \alpha_k] & 0 \\ * & A[\alpha_k] \end{bmatrix}.$$

Then the assumption that $A[\gamma_{i_k}]$ is an inverse M-matrix ensures the invertibility of $A[\alpha_k]$ for all k, and therefore P^TAP is invertible. Moreover,

$$P^TA^{-1}P = (P^TAP)^{-1} = \begin{bmatrix} (A[\alpha_1])^{-1} & 0 \\ * & * \end{bmatrix} = \begin{bmatrix} (A[\gamma_i])^{-1} & 0 \\ * & * \end{bmatrix}.$$

By the assumption, $A[\gamma_i]$ is an inverse M-matrix and hence $(A[\gamma_i])^{-1}$ has non-positive off-diagonal entries only. In particular, all off-diagonal entries in the ith row of A^{-1} are non-positive. As i is arbitrary, we conclude that A^{-1} has non-positive off-diagonal entries only. Therefore, A is an inverse M-matrix. □

A few remarks on Theorem 1. By (P1) and (P5), it is natural to assume in Theorem 1 that A is zero-pattern invariant and has positive diagonal entries. On the other hand, given an $n \times n$ matrix A with the above mentioned properties, to determine whether A is an inverse M-matrix, by applying Theorem 1, one only needs to check whether the n principal submatrices $A[\gamma_1], \ldots, A[\gamma_n]$ are inverse M-matrices. In particular, if $|\gamma_i| \le k < n$ for all $i \in \langle n \rangle$, one only has to consider n submatrices of A which are of size at most k. It will be definitely an advantage in computation if k is much smaller than n. To illustrate this, let us consider the following simple example.

Example Let

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

First it can be checked that A is zero-pattern invariant. Since

$$\gamma_1 = \{1,3,5\}, \quad \gamma_2 = \{2,3\}, \quad \gamma_3 = \{3\}, \quad \gamma_4 = \{2,3,4\}, \quad \text{and} \quad \gamma_5 = \{3,5\},$$

one suffices to check the submatrices

$$A[\{1,3,5\}] = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad A[\{2,3,4\}] = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

Observe that both these two matrices are inverse M-matrix matrices, so as A by Theorem 1. Indeed,

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{bmatrix}.$$

The following corollary is immediate from Theorem 1.

Corollary 2. *Suppose A is an $n \times n$ matrix with at most k nonzero entries in every row (column). Then A is an inverse M-matrix if and only if A is zero-pattern invariant and every $k \times k$ principal submatrix of A is an inverse M-matrix.*

3. *k*-diagonal matrices and triadic matrices

The sufficient condition in Theorem 1 can be further reformulated if certain special zero pattern is imposed. A matrix $A = [a_{ij}]$ is called *k-diagonal* if $a_{ij} = 0$ for all $|i - j| > \frac{k-1}{2}$. Obviously, we can always assume k is odd. Now we have the following series of results for k-diagonal matrices.

Theorem 3. *Suppose A is an $n \times n$ nonnegative k-diagonal matrix with $1 < k < n$. Then A is an inverse M-matrix if and only if A is zero-pattern invariant and the $(k-1) \times (k-1)$ principal submatrix*

$$A[\langle r\rangle \backslash \langle r-k+1\rangle]$$

is an inverse M-matrix for all $r = k-1, \ldots, n$.

Proof. The necessity part is trivial by (P2) and (p5). For the sufficiency part, note that for any $i \in \langle n\rangle$, there is $k \le r \le n$ such that $A[\gamma_i]$ is a principal submatrix of the $k \times k$ matrix $A[\langle r\rangle \backslash \langle r-k\rangle]$. By Theorem 1 and (P2), it suffices to show that $A[\langle r\rangle \backslash \langle r-k\rangle]$ is an inverse M-matrix for all $r = k, \ldots, n$.

Let $B = [b_{ij}] = A[\langle r\rangle \backslash \langle r-k\rangle]$ and $p = \frac{k+1}{2}$. Clearly, B is a $k \times k$ nonnegative zero-pattern invariant k-diagonal matrix. By considering $(1,k)$th entry of B^2 with the fact that $b_{1k} = 0$, we have

$$0 \le b_{1p} b_{pk} \le \sum_{j=1}^{k} b_{1j} b_{jk} = 0.$$

Then either $b_{1p} = 0$ or $b_{pk} = 0$. If $b_{pk} = 0$, then B has at most $k-1$ nonzero entries in every row. Define $\beta_i = \{\ell : b_{i\ell} > 0\}$ for $i \in \langle k\rangle$. Observe that $B[\beta_i]$ is a principal submatrix of either

$$B[\langle k-1\rangle] = A[\langle r-1\rangle \backslash \langle r-k\rangle] \quad \text{or} \quad B[\langle k\rangle \backslash \langle 1\rangle] = A[\langle r\rangle \backslash \langle r-k+1\rangle].$$

By assumption, both these two matrices are inverse M-matrices. Thus, $B[\beta_i]$ is an inverse M-matrix and the same conclusion occurs to B by Theorem 1. If $b_{1p} = 0$, then B has at most $k-1$ nonzero entries in every column. By a similar argument, the result follows by considering $\tau_j = \{\ell : b_{\ell j} > 0\}$. □

If A is also symmetric, then one only needs to consider submatrices with size $\frac{k+1}{2}$ as shown below.

Corollary 4. *Suppose $1 < k < n$ and A is an $n \times n$ nonnegative symmetric k-diagonal matrix. Then A is an inverse M-matrix if and only if A is zero-pattern invariant and the $p \times p$ principal submatrix*

$$A[\langle r\rangle \backslash \langle r-p\rangle]$$

is an inverse M-matrix for all $r = p, \ldots, n$, where $p = \frac{k+1}{2}$.

Proof. If A is an inverse M-matrix, obviously the conclusion is true by (P2) and (P5). Conversely, to get the result, it suffices to show that every $A[\gamma_i]$ is a principal submatrix of $A[\langle r\rangle\backslash\langle r-p\rangle]$ for some $p \leq r \leq n$.

To see this, suppose a_{is} and a_{it} are the first and the last nonzero entires in the ith row, respectively. Notice that the (s,t)th entry of A^2 is equal to

$$\sum_{\ell=1}^{n} a_{s\ell}a_{\ell t} \geq a_{si}a_{it} = a_{is}a_{it} > 0.$$

Because of the zero-pattern invariance property, A^2 is also k-diagonal and so $|t-s| \leq \frac{k-1}{2} < p$. Then $\gamma_i \subseteq \{s,\ldots,t\} \subseteq \langle t\rangle\backslash\langle t-p\rangle$, and therefore, $A[\gamma_i]$ is a principal submatrix of $A[\langle t\rangle\backslash\langle t-p\rangle]$. □

Notice that a 2×2 nonnegative matrix B is an inverse M-matrix if and only if the determinant of B is positive. Then Theorem 3 implies the following.

Corollary 5. *Suppose A is a nonnegative tridiagonal matrix. Then A is an inverse M-matrix if and only if A is a zero-pattern invariant matrix with all its principal minors of order* 2 *being positive.*

For 3×3 case, we have the following equivalent conditions for inverse M-matrix, which can be found in [4, 13].

Lemma 6. *Suppose $A = [a_{ij}]$ is a 3×3 nonnegative matrix with positive diagonal entries. Then the following are equivalent.*

(a) *A is an inverse M-matrix;*

(b) *For any distinct i, j and k,*

$$a_{ij}a_{ji} < a_{ii}a_{jj} \quad \textit{and} \quad a_{ik}a_{kj} \leq a_{ij}a_{kk}.$$

(c) *The Schur complements $A/[a_{11}]$, $A/[a_{22}]$, and $A/[a_{33}]$ are nonnegative with positive diagonal entires.*

Now Theorem 3 and Lemma 6 give the following result.

Corollary 7. *Suppose $A = [a_{ij}]$ is a nonnegative symmetric* 5*-diagonal matrix with positive diagonal entries. Then A is an inverse M-matrix if and only if the Schur complement $A/[a_{jj}]$ is nonnegative with positive diagonal entries for all $j \in \langle n\rangle$.*

Proof. The necessity part is clear by (P1) and (P4). For the sufficiency part, suppose the Schur complement $A/[a_{jj}]$ is nonnegative with positive diagonal entires for all $j \in \langle n\rangle$. Then for any distinct i, j and k,

$$a_{ij}a_{ji} < a_{ii}a_{jj} \quad \text{and} \quad a_{ik}a_{kj} \leq a_{ij}a_{kk}.$$

So $a_{ij} = 0$ implies $a_{ik}a_{kj} = 0$ and hence $\sum_{k=1}^{n} a_{ik}a_{kj} = 0$. Thus, A is zero-pattern invariant. Also by Lemma 6, the submatrix $A[\langle r\rangle\backslash\langle r-3\rangle]$ is an inverse M-matrix for all $r = 3,\ldots,n$. Then the result follows by Theorem 3. □

A matrix A is called a *triadic* matrix if each row of A has at most two nonzero off-diagonal entires. Obviously, a tridiagonal matrix is a special case. We remark that this definition is slightly different from the one given by Fang and O'leary in [2]. By a similar argument as in the proof of Corollary 7, we have the following result for triadic matrices.

Theorem 8. *Suppose $A = [a_{ij}]$ is a nonnegative triadic matrix with positive diagonal entries. Then A is an inverse M-matrix if and only if the Schur complement $A/[a_{jj}]$ is nonnegative with positive diagonal entires for all $j \in \langle n \rangle$.*

Corollary 9. *Suppose A is a triadic $(0,1)$-matrix. Then A is an inverse M-matrix if and only if A is a nonsingular zero-pattern invariant matrix.*

Proof. The necessity part is clear by (P5). Suppose A is nonsingular and zero-pattern invariant. Clearly, all its diagonal entries must be positive, i.e., $a_{jj} = 1$. In addition, if $a_{ik}a_{kj} \neq 0$, then zero-pattern invariant property ensures $a_{ij} \neq 0$ and by the fact that A is a $(0,1)$-matrix, we conclude $a_{ik}a_{kj} \leq a_{ij}a_{kk}$ for all distinct i, j and k.

We next claim that $a_{ij}a_{ji} = 0$ for all $i \neq j$. Suppose not, then $a_{ij} = a_{ji} = 1$. For any $k \neq i$ and j,

$$a_{ik} = 0 \quad \Rightarrow \quad a_{ij}a_{jk} = 0 \quad \Rightarrow \quad a_{jk} = 0 \quad \Rightarrow \quad a_{ji}a_{ik} = 0 \quad \Rightarrow \quad a_{ik} = 0.$$

Therefore, $a_{ik} = 0$ if and only if $a_{jk} = 0$. In this case, the ith and jth rows of A are the same as A is a $(0,1)$-matrix. But this contradicts that A is nonsingular. So $a_{ij}a_{ji} = 0$ and hence $a_{ij}a_{ji} < a_{ii}a_{jj}$. Since the above inequalities hold for any arbitrary distinct i, j and k, it can be concluded by Lemma 6 that any 3×3 principal submatrix of A is an inverse M-matrix. Then the result follows by Theorem 3. ☐

Back to the Example before Corollary 2. Indeed, the matrix A in the example is a triadic zero-pattern invariant $(0,1)$-matrix. One can conclude directly by Corollary 9 that A is an inverse M-matrix, and the examination of those principal submatrices $A[\gamma_i]$ is actually redundant.

However, it has to be noted that the sufficiency part of Corollary 9 is not true if one removes the triadic condition. This can be seen by considering the following counter-example.

$$B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad B^{-1} = \begin{bmatrix} 1 & -1 & -1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Notice that B is a nonsingular zero-pattern invariant $(0,1)$-matrix, but B is not an inverse M-matrix.

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