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CONVERGENCE OF BOUNDARY LAYERS OF CHEMOTAXIS MODELS WITH PHYSICAL BOUNDARY CONDITIONS I: DEGENERATE INITIAL DATA*

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Abstract. The celebrated experiment of Tuval et al. [*Proc. Natl. Acad. Sci. USA*, 102 (2005), pp. 2277–2282] showed that the bacteria living a water drop can form a thin layer near the air-water interface, where a so-called chemotaxis-fluid system with physical boundary conditions was proposed to interpret the mechanism underlying the pattern formation alongside numerical simulations. However, the rigorous proof for the existence and convergence of the boundary layer solutions to the proposed model still remains open. This paper shows that the model with physical boundary conditions proposed by Tuval et al. in one dimension can generate a boundary layer solution as the oxygen diffusion rate $\varepsilon > 0$ is small. Specifically, we show that the solution of the model with $\varepsilon > 0$ will converge to the solution with $\varepsilon = 0$ (outer-layer solution) plus the boundary layer profiles (inner-layer solution) with a sharp transition near the boundary as $\varepsilon \to 0$. There are two major difficulties in our analysis. First, the global well-posedness of the model is hard to prove since the Dirichlet boundary condition cannot contribute to the gradient estimates needed for the cross-diffusion structure in the model. Resorting to the technique of taking the antiderivative, we remove the cross-diffusion structure such that the Dirichlet boundary condition can facilitate the needed estimates. Second, the outer-layer profile of bacterial density is required to be degenerate at the boundary as $t \to 0^+$, which incapacitates the traditional cancellation technique. Here we employ the Hardy inequality and delicate weighted energy estimates to overcome this obstacle and derive the requisite uniform-in-ε estimates allowing us to pass the limit $\varepsilon \to 0$ to achieve our results.

Key words. boundary layers, chemotaxis, linear sensitivity, physical boundary conditions, antiderivative

MSC codes. 35B40, 35K57, 35Q92, 92C17

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1. Introduction.

The directional movement of cells in response to a chemical concentration gradient is referred to as chemotaxis, which is said to be endogenous if the chemical is secreted by the cell itself and exogenous if the chemical comes from an external source (like oxygen, light, or food). Chemotaxis is a common biological migration strategy occurring in various biological processes, such as aggregation of

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bacteria (cf. [50]), slime mold formation (cf. [23]), or tumor angiogenesis (cf. [7, 10]). The mathematical models of chemotaxis mostly studied nowadays are of the Keller–Segel type originally proposed in [31, 32]. The prototype of the Keller–Segel model describing the exogenous chemotaxis reads as

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla \phi(v)), \\ v_t = \varepsilon \Delta v - uv, \end{cases} \tag{1.1}$$

where u and v denote the cell density and chemical concentration, respectively, at position $x \in \Omega$ and time $t > 0$. $\varepsilon > 0$ denotes the chemical diffusivity, and $\phi(v)$ is called the chemotactic sensitivity function which has two prototypes: $\phi(v) = \ln v$ (logarithmic sensitivity) and $\phi(v) = v$ (linear sensitivity). The logarithmic sensitivity was first proposed in [32] based on the Weber–Fechner law (the sensory response to a stimulus is logarithmic) which has various prominent biological applications (cf. [12, 29, 36]). It was mentioned in [32, p. 241] that the chemical (i.e., oxygen) diffusion rate ε is negligible (i.e., $0 < \varepsilon \ll 1$) compared to the bacterial diffusion rate. The most important application of the logarithmic sensitivity lies in its capability of producing traveling waves to interpret the experiment findings (cf. [30]), motivating a great deal of interesting mathematical works on the study of existence and stability of traveling wave solutions including [5, 9, 11, 38, 39], just to mention a few. The system (1.1) with linear sensitivity $\phi(v) = v$ was employed in a chemotaxis-fluid model proposed in [49] to interpret the boundary accumulation layer of aerobic bacterial chemotaxis towards the drop edge (air-water interface) in a sessile drop mixed with *Bacillus subtilis* bacteria. The model in [49] reads as

$$\begin{cases} u_t + \mathbf{w} \cdot \nabla u = \Delta u - \nabla \cdot (u \nabla v) & \text{in } \Omega, \\ v_t + \mathbf{w} \cdot \nabla v = D \Delta v - uv & \text{in } \Omega, \\ \rho(\mathbf{w}_t + \mathbf{w} \cdot \nabla \mathbf{w}) = \mu \Delta \mathbf{w} + \nabla p - V_b g u (\rho_b - \rho)\, \mathbf{z} & \text{in } \Omega, \\ \nabla \cdot \mathbf{w} = 0, \end{cases} \tag{1.2}$$

with the following physical zero-flux Dirichlet no-slip mixed boundary conditions:

$$(\nabla u - u \nabla v) \cdot \nu = 0, \quad v = v_*, \ \mathbf{w} = 0 \text{ on } \partial\Omega, \tag{1.3}$$

where u and v denote the bacterial and oxygen concentrations at $x \in \Omega$ and $t > 0$, respectively, and $\mathbf{w}$ is the fluid velocity governed by the incompressible Navier–Stokes equations with the pure fluid density ρ and viscosity μ. p is a pressure function, $V_b g u(\rho_b - \rho)\, \mathbf{z}$ denotes the buoyant force along the upward unit vector $\mathbf{z}$ where V_b and ρ_b are the bacterial volume and density, respectively, and g is the gravitational constant. In (1.3), ν denotes the outward unit normal vector of $\partial\Omega$, and $v_* > 0$ is a constant representing the saturation of oxygen at the air-water interface (i.e., boundary). The numerical simulations in works [8, 35, 49] have shown that the system (1.2) can reproduce the key features of boundary layer formation observed in the experiment of [49] in two and three dimensions under the physical boundary conditions (1.3). Therefore, justifying that (1.2)–(1.3) admits boundary layer solutions becomes an imperative question, which has remained open for a long time and, as far as we know, has not made any good progress. Indeed, the boundary layer problem has been a fundamental topic in fluid mechanics due to the distortion of nonviscous flow by surrounding viscous forces, as observed by Prandtl in 1904 [44], and has attracted extensive studies (cf. [1, 18, 19, 20, 21, 28, 53, 55], just to mention a few). Though the model (1.2) contains the fluid dynamics, the boundary layer was formed due to the aggregation of bacteria attracted by the oxygen near the air-water interface (cf. [13, 49]), and thus the fluid dynamics will play minor roles as can be glimpsed from the boundary conditions (1.3). Since the Dirichlet boundary condition for v cannot directly contribute to the estimate of ∇v required by the first equation of (1.1) for the estimate of u, many basic questions on (1.2)–(1.3), such as the global well-posedness, still remain poorly understood so far apart from the boundary layer solutions. To the best of our knowledge, there are fewer analytical results for problem (1.2)–(1.3), the local existence of weak solutions with large initial data was obtained in [41], and recently Wang, Winkler, and Xiang [51] proved the global existence of smooth solutions with small initial data; the reader can also see [43] for a global existence result of the system with slightly different boundary conditions. If the domain Ω is radially symmetric (say a ball) and the solutions are also radially symmetric, then the incompressibility condition $\nabla\cdot\ \mathbf{w}=0$ on Ω and no-slip boundary condition $\mathbf{w}|_{\partial\Omega}=0$ imply that $\mathbf{w}=0$, and as a result, (1.2)–(1.3) is simplified as

$$\begin{cases} u_t=\Delta u-\nabla\cdot(u\nabla v) & \text{in } \ \Omega, \\ v_t=\varepsilon\Delta v-uv & \text{in } \ \Omega, \end{cases} \tag{1.4}$$

with boundary conditions

$$(\nabla u-u\nabla v)\cdot\nu=0, \quad v=v_* \quad \text{on} \quad \partial\Omega. \tag{1.5}$$

Regarding the boundary layer solutions, it was first shown in [34] that the problem (1.4)–(1.5) has a unique stationary solution in all dimensions, which possesses a boundary layer profile with thickness of order $\varepsilon^{1/2}$ as $\varepsilon\to 0$. Subsequently, the nonlinear local time-asymptotic stability of stationary solutions of (1.4)–(1.5) in one dimension was established recently in [25]. However, whether the time-dependent problem (1.4)–(1.5) can develop boundary layer profiles as $\varepsilon\to 0$ remains unknown. To see the possibility, we integrate the second equation of (1.4) with $\varepsilon=0$ and get $v(x,t)=v_0(x)\,\mathrm{e}^{-\int_0^t u(x,\tau)\mathrm{d}\tau}$, which gives rise to

$$v|_{\partial\Omega}=v_0|_{\partial\Omega}\mathrm{e}^{-\int_0^t u|_{\partial\Omega}\mathrm{d}\tau}. \tag{1.6}$$

This implies that the boundary value of v as $\varepsilon=0$ is intrinsically determined by (1.6), which may mismatch the prescribed boundary value of v for $\varepsilon>0$. If this occurs, boundary layers will arise as $\varepsilon\to 0$, and the zero-diffusion limit of (1.4)–(1.5) as $\varepsilon\to 0$ becomes a singular problem. However, how to justify the convergence of solutions of the singular problem (1.4)–(1.5) as $\varepsilon\to 0$ still remains an outstanding open question as far as we know. The goal of this paper is to investigate the zero-diffusion limit of the problem (1.4)–(1.5) in a one-dimensional domain $\mathcal{I}=(0,1)$ as $\varepsilon\to 0$, reading as

$$\begin{cases} u_t=u_{xx}-(uv_x)_x\,, & x\in\mathcal{I},\ t>0, \\ v_t=\varepsilon v_{xx}-uv, & x\in\mathcal{I},\ t>0, \\ (u,v)(x,0)=(u_0,v_0)(x), & x\in\overline{\mathcal{I}}, \end{cases} \tag{1.7}$$

with boundary conditions

$$\begin{cases} (u_x-uv_x)|_{\partial\mathcal{I}}=0, \quad v|_{\partial\mathcal{I}}=v_* & \text{if } \varepsilon>0, \\ (u_x-uv_x)|_{\partial\mathcal{I}}=0 & \text{if } \varepsilon=0, \end{cases} \tag{1.8}$$

where $\overline{\mathcal{I}}=[0,1]$ and $\partial\mathcal{I}=\{0,1\}$.

The zero-diffusion limit of problem (1.7)–(1.8) as $\varepsilon \to 0$ is a multiscale problem involving sophisticated formal and rigorous analysis with complex compatibility conditions. In this paper, we shall prove that the solution of (1.7)–(1.8) is not uniformly convergent in L^∞ with respect to $\varepsilon > 0$ but stabilizes to the outer layer profile (solution with $\varepsilon = 0$) plus an inner (boundary) layer profile as $\varepsilon \to 0$ where governing equations for both outer- and inner-layer profiles can be precisely derived. There are two major difficulties encountered in our analysis: (1) how to employ the Dirichlet boundary condition of v to obtain the estimates of v_x in order to gain requisite regularity of solutions for the global well-posedness due to the cross-diffusion structure in the first equation of (1.7); (2) how to derive the uniform-in-ε estimates in order to pass to the limit $\varepsilon \to 0$. To overcome the former one, with the mass conservation of u resulting from the zero-flux boundary condition, we make a change of variable (see (2.1)) based on the technique of taking the antiderivative as used in our previous works [4, 25] to reformulate (1.7)–(1.8) into a new Dirichlet problem (2.2)–(2.3) without cross-diffusion structure, for which the Dirichlet boundary condition on v can contribute to derive desired estimates. In doing so, we pay a price by requiring $\inf_{x\in\overline{\mathcal{I}}} u_0(x) = 0$ (i.e., the initial value is degenerate) in the compatibility conditions for the reformulated problem, which leads to the failure of the cancellation technique used in the existing work [25] dealing with the reformulated problem. In this paper, we shall develop a new idea with the help of the Hardy inequality to derive requisite uniform-in-ε estimates and finally prove our main results. However, our results cannot cover the case $\inf_{x\in\overline{\mathcal{I}}} u_0 > 0$ for which initial layers will be present (see Remark 2.1), and new ideas are needed to overcome this barrier. This case will be investigated in a separate work. We stress that the zero-flux boundary condition of u given in (1.8) cannot extrapolate the boundary profile of u. While showing that the solution component v has boundary layer profiles as expected, we also prove that u has boundary layer profiles as $\varepsilon \to 0$ (see Theorem 2.2). As far as we know, this is the first result showing that the time-dependent chemotaxis models with physical boundary conditions in (1.5) have boundary layer profiles for both cell density and oxygen concentration. Our results hence assert that the chemotaxis-fluid model (1.2) is capable of generating boundary layer profiles in one dimension, though the higher dimensional case is yet to be proved. Since the technique of taking the antiderivative is not directly applicable in multidimensions, the boundary layer problem of (1.2)–(1.3) or (1.4)–(1.5) in multidimensions has to be left out for future efforts with new ideas and techniques.

Apart from the boundary layer problem, when Ω is a radially symmetric domain in $\mathbb{R}^n (n \geq 2)$, the existence of global classical solutions of (1.4)–(1.5) with $\varepsilon > 0$ in two dimension ($n = 2$) and global weak solutions in higher dimensions ($n = 3, 4, 5$) were established in [33]. In the case when u and v satisfy zero-flux and Robin boundary conditions, respectively, the global classical solutions of (1.4) were obtained in [2] for any $n \geq 1$, and the existence of boundary layer solutions as $\varepsilon \to 0$ was established recently in [26]. With homogeneous Neumann boundary conditions, the global dynamics of (1.4) have been well understood (cf. [17, 46, 47]) by employing a clever cancelling idea which is, unfortunately, inherently restricted to Neumann boundary conditions. For the time-dependent problem (1.4)–(1.5), aside from the local stability of stationary solutions shown in [25], a slightly modified model of (1.4) subject to (1.3) was recently considered in [52], where the global generalized (weak) solution was obtained in a three-dimensional domain ($n = 3$). In the case when homogeneous Neumann boundary conditions for u and v and Dirichlet boundary conditions for $\mathbf{w}$ are imposed or the domain is the whole space $\mathbb{R}^n (n \geq 1)$, the chemotaxis-fluid model

(1.2) and its variants have been widely studied in the literature [6, 14, 15, 40, 54]; we mention only a few due to limited spaces.

The rest of the paper is organized as follows: In section 2, we first reformulate our problem by taking the antiderivative of a perturbed function against the cell mass and derive the equations for the outer- and boundary (or inner-) layer profiles. Then we state our main results on the convergence of boundary layer solutions. In section 3, we are devoted to deriving the regularity of outer and boundary layer profiles. Finally, in section 4, we prove our main results.

2. Statement of main results. In this section, we shall first derive the equations that outer-layer and boundary layer profiles satisfy by the Wentzel-Kramers-Brillouin (WKB) method (cf. [22, 24, 45]), and then we state our main results on the convergence of boundary layers as $\varepsilon \to 0$. For clarity, we first introduce some notation used throughout the paper.

Notation.

- Denote $\mathbb{R}_+ := (0,\infty)$ and $\mathbb{R}_- := (-\infty,0)$. $\mathbb{N}$ represents the set of nonnegative integers. Let L^p with $1 \le p \le \infty$ denote the Lebesgue space $L^p(\mathcal{I})$ in which functions are defined with respect to (w.r.t.) the variable $x \in (0,1)$. L^p_z denotes the space $L^p(0,\infty)$ for functions defined w.r.t. $z \in (0,\infty)$, and L^p_ξ denotes $L^p(-\infty,0)$ for functions defined w.r.t. $\xi \in (-\infty,0)$, respectively. Accordingly, we denote by H^k, H^k_z, and H^k_ξ the standard Sobolev spaces $W^{k,2}$ for functions defined w.r.t. $x \in \mathcal{I}$, $z \in (0,\infty)$, and $\xi \in (-\infty,0)$, respectively. We also write $L^p_T Y := L^p(0,T;Y)$ (e.g., $L^\infty_T L^\infty_z := L^\infty(0,T;L^\infty_z)$) for convenience when no confusion is caused.
- Denote $\langle z\rangle = \sqrt{1+z^2}$ for $z \in [0,\infty)$ and $\langle \xi\rangle = \sqrt{1+\xi^2}$ for $\xi \in (-\infty,0]$.
- $C(T) > 0$ represents a generic constant depending on T but independent of v_* such that $C(T) \to 0$ as $T \to 0^+$. $C(v_*,T)$ denotes a generic positive constant depending on v_* and T such that $C(v_*,T) \to 0$ as $(v_*,T) \to (0,0)$ and $C(v_*,T) \to +\infty$ as $v_* \to +\infty$ or $T \to +\infty$. Moreover we denote $c(T) := c_0 + C(T)$ and $c(v_*,T) := c_0 + C(v_*,T)$, where $c_0 > 0$ denotes a generic constant independent of v_* and T.
- We often use $(*)_i$ to denote the ith equation of the system $(*)$ for brevity.

2.1. Construction of outer- and boundary (inner-) layer profiles. In this subsection, we shall first reformulate our target system (1.7)–(1.8) and then derive the equations for the outer-layer and boundary layer profiles of the reformulated problem with small $\varepsilon > 0$ based on the WKB method. Notice that the zero-flux boundary condition for u gives rise to the conservation of mass

$$\int_{\mathcal{I}} u(x,t)\mathrm{d}x = \int_{\mathcal{I}} u_0 \mathrm{d}x =: M,$$

where the constant $M > 0$ denotes the cell total mass. By defining

$$\varphi(x,t) = \int_0^x (u(y,t) - M)\mathrm{d}y \quad \text{with} \quad \varphi(x,0) = \int_0^x (u_0(y) - M)\mathrm{d}y =: \varphi_0(x), \tag{2.1}$$

we reformulate the problem (1.7)–(1.8) as

$$\begin{cases} \varphi_t = \varphi_{xx} - (\varphi_x + M)v_x, & x \in \mathcal{I}, \\ v_t = \varepsilon v_{xx} - (\varphi_x + M)v, & x \in \mathcal{I}, \\ (\varphi, v)(x,0) = (\varphi_0, v_0), \end{cases} \tag{2.2}$$

subject to boundary conditions

$$\begin{cases} \varphi(0,t)=\varphi(1,t)=0, \quad v(0,t)=v(1,t)=v_* & \text{if } \varepsilon>0, \\ \varphi(0,t)=\varphi(1,t)=0 & \text{if } \varepsilon=0. \end{cases} \tag{2.3}$$

We proceed to derive the equations for outer-layer and boundary layer profiles of the problem (2.2)–(2.3) with small $\varepsilon > 0$. As we will see later, once these profiles are determined, one can easily recover outer-layer and boundary layer profiles of the original problem (1.7)–(1.8). To this end, we define the so-called boundary layer coordinates

$$z=\frac{x}{\sqrt{\varepsilon}}, \quad \xi=\frac{x-1}{\sqrt{\varepsilon}}, \quad x\in[0,1], \tag{2.4}$$

where $\sqrt{\varepsilon}$ is the thickness of boundary layers which can be determined by the asymptotic matching method (cf. [24, 27]). Clearly $z\in[0,\infty)$ and $\xi\in(-\infty,0]$. The equations governing outer-layer and boundary layer profiles of (2.2) can be derived in four successive steps.

Step 1. Asymptotic expansions. By the method of perturbation (cf. [22, 24, 45]), we assume that the solution of problem (2.2)–(2.3) with $\varepsilon>0$ formally has the following expansions for $j\in\mathbb{N}$:

$$\begin{cases} \varphi^\varepsilon(x,t)=\sum_{j=0}^{\infty}\varepsilon^{\frac{j}{2}}\left(\varphi^{I,j}(x,t)+\varphi^{B,j}(z,t)+\varphi^{b,j}(\xi,t)\right), \\ v^\varepsilon(x,t)=\sum_{j=0}^{\infty}\varepsilon^{\frac{j}{2}}\left(v^{I,j}(x,t)+v^{B,j}(z,t)+v^{b,j}(\xi,t)\right), \end{cases} \tag{2.5}$$

where the boundary layer profiles $(\varphi^{B,j},v^{B,j})$ and $(\varphi^{b,j},v^{b,j})$ are smooth and satisfy the following asymptotic behavior for $j\geq 0$:

$$\begin{cases} \varphi^{B,j} \text{ and } v^{B,j} \text{ decay to zero exponentially as } z\to\infty, \\ \varphi^{b,j} \text{ and } v^{b,j} \text{ decay to zero exponentially as } \xi\to-\infty. \end{cases} \tag{2.6}$$

Step 2. Initial and boundary conditions. For initial conditions, setting $t=0$ in (2.5) and noticing that the initial value (φ_0,v_0) is independent of $\varepsilon>0$, we immediately get

$$\begin{cases} \varphi^{I,0}(x,0)=\varphi_0(x), \quad \varphi^{B,0}(z,0)=\varphi^{b,0}(\xi,0)=0, \\ v^{I,0}(x,0)=v_0(x), \quad v^{B,0}(z,0)=v^{b,0}(\xi,0)=0 \end{cases}$$

and

$$\begin{cases} \varphi^{I,j}(x,0)=\varphi^{B,j}(z,0)=\varphi^{b,j}(\xi,0)=0, \quad j\geq 1, \\ v^{I,j}(x,0)=v^{B,j}(z,0)=v^{b,j}(\xi,0)=0, \quad j\geq 1. \end{cases}$$

To match boundary conditions, we substitute (2.5) into (2.3) and use the asymptotic matching method to get

$$\begin{cases} \varphi^{I,j}(0,t)+\varphi^{B,j}(0,t)=0, \quad \varphi^{I,j}(1,t)+\varphi^{b,j}(0,t)=0, \quad j\geq 0, \\ v^{I,0}(0,t)+v^{B,0}(0,t)=v_*, \quad v^{I,0}(1,t)+v^{b,0}(0,t)=v_*, \\ v^{I,j}(0,t)+v^{B,j}(0,t)=0, \quad v^{I,j}(1,t)+v^{b,j}(0,t)=0, \quad j\geq 1, \end{cases}$$

where we have neglected $(\varphi^{b,j}(-\frac{1}{\varepsilon^{1/2}},t), v^{b,j}(-\frac{1}{\varepsilon^{1/2}},t))$ at $x=0$ and $(\varphi^{B,j}(\frac{1}{\varepsilon^{1/2}},t), v^{B,j}(\frac{1}{\varepsilon^{1/2}},t))$ at $x=1$ based on the decay properties in (2.6) since $\varepsilon>0$ is small.

Step 3. Equations for outer-layer profiles $(\varphi^{I,j}, v^{I,j})$. Substituting (2.5) without boundary layer profiles into the equations in (2.2), we get equations for the outer-layer profiles $\varphi^{I,j}$,

$$\varphi_t^{I,j} = \varphi_{xx}^{I,j} - M v_x^{I,j} - \sum_{k=0}^{j} \varphi_x^{I,k} v_x^{I,j-k}, \quad j\geq 0, \tag{2.7}$$

and equations for the outer-layer profiles $v^{I,j}$,

$$\begin{cases} v_t^{I,0} = -(\varphi_x^{I,0}+M)v^{I,0}, & j=0,\\ v_t^{I,1} = -(\varphi_x^{I,0}+M)v^{I,1} - \varphi_x^{I,1}v^{I,0}, & j=1,\\ v_t^{I,j} = v_{xx}^{I,j-2} - Mv^{I,j} - \displaystyle\sum_{k=0}^{j}\varphi_x^{I,k}v^{I,j-k}, & j\geq 2. \end{cases}$$

Step 4. Equations for boundary layer profiles $(\varphi^{B,j}, \varphi^{b,j}, v^{B,j}, v^{b,j})$. Using (2.7), we neglect the right boundary layer profiles $\varphi^{b,j}$ and $v^{b,j}$, and then insert the remaining terms of (2.5) into the first equation in (2.2) to derive the equations for the left boundary layer profiles $\varphi^{B,j}$,

$$\sum_{i\geq -2} \varepsilon^{\frac{i}{2}} G_i = 0 \quad \text{for} \quad i\geq -2, \tag{2.8}$$

where

$$\begin{cases} G_{-2} := \varphi_{zz}^{B,0} - \varphi_z^{B,0}v_z^{B,0},\\ G_{-1} := \varphi_{zz}^{B,1} - (\partial_x\varphi^{I,0}(0,t)+M)v_z^{B,0} - \varphi_z^{B,1}v_z^{B,0} - \varphi_z^{B,0}(v_x^{I,0}(0,t)+v_z^{B,1}),\\ G_0 := \varphi_t^{B,0} - \varphi_{zz}^{B,2} + v_z^{B,0}(\partial_x^2\varphi^{I,0}(0,t)z + \partial_x\varphi^{I,1}(0,t) + \varphi_z^{B,2})\\ \qquad + v_z^{B,1}(\partial_x\varphi^{I,0}(0,t)+M+\varphi_z^{B,1}) + \varphi_z^{B,1}\partial_x v^{I,0}(0,t) + (\partial_x^2 v^{I,0}(0,t)z\\ \qquad + \partial_x v^{I,1}(0,t) + v_z^{B,2})\varphi_z^{B,0},\\ G_1 := \cdots\\ \cdots\ \cdots. \end{cases}$$

Similarly, the right boundary layer profiles $\varphi^{b,j}$ satisfy

$$\sum_{i\geq -2} \varepsilon^{\frac{i}{2}} \tilde{G}_i = 0 \quad \text{for} \quad i\geq -2, \tag{2.9}$$

where, for each $i\geq -2$, $\tilde{G}_i$ is given by G_i with $(\partial_x^{\ell+1}\varphi^{I,k}(0,t), \partial_x^{\ell}v^{I,k}(0,t))$ $(\ell\geq 0)$ and $(\partial_z^l\varphi^{B,k}, \partial_z^{\lambda}v^{B,k})$ replaced by $(\partial_x^{\ell+1}\varphi^{I,k}(1,t), \partial_x^{\ell}v^{I,k}(1,t))$ $(\ell\geq 0)$ and $(\partial_\xi^l\varphi^{b,k}, \partial_\xi^{\lambda}v^{b,k})$ $(l,\lambda\geq 0)$, respectively.

By the same procedure used to derive the equations for $\varphi^{B,j}$ and $\varphi^{b,j}$ above, we obtain the equations for the left boundary layer profiles $v^{B,j}$ as

$$\begin{cases} \varphi_z^{B,0}(v^{B,0}+v^{I,0}(0,t)) = 0,\\ v_t^{B,0} - v_{zz}^{B,0} + \varphi_z^{B,0}(v_x^{I,0}(0,t)z + v^{I,1}(0,t) + v^{B,1})\\ \qquad +(\varphi_x^{I,0}(0,t)+M)v^{B,0} + \varphi_z^{B,1}(v^{B,0}+v^{I,0}(0,t)) = 0,\\ v_t^{B,1} - v_{zz}^{B,1} + (\varphi_x^{I,0}(0,t)+M)v^{B,1} + \varphi_z^{B,1}(v_x^{I,0}(0,t)z + v^{I,1}(0,t) + v^{B,1})\\ \qquad +(\varphi_{xx}^{I,0}(0,t)z + \varphi_x^{I,1}(0,t))v^{B,0} + \varphi_z^{B,2}(v^{I,0}(0,t)+v^{B,0})\\ \qquad +\varphi_z^{B,0}(\frac{z^2}{2}v_{xx}^{I,0}(0,t) + v_x^{I,1}(0,t)z + v^{B,2} + v^{I,2}(0,t)) = 0,\\ \cdots\ \cdots, \end{cases}$$

and the equations for the right boundary layer profiles $v^{b,j}$ as

$$\begin{cases}\varphi_\xi^{b,0}(v^{b,0}+v^{I,0}(1,t))=0,\\ v_t^{b,0}-v_{\xi\xi}^{b,0}+\varphi_\xi^{b,0}(v_x^{I,0}(1,t)\xi+v^{I,1}(1,t)+v^{b,1})\\ \qquad +(\varphi_x^{I,0}(1,t)+M)v^{b,0}+\varphi_\xi^{b,1}(v^{b,0}+v^{I,0}(1,t))=0,\\ v_t^{b,1}-v_{\xi\xi}^{b,1}+(\varphi_x^{I,0}(1,t)+M)v^{b,1}+\varphi_\xi^{b,1}(v_x^{I,0}(1,t)\xi+v^{I,1}(1,t)+v^{b,1})\\ \qquad +(\varphi_{xx}^{I,0}(1,t)\xi+\varphi_x^{I,1}(1,t))v^{b,0}+\varphi_\xi^{b,2}(v^{I,0}(1,t)+v^{b,0})\\ \qquad +\varphi_z^{b,0}(\frac{z^2}{2}v_{xx}^{I,0}(1,t)+v_x^{I,1}(1,t)z+v^{b,2}+v^{I,2}(1,t))=0,\\ \cdots\ \cdots.\end{cases}$$

Finally, from the above Steps 1–4, we derive initial-boundary value problems satisfied by the profiles $(\varphi^{I,j},\varphi^{B,j},\varphi^{b,j})\,(0\le j\le 2)$ and $(v^{I,j},v^{B,j},v^{b,j})\,(0\le j\le 1)$ for later use. First, the leading-order outer-layer profile $(\varphi^{I,0},v^{I,0})$ satisfies the problem

$$\begin{cases}\varphi_t^{I,0}=\varphi_{xx}^{I,0}-(\varphi_x^{I,0}+M)v_x^{I,0}, & x\in\mathcal{I},t>0,\\ v_t^{I,0}=-(\varphi_x^{I,0}+M)v^{I,0}, & x\in\mathcal{I},t>0,\\ \varphi^{I,0}(0,t)=\varphi^{I,0}(1,t)=0,\\ (\varphi^{I,0},v^{I,0})(x,0)=(\varphi_0,v_0),\end{cases}\tag{2.10}$$

which is nothing but the zero-diffusion problem of (2.2). We note that the stability of the unique nonconstant steady state to the problem (2.10) has been established in our previous work [25]. We further remark that, as will be stated in section 3, if the initial value is compatible with boundary conditions and smooth enough, one can prove the global existence of unique classical solutions to (2.10) with large initial data due to the dissipation effect. The first-order outer-layer profile $(\varphi^{I,1},v^{I,1})$ satisfies the following problem:

$$\begin{cases}\varphi_t^{I,1}=\varphi_{xx}^{I,1}-(\varphi_x^{I,0}+M)v_x^{I,1}-\varphi_x^{I,1}v_x^{I,0}, & x\in\mathcal{I},t>0,\\ v_t^{I,1}=-(\varphi_x^{I,0}+M)v^{I,1}-\varphi_x^{I,1}v^{I,0}, & x\in\mathcal{I},t>0,\\ \varphi^{I,1}(0,t)=-\varphi^{B,1}(0,t),\quad \varphi^{I,1}(1,t)=-\varphi^{b,1}(0,t),\\ (\varphi^{I,1},v^{I,1})(x,0)=(0,0).\end{cases}\tag{2.11}$$

The leading-order boundary layer profile $\varphi^{B,0}$ near the left boundary solves

$$\begin{cases}\varphi_{zz}^{B,0}-\varphi_z^{B,0}v_z^{B,0}=0, & z\in\mathbb{R}_+,\\ \varphi^{B,0}(0,t)=0,\quad \varphi^{B,0}(+\infty,t)=0,\\ \varphi^{B,0}(z,0)=0,\end{cases}$$

and thus $\varphi^{B,0}\equiv 0$. The boundary layer profile $v^{B,0}$ near the left boundary solves

$$\begin{cases}v_t^{B,0}=v_{zz}^{B,0}-(\varphi_x^{I,0}(0,t)+M)v^{I,0}(0,t)(\mathrm{e}^{v^{B,0}}-1)-(\varphi_x^{I,0}(0,t)+M)\mathrm{e}^{v^{B,0}}v^{B,0}, & z\in\mathbb{R}_+,\\ v^{B,0}(0,t)=v_*-v^{I,0}(0,t),\quad v^{B,0}(+\infty,t)=0,\\ v^{B,0}(z,0)=0,\end{cases}\tag{2.12}$$

and $\varphi^{B,1}$ is determined by $v^{B,0}$ through

$$\varphi^{B,1}=-\int_z^\infty(\varphi_x^{I,0}(0,t)+M)\left(\mathrm{e}^{v^{B,0}(y,t)}-1\right)\mathrm{d}y.\tag{2.13}$$

The boundary layer profile $v^{b,0}$ near the right boundary satisfies

(2.14)
$$\begin{cases} v_t^{b,0} = v_{\xi\xi}^{b,0} - (\varphi_x^{I,0}(1,t) + M)v^{I,0}(1,t)(\mathrm{e}^{v^{b,0}} - 1) - (\varphi_x^{I,0}(1,t) + M)\mathrm{e}^{v^{b,0}} v^{b,0}, & \xi \in \mathbb{R}_-, \\ v^{b,0}(0,t) = v_* - v^{I,0}(1,t), \quad v^{b,0}(-\infty,t) = 0, \\ v^{b,0}(\xi,0) = 0. \end{cases}$$

Furthermore, we have $\varphi^{b,0} \equiv 0$, and $\varphi^{b,1}$ is given by

$$\varphi^{b,1} = \int_{-\infty}^{\xi} (\varphi_x^{I,0}(1,t) + M)\left(\mathrm{e}^{v^{b,0}(y,t)} - 1\right) \mathrm{d}y. \tag{2.15}$$

Although we focus only on the convergence result for leading-order approximation, some estimates of the higher-order outer-layer and boundary layer profiles are also needed in our analysis. The problem formed by equations for $\varphi^{B,2}$ and $v^{B,1}$ reads

(2.16)
$$\begin{cases} -\varphi_{zz}^{B,2} + v_z^{B,0}(\varphi_{xx}^{I,0}(0,t)z + \varphi_x^{I,1}(0,t) + \varphi_z^{B,2}) \\ \qquad + v_z^{B,1}(\varphi_x^{I,0}(0,t) + M + \varphi_z^{B,1}) + \varphi_z^{B,1} v_x^{I,0}(0,t) = 0, & z \in \mathbb{R}_+, \\ v_t^{B,1} - v_{zz}^{B,1} + (\varphi_x^{I,0}(0,t) + M)v^{B,1} + \varphi_z^{B,1}(v_x^{I,0}(0,t)z + v^{I,1}(0,t) + v^{B,1}) \\ \qquad + (\varphi_{xx}^{I,0}(0,t)z + \varphi_x^{I,1}(0,t))v^{B,0} + \varphi_z^{B,2}(v^{I,0}(0,t) + v^{B,0}) = 0, & z \in \mathbb{R}_+, \\ v^{B,1}(0,t) = -v^{I,1}(0,t), \quad \varphi^{B,2}(+\infty,t) = v^{B,1}(+\infty,t) = 0, \\ (\varphi^{B,2}, v^{B,1})(z,0) = (0,0), \end{cases}$$

and the problem for $(\varphi^{b,2}, v^{b,1})$ can be stated as

(2.17)
$$\begin{cases} -\varphi_{\xi\xi}^{b,2} + v_\xi^{b,0}(\varphi_{xx}^{I,0}(1,t)\xi + \varphi_x^{I,1}(1,t) + \varphi_\xi^{b,2}) \\ \qquad + v_\xi^{b,1}(\varphi_x^{I,0}(1,t) + M + \varphi_\xi^{b,1}) + \varphi_\xi^{b,1} v_x^{I,0}(1,t) = 0, & \xi \in \mathbb{R}_-, \\ v_t^{b,1} - v_{\xi\xi}^{b,1} + (\varphi_x^{I,0}(1,t) + M)v^{b,1} + \varphi_\xi^{b,1}(v_x^{I,0}(1,t)\xi + v^{I,1}(1,t) + v^{b,1}) \\ \qquad + (\varphi_{xx}^{I,0}(1,t)\xi + \varphi_x^{I,1}(1,t))v^{b,0} + \varphi_\xi^{b,2}(v^{I,0}(1,t) + v^{b,0}) = 0, & \xi \in \mathbb{R}_-, \\ v^{b,1}(0,t) = -v^{I,1}(1,t), \quad \varphi^{b,2}(-\infty,t) = v^{b,1}(-\infty,t) = 0, \\ (\varphi^{b,2}, v^{b,1})(\xi,0) = (0,0). \end{cases}$$

Finally, we remark that the global existence and regularity of solutions to problems (2.11), (2.12), (2.14), (2.16), and (2.17) will be detailed in section 3.

2.2. Statement of main results. To prove the convergence of boundary layer profiles deduced in the preceding subsection, we require that the initial data (φ_0, v_0) satisfy compatibility conditions at the boundary as follows:

$$\partial_t^i \varphi^{I,0}|_{t=0} = 0, \ i = 1,2,3, \qquad \text{on } \partial\mathcal{I}, \tag{2.18a}$$

$$v_0 = v_*, \ \partial_t^j v^{I,0}|_{t=0} = 0, \ j = 1,2, \qquad \text{on } \partial\mathcal{I}, \tag{2.18b}$$

where $\partial_t^i \varphi^{I,0}|_{t=0}$ and $\partial_t^i v^{I,0}|_{t=0}$ can be inductively determined from the equations in (2.10) as

(2.19)
$$\begin{cases}\partial_t\varphi^{I,0}|_{t=0} := \varphi_{0xx} - (\varphi_{0x}+M)v_{0x},\\ \partial_t^2\varphi^{I,0}|_{t=0} := (\partial_t\varphi^{I,0}|_{t=0})_{xx} + (\varphi_{0x}+M)((\varphi_{0x}+M)v_0)_x - (\partial_t\varphi^{I,0}|_{t=0})_x v_{0x},\\ \partial_t^3\varphi^{I,0}|_{t=0} := (\partial_t^2\varphi^{I,0}|_{t=0})_{xx} - (\partial_t^2\varphi^{I,0}|_{t=0})_x v_{0x} + 2(\partial_t\varphi^{I,0}|_{t=0})_x((\varphi_{0x}+M)v_0)_x\\ \qquad\qquad +(\varphi_{0x}+M)((\partial_t\varphi^{I,0}|_{t=0})_x v_0)_x - (\varphi_{0x}+M)((\varphi_{0x}+M)^2 v_0)_x = 0,\\ \partial_t v^{I,0}|_{t=0} = -(\varphi_{0x}+M)v_0,\\ \partial_t^2 v^{I,0}|_{t=0} = [-\varphi_{0xxx} + ((\varphi_{0x}+M)v_{0x})_x + (\varphi_{0x}+M)^2]v_0.\end{cases}$$

We say that the initial value $\varphi^{I,0}|_{t=0}$ of the problem (2.10) is compatible with boundary conditions up to order three if it fulfills (2.18a), while the initial values of problems (2.12) and (2.14) are compatible with boundary conditions up to order two if the conditions in (2.18b) hold. The compatibility conditions for other initial-boundary value problems mentioned in what follows are defined similarly. In terms of the initial data (φ_0, v_0), we can write the compatibility conditions given by (2.18)–(2.19) more explicitly as

(2.20)
$$\begin{cases} v_0 = v_*,\ \varphi_{0x}+M=0,\ \varphi_{0xx}v_{0x} - \varphi_{0xxx} = 0, & \text{on } \partial\mathcal{I},\\ (\partial_t\varphi^{I,0}|_{t=0})_{xx} - (\partial_t\varphi^{I,0}|_{t=0})_x v_{0x} = 0, & \text{on } \partial\mathcal{I},\\ (\partial_t^2\varphi^{I,0}|_{t=0})_{xx} - (\partial_t^2\varphi^{I,0}|_{t=0})_x v_{0x} + 2(\partial_t\varphi^{I,0}|_{t=0})_x(\varphi_{0x}+M)v_{0x} = 0, & \text{on } \partial\mathcal{I},\end{cases}$$

where for brevity we have not explicitly expressed $\partial_t\varphi^{I,0}|_{t=0}$ and $\partial_t^2\varphi^{I,0}|_{t=0}$ as given in (2.19).

We underline that the condition $(\varphi_{0x}+M)|_{\partial\mathcal{I}} = 0$ in (2.20) implies that $\inf_{x\in\mathcal{I}} u_0 = 0$ (i.e., the initial value u_0 is degenerate on $\bar{\mathcal{I}}$) and hence $\inf_{(x,t)\in\mathcal{I}\times(0,T]} u^{I,0}(x,t) = 0$, where $u^{I,0}$ is the leading outer-layer profile of u satisfying $u^{I,0}(x,0) = u_0(x)$; see (3.1).

The main results of this paper concerning the convergence of boundary layers for the reformulated problem (2.2)–(2.3) as $\varepsilon\to 0$ are stated in the following.

THEOREM 2.1. *Assume that* $(\varphi_0, v_0)\in H^7\times H^7$ *and* $(\sqrt{v_0})_x\in L^2$ *with* $\varphi_{0x}+M\geq,\not\equiv 0$ *satisfying* (2.20). *Then for any* $v_*>0$, *there exist constants* $T_0(v_*)>0$ *and* $\varepsilon_0>0$, *where* $T_0(v_*)\to\infty$ *as* $v_*\to 0$, *such that for any* $\varepsilon\in(0,\varepsilon_0)$, *the problem* (2.2)–(2.3) *admits a unique solution* $(\varphi^\varepsilon, v^\varepsilon)\in L^\infty(0,T_0;H^2\times H^2)$ *satisfying the following asymptotic expansions for any* $x\in[0,1]$:

$$\varphi^\varepsilon(x,t) = \varphi^{I,0}(x,t) + \varepsilon^{1/2}\left[\varphi^{I,1}(x,t) + \varphi^{B,1}(z,t) + \varphi^{b,1}(\xi,t)\right] + O(\varepsilon^{5/8}), \tag{2.21a}$$

$$\varphi_x^\varepsilon(x,t) = \varphi_x^{I,0}(x,t) + \left[\varphi_z^{B,1}(z,t) + \varphi_\xi^{b,1}(\xi,t)\right] + O(\varepsilon^{1/4}), \tag{2.21b}$$

$$v^\varepsilon(x,t) = v^{I,0}(x,t) + v^{B,0}(z,t) + v^{b,0}(\xi,t) + O(\varepsilon^{1/2}), \tag{2.21c}$$

with $z := \frac{x}{\varepsilon^{1/2}}$ *and* $\xi := \frac{x-1}{\varepsilon^{1/2}}$, *where* $(\varphi^{I,0}, v^{I,0})$, $v^{B,0}$, *and* $v^{b,0}$ *are solutions of problems* (2.10), (2.12),*and* (2.14), *respectively;* $\varphi^{I,1}$ *is determined by* (2.11); *and* $\varphi^{B,1}$ *and* $\varphi^{b,1}$ *are given by* (2.13) *and* (2.15), *respectively.*

With the transformation (2.1), we can transfer the results of (2.2)–(2.3) stated in Theorem 2.1 to the original problem (1.7)–(1.8). Indeed, from (2.1) we have

$$u^\varepsilon = \varphi_x^\varepsilon + M,\quad u^{I,0} = \varphi_x^{I,0} + M \tag{2.22}$$

with φ^ε and $\varphi^{I,0}$ being the solutions to the problem (2.2)–(2.3) and the problem (2.10), respectively. Then $(u^\varepsilon, v^\varepsilon)$ and $(u^{I,0}, v^{I,0})$ solve the problem (1.7)–(1.8) for $\varepsilon>0$ and $\varepsilon=0$, respectively. With (2.13) and (2.15), we have

$$u^{B,0}(z,t) = \varphi_z^{B,1}(z,t) = (\varphi_x^{I,0}(0,t)+M)(\mathrm{e}^{v^{B,0}(z,t)}-1),$$
$$u^{b,0}(\xi,t) = \varphi_\xi^{b,1}(\xi,t) = (\varphi_x^{I,0}(1,t)+M)(\mathrm{e}^{v^{b,0}(\xi,t)}-1). \tag{2.23}$$

Then the convergence of boundary layer solutions of the original problem (1.7)–(1.8) is stated in the following theorem.

THEOREM 2.2. *Assume that* $(u_0, v_0) \in H^6 \times H^7$ *with* $u_0 \geq, \not\equiv 0, v_0 \geq 0,$ *and* $(\sqrt{v_0})_x \in L^2$ *satisfying the compatibility conditions* (2.20) *with* $\varphi_{0x} = u_0 - M$. *Then for any* $v_* > 0$, *there exists constants* $T_0(v_*) > 0$ *and* $\varepsilon_0 > 0$, *where* $T_0(v_*) \to \infty$ *as* $v_* \to 0$, *such that for any* $\varepsilon \in (0, \varepsilon_0)$, *the problem* (1.7)–(1.8) *admits a unique solution* $(u^\varepsilon, v^\varepsilon) \in L^\infty(0, T_0; H^1 \times H^2)$ *which satisfies, for any* $x \in [0,1]$,

$$u^\varepsilon(x,t) = u^{I,0}(x,t) + u^{B,0}\Big(\frac{x}{\sqrt{\varepsilon}},t\Big) + u^{b,0}\Big(\frac{1-x}{\sqrt{\varepsilon}},t\Big) + O(\varepsilon^{1/4}),$$
$$v^\varepsilon(x,t) = v^{I,0}(x,t) + v^{B,0}\Big(\frac{x}{\sqrt{\varepsilon}},t\Big) + v^{b,0}\Big(\frac{1-x}{\sqrt{\varepsilon}},t\Big) + O(\varepsilon^{1/2}), \tag{2.24}$$

where $u^{I,0}$ *and* $(u^{B,0}, b^{b,0})$ *are given as in* (2.22) *and* (2.23), *respectively, while* $(\varphi^{I,0}, v^{I,0})$, $v^{B,0}$, *and* $v^{b,0}$ *are solutions of problems* (2.10), (2.12), *and* (2.14), *respectively.*

Remark 2.1. We give several remarks to enhance the understanding of our results.

- The $O(\varepsilon^r)$, for some $r > 0$, notation used in the main results is a shortcut whose exact meaning is that the difference of the two sides of the identities (2.21) and (2.24) in $L_T^\infty L_x^\infty$, for any $0 < T < T_0$, is bounded by ε^r modulo a constant depending only on the initial data and v_*.
- The conditions of (u_0, v_0) assumed in Theorem 2.2 can be fulfilled by many functions, for instance, $u_0 = x^8(1-x)^8$ and $v_0 = v_* + x^6(1-x)^6$. Furthermore, if (u_0, v_0) satisfies some higher-order compatibility conditions, by the standard energy method (cf. [16, Chap. 7]), one can prove that the solutions $(\varphi^\varepsilon, v^\varepsilon)$ and $(u^\varepsilon, v^\varepsilon)$ obtained in Theorem 2.1 and Theorem 2.2 are indeed classical. We skip the details here since this is not the main goal of this paper.
- From the refined solution structure given in (2.24), without difficulty we can show, for any $\delta = O(\varepsilon^\alpha) > 0$ $(0 < \alpha < 1/2)$, it holds that

$$\lim_{\varepsilon\to 0} \left\|u^\varepsilon - u^{I,0}\right\|_{L^\infty([\delta,1-\delta]\times[0,T_0])} = 0, \ \liminf_{\varepsilon\to 0} \left\|u^\varepsilon - u^{I,0}\right\|_{L^\infty([0,1]\times[0,T_0])} > 0,$$
$$\lim_{\varepsilon\to 0} \left\|v^\varepsilon - v^{I,0}\right\|_{L^\infty([\delta,1-\delta]\times[0,T_0])} = 0, \ \liminf_{\varepsilon\to 0} \left\|v^\varepsilon - v^{I,0}\right\|_{L^\infty([0,1]\times[0,T_0])} > 0,$$

  which indicates that the solution $(u^\varepsilon, v^\varepsilon)$ of (1.7)–(1.8) will develop a boundary layer profile with thickness of order $\varepsilon^{1/2}$ as $\varepsilon \to 0$, which consists of outer-layer profile $(u^{I,0}, v^{I,0})$ (i.e., the solution of (1.7)–(1.8) with $\varepsilon = 0$) and boundary (inner-) layer profiles $(u^{B,0}, v^{B,0})$ at the left boundary $x = 0$ and $(u^{b,0}, v^{b,0})$ at the right boundary $x = 1$, with an error at the order of $\varepsilon^{1/4}$ for u^ε and of $\varepsilon^{1/2}$ for v^ε as $\varepsilon \to 0$.
- Though the boundary values of u^ε are elusive in the zero-flux boundary condition of u prescribed for u in (1.8), the expansion (2.24) not only indicates that $u^\varepsilon(x,t)$ has boundary layer profiles $u^{B,0}(z,t)$ near $x = 0$ and $u^{b,0}(z,t)$ near $x = 1$, but also gives the approximate boundary value of u for $0 < \varepsilon \ll 1$,

$$u^\varepsilon(0,t) = u^{I,0}(0,t)\exp\big(v_* - v^{I,0}(0,t)\big) + O(\varepsilon^{1/4}),$$
$$u^\varepsilon(1,t) = u^{I,0}(1,t)\exp\big(v_* - v^{I,0}(1,t)\big) + O(\varepsilon^{1/2}),$$

  where $u^{I,0}(x,t) = \varphi_x^{I,0} + M$; see (2.23).

- When $v_* = 0$, according to our analysis, the boundary layer profiles in (2.24) will vanish, which leads to $(u^\varepsilon, v^\varepsilon) \to (u^{I,0}, v^{I,0})$ in L^∞ as $\varepsilon \to 0$, where $(u^{I,0}, v^{I,0})$ is the solution of the problem (1.7)–(1.8) with $\varepsilon = 0$.
- The compatibility condition $(\varphi_{0x} + M)|_{\partial\mathcal{I}} = 0$ implies $\min_{x\in\bar{\mathcal{I}}} u_0 = 0$. If we assume $\min_{x\in\bar{\mathcal{I}}} u_0 > 0$, by the maximum principle we can find some constant $c > 0$ which may depend on T_0 such that $0 < c^{-1} \le u^{I,0}(x,t) \le c$ for any $t \in [0, T_0]$ and $x \in (0,1)$. In this case the condition $(\varphi_{0x} + M)|_{\partial\mathcal{I}} = 0$ in (2.20) will fail, and consequently the initial values of (2.12) and (2.14) only satisfy the zero-order compatibility conditions, for which initial layers will be present and the key analyses in this paper are inapplicable. We shall investigate this case in a separate paper using different approaches.

3. Regularity of the outer-/boundary layer profiles. In this section, we shall derive the regularity of solutions to problems (2.10), (2.11), (2.12), (2.14), (2.16), and (2.17), respectively. Let us begin with the problem (2.10) for the leading-order outer-layer profile $(\varphi^{I,0}, v^{I,0})$. As mentioned before, this problem is exactly the zero-diffusion problem of (2.2) which, in the sense of classical solutions, is equivalent to the zero-diffusion problem of (1.7)–(1.8). Denote by $(u^{I,0}, v^{I,0})$ the solution to the zero-diffusion problem of (1.7)–(1.8). Then we have

$$
\begin{cases}
u_t^{I,0} = \left(u_x^{I,0} - u^{I,0} v_x^{I,0}\right)_x, & x \in \mathcal{I},\\
v_t^{I,0} = -u^{I,0} v^{I,0}, & x \in \mathcal{I},\\
(u_x^{I,0} - u^{I,0} v_x^{I,0})|_{\partial\mathcal{I}} = 0,\\
(u^{I,0}, v^{I,0})(x,0) = (u_0, v_0)(x).
\end{cases} \tag{3.1}
$$

We will first establish the global existence of solutions to the problem (3.1), and then transfer the result to problem (2.10).

LEMMA 3.1. *Assume that* $(u_0, v_0) \in H^6 \times H^7$ *with* $u_0 \ge, \not\equiv 0, v_0 \ge 0$, *and* $(\sqrt{v_0})_x \in L^2$ *subject to compatibility conditions in* (2.20) *with* $\varphi_0 = \int_0^x (u_0 - M)\mathrm{d}y$ *and* $M = \int_{\mathcal{I}} u_0 \mathrm{d}x$. *Then for any* $T > 0$, *the problem* (3.1) *admits a unique classical solution on* $[0, T]$ *such that*

$$u^{I,0} \ge 0, \quad \|\partial_t^k u^{I,0}\|_{L_T^2 H^{7-2k}} \le c(T), \quad k = 0,1,2,3,4, \tag{3.2a}$$

$$\|v^{I,0}\|_{L_T^\infty H^7} + \|\partial_t^k v^{I,0}\|_{L_T^2 H^{9-2k}} \le c(T), \quad k = 1,2,3,4. \tag{3.2b}$$

Proof. The proof of local existence and uniqueness of classical solutions to the problem (3.1) is standard based on Banach's fixed point theorem (cf. Chapter 9 in [16]), so is the property $u^{I,0} \ge 0$ in its lifespan if $u_0 \ge 0$ (cf. [17]). In the following, we are devoted to deriving the a priori estimates of solutions by which the local solutions can be extended to global ones. To begin with, for any $T > 0$, we assume that $(u^{I,0}, v^{I,0})$ is a classical solution to the problem (3.1) on $[0, T]$ satisfying the following a priori assumption:

$$\int_0^t \|v_x^{I,0}\|_{L^\infty}^2 \mathrm{d}\tau \le C_1, \quad t \in [0, T], \tag{3.3}$$

for some constant $C_1 > 0$ to be determined later. Testing the equation $(3.1)_1$ against $u_-^{I,0} := -\max\{-u^{I,0}, 0\}$, we get

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{I}}|u_-^{I,0}|^2\mathrm{d}x+\int_{\mathcal{I}}|(u_-^{I,0})_x|^2\mathrm{d}x&=\int_{\{u^{I,0}<0\}}u^{I,0}v_x^{I,0}u_x^{I,0}\mathrm{d}x\\&\le\frac{1}{2}\int_{\mathcal{I}}|(u_-^{I,0})_x|^2\mathrm{d}x+c_0\|v_x^{I,0}\|_{L^\infty}^2\int_{\mathcal{I}}|u_-^{I,0}|^2\mathrm{d}x,\end{aligned}$$

where the Cauchy–Schwarz inequality has been used, and the constant $c_0>0$ is independent of C_1. This, along with (3.3) and the Gronwall inequality, gives

$$\int_{\mathcal{I}}|u_-^{I,0}|^2\mathrm{d}x\le \mathrm{e}^{C_1 t}\int_{\{u_0<0\}}|u_0|^2\mathrm{d}x=0$$

for any $t\in(0,T]$, where $u_0\ge 0$ has been used. Therefore it holds that

$$u^{I,0}(x,t)\ge 0,\quad t\in(0,T]. \tag{3.4}$$

With (3.4), we have from $(3.1)_2$ that $v^{I,0}\le v_0$. Testing $(3.1)_1$ against $\ln u^{I,0}$, one has

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{I}}u^{I,0}\ln u^{I,0}\mathrm{d}x+\int_{\mathcal{I}}\frac{|u_x^{I,0}|^2}{u^{I,0}}\mathrm{d}x=\int_{\mathcal{I}}u_x^{I,0}v_x^{I,0}\mathrm{d}x, \tag{3.5}$$

where $\int_{\mathcal{I}}u^{I,0}\mathrm{d}x=\int_{\mathcal{I}}u_0\mathrm{d}x=M$ due to the zero-flux boundary condition. Differentiating $(3.1)_2$ with respect to x, and testing the resulting equation against $v_x^{I,0}/v^{I,0}$, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{I}}\frac{|v_x^{I,0}|^2}{v^{I,0}}\mathrm{d}x+\frac{1}{2}\int_{\mathcal{I}}\frac{u^{I,0}|v_x^{I,0}|^2}{v^{I,0}}\mathrm{d}x=-\int_{\mathcal{I}}u_x^{I,0}v_x^{I,0}\mathrm{d}x. \tag{3.6}$$

Combining (3.5) with (3.6), and integrating the resulting identity over $[0,t]$ for any $t\in(0,T]$, we have

$$\int_{\mathcal{I}}u^{I,0}\ln u^{I,0}\mathrm{d}x+\frac{1}{2}\int_{\mathcal{I}}\frac{|v_x^{I,0}|^2}{v^{I,0}}\mathrm{d}x+\int_0^t\int_{\mathcal{I}}\left(\frac{|u_x^{I,0}|^2}{u^{I,0}}+\frac{1}{2}\frac{u^{I,0}|v_x^{I,0}|^2}{v^{I,0}}\right)\mathrm{d}x\mathrm{d}\tau\le c_0,$$

which, along with the basic inequality $-x\ln x\le \mathrm{e}^{-1}$ for $x\ge 0$, and $v^{I,0}\le v_0$, gives

$$\frac{1}{2}\int_{\mathcal{I}}|v_x^{I,0}|^2\mathrm{d}x+\int_0^t\int_{\mathcal{I}}\left(\frac{|u_x^{I,0}|^2}{u^{I,0}}+\frac{1}{2}\frac{u^{I,0}|v_x^{I,0}|^2}{v^{I,0}}\right)\mathrm{d}x\mathrm{d}\tau\le c_0 \tag{3.7}$$

for any $t\in[0,T]$, where the constant $c_0>0$ is independent of C_1. Furthermore, it holds from (3.4), (3.8), the basic inequality $\|f\|_{L^\infty}\le c_0\|f\|_{W^{1,1}}$, and the Hölder inequality that

$$\begin{aligned}\int_0^T\|u^{I,0}\|_{L^\infty}\mathrm{d}\tau&\le c_0\int_0^T\|u_x^{I,0}\|_{L^1}\mathrm{d}\tau+c_0\int_0^T\|u^{I,0}\|_{L^1}\mathrm{d}\tau\\&\le\int_0^T\left(\int_{\mathcal{I}}\frac{|u_x^{I,0}|^2}{u^{I,0}}\mathrm{d}x\right)^{1/2}\left(\int_{\mathcal{I}}u^{I,0}\mathrm{d}x\right)^{1/2}\mathrm{d}\tau+c(T)\le c(T),\end{aligned} \tag{3.8}$$

where $c(T)>0$ is as stated in section 2, and it is independent of C_1. To proceed, multiplying $(3.1)_1$ by $u_t^{I,0}$ followed by an integration over $\mathcal{I}$, we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{I}}|u_x^{I,0}|^2\mathrm{d}x+\int_{\mathcal{I}}|u_t^{I,0}|^2\mathrm{d}x\\&\quad=\int_{\mathcal{I}}u^{I,0}v_x^{I,0}u_{tx}^{I,0}\mathrm{d}x=\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{I}}u^{I,0}v_x^{I,0}u_x^{I,0}\mathrm{d}x-\int_{\mathcal{I}}u_t^{I,0}v_x^{I,0}u_x^{I,0}\mathrm{d}x-\int_{\mathcal{I}}u^{I,0}v_{xt}^{I,0}u_x^{I,0}\mathrm{d}x.\end{aligned} \tag{3.9}$$

By (3.7), Sobolev inequality (C.2) and equation $(3.1)_1$, we deduce that

$$\begin{aligned}
&\int_{\mathcal{I}} |u_x^{I,0}|^2 |v_x^{I,0}|^2 \mathrm{d}x \\
&\leq \|u_x^{I,0}\|_{L^\infty}^2 \|v_x^{I,0}\|_{L^2}^2 \\
&\leq c_0 \|u_x^{I,0}\|_{L^2} \|u_{xx}^{I,0}\|_{L^2} + c_0 \|u_x^{I,0}\|_{L^2}^2 \\
&\leq c_0 \|u_x^{I,0}\|_{L^2} (\|u_t^{I,0}\|_{L^2} + \|u_x^{I,0} v_x^{I,0}\|_{L^2} + \|u^{I,0} v_{xx}^{I,0}\|_{L^2}) + c_0 \|u_x^{I,0}\|_{L^2}^2 \\
&\leq \frac{1}{2} \|u_x^{I,0} v_x^{I,0}\|_{L^2}^2 + \frac{1}{32} \|u_t^{I,0}\|_{L^2}^2 + c_0 \|u^{I,0}\|_{L^\infty} \|v_{xx}^{I,0}\|_{L^2} \|u_x^{I,0}\|_{L^2} + c_0 \|u_x^{I,0}\|_{L^2}^2.
\end{aligned}$$

That is,

$$\int_{\mathcal{I}} |u_x^{I,0}|^2 |v_x^{I,0}|^2 \mathrm{d}x \leq \frac{1}{16} \|u_t^{I,0}\|_{L^2}^2 + c_0 \|u^{I,0}\|_{L^\infty} \|v_{xx}^{I,0}\|_{L^2} \|u_x^{I,0}\|_{L^2} + c_0 \|u_x^{I,0}\|_{L^2}^2 \tag{3.10}$$

for some constant $c_0 > 0$ independent of C_1. This, along with the Cauchy–Schwarz inequality, gives

$$\begin{aligned}
-\int_{\mathcal{I}} u_t^{I,0} v_x^{I,0} u_x^{I,0} \mathrm{d}x &\leq \frac{1}{4} \int_{\mathcal{I}} |u_t^{I,0}|^2 \mathrm{d}x + 4 \int_{\mathcal{I}} |u_x^{I,0}|^2 |v_x^{I,0}|^2 \mathrm{d}x \\
&\leq \frac{1}{2} \|u_t^{I,0}\|_{L^2}^2 + c_0 \|u^{I,0}\|_{L^\infty} \left(\|v_{xx}^{I,0}\|_{L^2}^2 + \|u_x^{I,0}\|_{L^2}^2 \right) + c_0 \|u_x^{I,0}\|_{L^2}^2.
\end{aligned}$$

Noticing from $(3.1)_2$ that $v_{tx}^{I,0} = -u_x^{I,0} v^{I,0} - u^{I,0} v_x^{I,0}$, we estimate the last term on the right-hand side of (3.9) as follows:

$$\begin{aligned}
&-\int_{\mathcal{I}} u^{I,0} v_{xt}^{I,0} u_x^{I,0} \mathrm{d}x \\
&= \int_{\mathcal{I}} u^{I,0} (u_x^{I,0} v^{I,0} + v_x^{I,0} u^{I,0}) u_x^{I,0} \mathrm{d}x \\
&\leq \|v^{I,0}\|_{L^\infty} \|u^{I,0}\|_{L^\infty} \|u_x^{I,0}\|_{L^2}^2 + \|u^{I,0}\|_{L^\infty} \left(\|u^{I,0}\|_{L^\infty}^2 \|v_x^{I,0}\|_{L^2}^2 + \|u_x^{I,0}\|_{L^2}^2 \right) \\
&\leq c_0 \|u^{I,0}\|_{L^\infty} \|u_x^{I,0}\|_{L^2}^2 + c_0 \|u^{I,0}\|_{L^\infty} \left[(\|u_x^{I,0}\|_{L^2} + \|u^{I,0}\|_{L^1})^2 + \|u_x^{I,0}\|_{L^2}^2 \right] \\
&\leq c_0 \|u^{I,0}\|_{L^\infty} \left(\|u_x^{I,0}\|_{L^2}^2 + 1 \right),
\end{aligned}$$

where we have used (3.7), $v^{I,0} \leq v_0$, $\|u^{I,0}\|_{L^1} = M$, (C.1), and the Cauchy–Schwarz inequality. Therefore we have from (3.9) that

$$\begin{aligned}
&\frac{1}{2} \int_{\mathcal{I}} |u_x^{I,0}|^2 \mathrm{d}x - \int_{\mathcal{I}} u^{I,0} v_x^{I,0} u_x^{I,0} \mathrm{d}x + \frac{1}{2} \int_0^t \int_{\mathcal{I}} |u_\tau^{I,0}|^2 \mathrm{d}x \mathrm{d}\tau \\
&\quad \leq c(T) + c_0 \int_0^t \|u^{I,0}\|_{L^\infty} \left(\|v_{xx}^{I,0}\|_{L^2}^2 + \|u_x^{I,0}\|_{L^2}^2 \right) \mathrm{d}\tau + c_0 \int_0^t \|u_x^{I,0}\|_{L^2}^2 \mathrm{d}\tau,
\end{aligned} \tag{3.11}$$

where (3.8) has been used, and the constant $c_0 > 0$ is independent of C_1. Noting that

$$\begin{aligned}
\int_{\mathcal{I}} u^{I,0} v_x^{I,0} u_x^{I,0} \mathrm{d}x &\leq \frac{1}{8} \int_{\mathcal{I}} |u_x^{I,0}|^2 \mathrm{d}x + \int_{\mathcal{I}} |u^{I,0}|^2 |v_x^{I,0}|^2 \mathrm{d}x \\
&\leq \frac{1}{8} \int_{\mathcal{I}} |u_x^{I,0}|^2 \mathrm{d}x + c_0 \|u^{I,0}\|_{L^\infty}^2 \leq \frac{1}{4} \int_{\mathcal{I}} |u_x^{I,0}|^2 \mathrm{d}x + c_0
\end{aligned}$$

due to (3.7), (C.1), and $\|u^{I,0}\|_{L^1} = M$, we further update (3.11) as

$$\begin{aligned}
&\int_{\mathcal{I}} |u_x^{I,0}|^2 \mathrm{d}x + \int_0^t \int_{\mathcal{I}} |u_\tau^{I,0}|^2 \mathrm{d}x \mathrm{d}\tau \\
&\quad \leq c(T) + c_0 \int_0^t \|u^{I,0}\|_{L^\infty} \left(\|v_{xx}^{I,0}\|_{L^2}^2 + \|u_x^{I,0}\|_{L^2}^2 \right) \mathrm{d}\tau + c_0 \int_0^t \|u_x^{I,0}\|_{L^2}^2 \mathrm{d}\tau.
\end{aligned} \tag{3.12}$$

On the other hand, differentiating the equation $(3.1)_2$ with respect to x twice gives

$$v_{txx}^{I,0} = -u_{xx}^{I,0} v^{I,0} - 2u_x^{I,0} v_x^{I,0} - u^{I,0} v_{xx}^{I,0}.$$

Testing the above equation against $v_{xx}^{I,0}$, thanks to $(3.1)_1$, (3.10), the fact $v^{I,0} \le v_0$, and the Cauchy–Schwarz inequality, it follows that

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{I}} |v_{xx}^{I,0}|^2 \mathrm{d}x + \int_{\mathcal{I}} u^{I,0}|v_{xx}^{I,0}|^2 \mathrm{d}x &= -\int_{\mathcal{I}} u_{xx}^{I,0} v^{I,0} v_{xx}^{I,0} \mathrm{d}x - \int_{\mathcal{I}} 2u_x^{I,0} v_x^{I,0} v_{xx}^{I,0} \mathrm{d}x \\
&= -\int_{\mathcal{I}} (u_t^{I,0} + u_x^{I,0} v_x^{I,0} + u^{I,0} v_{xx}^{I,0}) v_{xx}^{I,0} v^{I,0} \mathrm{d}x + \|v_{xx}^{I,0}\|_{L^2} \|u_x^{I,0} v_x^{I,0}\|_{L^2} \\
&\le \frac{1}{8}\int_{\mathcal{I}} |u_t^{I,0}|^2 \mathrm{d}x + c_0 \|u^{I,0}\|_{L^\infty} (\|u_x^{I,0}\|_{L^2}^2 + \|v_{xx}^{I,0}\|_{L^2}^2) + c_0 \|u_x^{I,0}\|_{L^2}^2 + c_0 \|v_{xx}^{I,0}\|_{L^2}^2,
\end{aligned} \tag{3.13}$$

where $c_0 > 0$ is independent of C_1. Integrating (3.13) over $(0,t)$ for any $t \in (0,T]$ yields that

$$\begin{aligned}
&\int_{\mathcal{I}} |v_{xx}^{I,0}|^2 \mathrm{d}x + \int_0^t \int_{\mathcal{I}} u^{I,0} |v_{xx}^{I,0}|^2 \mathrm{d}x\mathrm{d}\tau \\
&\quad \le \frac{1}{8}\int_0^t \int_{\mathcal{I}} |u_\tau^{I,0}|^2 \mathrm{d}x\mathrm{d}\tau + c_0 \int_0^t (\|u^{I,0}\|_{L^\infty} + 1)(\|u_x^{I,0}\|_{L^2}^2 + \|v_{xx}^{I,0}\|_{L^2}^2)\mathrm{d}\tau.
\end{aligned}$$

This, combined with (3.4) and (3.12), implies that

$$\begin{aligned}
&\int_{\mathcal{I}} \left(|u_x^{I,0}|^2 + |v_{xx}^{I,0}|^2\right) \mathrm{d}x + \int_0^t \int_{\mathcal{I}} |u_\tau^{I,0}|^2 \mathrm{d}x\mathrm{d}\tau \\
&\quad \le c_0 \int_0^t (\|u^{I,0}\|_{L^\infty} + 1)(\|u_x^{I,0}\|_{L^2}^2 + \|v_{xx}^{I,0}\|_{L^2}^2)\mathrm{d}\tau + c(T).
\end{aligned} \tag{3.14}$$

Therefore an application of the Gronwall inequality along with (3.8) gives

$$\int_{\mathcal{I}} \left(|u_x^{I,0}|^2 + |v_{xx}^{I,0}|^2\right) \mathrm{d}x + \int_0^t \int_{\mathcal{I}} |u_\tau^{I,0}|^2 \mathrm{d}x\mathrm{d}\tau \le c(T) \tag{3.15}$$

for any $t \in (0,T]$, where the constant $c(T) > 0$ is independent of C_1. Furthermore, by virtue of (3.7), (3.8), (3.15), (C.2), and the equations in (3.1), we have

$$\int_0^T \left(\|u_{xx}^{I,0}\|_{L^2}^2 + \|v_t^{I,0}\|_{H^2}^2\right) \mathrm{d}t \le c(T). \tag{3.16}$$

Using (3.7), (3.14), and the Sobolev inequality $\|f\|_{L^\infty} \le c_0 \|f\|_{W^{1,2}}$, we get

$$\int_0^t \|v_x^{I,0}\|_{L^\infty}^2 \mathrm{d}t \le c(T),$$

where the constant $c(T) > 0$ depends on the initial data and T but is independent of C_1. Therefore the a priori assumption (3.3) is closed provided that $C_1 > 0$ is chosen to be large such that $C_1 > c(T)$, and thus the estimates (3.4), (3.7), (3.8), (3.15), and (3.16) subsequently follow. Next, we shall derive some higher-order estimates for the solution. The proof is based on the standard energy method (cf. [16, pp. 387–388]), namely, recovering the estimates on spatial derivatives from those on time derivatives. For brevity, we will establish the estimates on the second-order time derivatives of the solution only and their implications in the estimates of spatial derivatives, while estimates on the higher-order time derivatives can be obtained in the same spirit. To this end, we differentiate the equations in (3.1) with respect to t and get

$$\begin{cases} u_{tt}^{I,0} = \left(u_{tx}^{I,0} - u^{I,0}v_{tx}^{I,0} - u_t^{I,0}v_x^{I,0}\right)_x, \\ v_{tt}^{I,0} = -u_t^{I,0}v^{I,0} - u^{I,0}v_t^{I,0}. \end{cases} \tag{3.17}$$

Multiplying $(3.17)_1$ by $u_t^{I,0}$ and integrating the resulting equation over $\mathcal{I}$, we have

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{I}}|u_t^{I,0}|^2\mathrm{d}x + \int_{\mathcal{I}}|u_{xt}^{I,0}|^2\mathrm{d}x &= \int_{\mathcal{I}} u^{I,0}v_{tx}^{I,0}u_{xt}^{I,0}\mathrm{d}x + \int_{\mathcal{I}} u_t^{I,0}v_x^{I,0}u_{xt}^{I,0}\mathrm{d}x \\ &\le \frac{1}{4}\int_{\mathcal{I}}|u_{xt}^{I,0}|^2\mathrm{d}x + c_0\int_{\mathcal{I}}|u^{I,0}v_{tx}^{I,0}|^2\mathrm{d}x + c_0\int_{\mathcal{I}}|u_t^{I,0}v_x^{I,0}|^2\mathrm{d}x \\ &\le \frac{1}{4}\int_{\mathcal{I}}|u_{xt}^{I,0}|^2\mathrm{d}x + c_0\|u^{I,0}\|_{L^\infty}^2\|v_{tx}^{I,0}\|_{L^2}^2 + c_0\|v_x^{I,0}\|_{L^\infty}^2\|u_t^{I,0}\|_{L^2}^2 \\ &\le \frac{1}{4}\int_{\mathcal{I}}|u_{xt}^{I,0}|^2\mathrm{d}x + c(T)\left(\|v_{tx}^{I,0}\|_{L^2}^2 + \|u_t^{I,0}\|_{L^2}^2\right), \end{aligned}$$

where we have used the Cauchy–Schwarz inequality and $\|u^{I,0}\|_{L_T^\infty L^\infty} + \|v_x^{I,0}\|_{L_T^\infty L^\infty} \le c(T)$ ensured by (3.7), (3.15), and (C.2). Therefore we get, thanks to (3.15) and (3.16),

$$\int_{\mathcal{I}}|u_t^{I,0}|^2(\cdot,t)\mathrm{d}x + \int_0^t\int_{\mathcal{I}}|u_{x\tau}^{I,0}|^2\mathrm{d}x\mathrm{d}\tau \le c(T) \tag{3.18}$$

for any $t \in [0,T]$. This, along with (3.15), (3.16), and the equations in (3.1), further implies that

$$\|u_{xx}^{I,0}\|_{L_T^\infty L^2}^2 + \|v_t^{I,0}\|_{L_T^\infty H^2} + \|v_{tt}^{I,0}\|_{L_T^\infty L^2} + \|v_{ttx}^{I,0}\|_{L_T^2 L^2} \le c(T). \tag{3.19}$$

Next, testing $(3.17)_1$ against $u_{tt}^{I,0}$, we have

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{I}}|u_{tx}^{I,0}|^2\mathrm{d}x + \int_{\mathcal{I}}|u_{tt}^{I,0}|^2\mathrm{d}x = \int_{\mathcal{I}}\left(u^{I,0}v_{tx}^{I,0} + u_t^{I,0}v_x^{I,0}\right)u_{ttx}^{I,0}\mathrm{d}x \\ &\quad = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{I}}\left(u^{I,0}v_{tx}^{I,0} + u_t^{I,0}v_x^{I,0}\right)u_{tx}^{I,0}\mathrm{d}x \\ &\qquad - \int_{\mathcal{I}}\left(u_t^{I,0}v_{tx}^{I,0} + u^{I,0}v_{ttx}^{I,0} + u_{tt}^{I,0}v_x^{I,0} + u_t^{I,0}v_{xt}^{I,0}\right)u_{tx}^{I,0}\mathrm{d}x \\ &\quad \le \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{I}}\left(u^{I,0}v_{tx}^{I,0} + u_t^{I,0}v_x^{I,0}\right)u_{tx}^{I,0}\mathrm{d}x \\ &\qquad + \left(\|u_t^{I,0}\|_{L^\infty}\|v_{tx}^{I,0}\|_{L^2} + \|u^{I,0}\|_{L^\infty}\|v_{ttx}^{I,0}\|_{L^2}\right)\|u_{tx}^{I,0}\|_{L^2} \\ &\qquad + c_0\left(\|u_{tt}^{I,0}\|_{L^2}\|v_x^{I,0}\|_{L^\infty} + \|u_t^{I,0}\|_{L^\infty}\|v_{tx}^{I,0}\|_{L^2}\right)\|u_{tx}^{I,0}\|_{L^2} \\ &\quad \le \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{I}}\left(u^{I,0}v_{tx}^{I,0} + u_t^{I,0}v_x^{I,0}\right)u_{tx}^{I,0}\mathrm{d}x + \frac{1}{8}\|u_{tt}^{I,0}\|_{L^2}^2 + c(T)\|u_{tx}^{I,0}\|_{L^2}^2 + c(T)\|v_{ttx}^{I,0}\|_{L^2}^2, \end{aligned} \tag{3.20}$$

where we have used (3.15), (3.16), (3.18), (3.19), the fact $\|u^{I,0}\|_{L_T^\infty L^\infty} + \|v_x^{I,0}\|_{L_T^\infty L^\infty} \le c(T)$, and the Sobolev inequality (C.2). Noting that

$$\begin{aligned} &\int_{\mathcal{I}}\left(u^{I,0}v_{tx}^{I,0} - u_t^{I,0}v_x^{I,0}\right)u_{tx}^{I,0}\mathrm{d}x \\ &\quad \le \frac{1}{4}\|u_{tx}^{I,0}\|_{L^2}^2 + c_0\left(\|u^{I,0}\|_{L^\infty}^2\|v_{tx}^{I,0}\|_{L^2}^2 + \|v_x^{I,0}\|_{L^\infty}^2\|u_t^{I,0}\|_{L^2}^2\right) \le \frac{1}{4}\|u_{tx}^{I,0}\|_{L^2}^2 + c(T) \end{aligned}$$

due to (3.8), (3.15), (3.18), and (3.19), we get, after integrating (3.20) over $[0,t]$ for any $t\in(0,T]$,

$$\int_{\mathcal{I}}|u_{tx}^{I,0}|^2(\cdot,t)\mathrm{d}x+\int_0^t\int_{\mathcal{I}}|u_{\tau\tau}^{I,0}|^2\mathrm{d}x\mathrm{d}\tau\le c(T),$$

where (3.18) has been used. This combined with (3.16), $(3.17)_1$, (3.18), and (3.19) entails that

$$\|u_t^{I,0}\|_{L_T^\infty L^\infty}+\|u_t^{I,0}\|_{L_T^2H^2}\le c(T). \tag{3.21}$$

Applying ∂_x^3 to the equation $(3.17)_2$, we get

$$\partial_x^3 v_t^{I,0}=-\sum_{k=0}^{3}\partial_x^k u^{I,0}\partial_x^{3-k}v^{I,0}.$$

Multiplying this equation by $\partial_x^3 v^{I,0}$ followed by an integration over $\mathcal{I}$, we have

$$\begin{aligned}\frac12\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{I}}|\partial_x^3v^{I,0}|^2\mathrm{d}x&+\int_{\mathcal{I}}u^{I,0}|\partial_x^3v^{I,0}|^2\mathrm{d}x\\&\le\sum_{k=0}^{1}\|\partial_x^ku^{I,0}\|_{L^\infty}\|\partial_x^{3-k}v^{I,0}\|_{L^2}\|\partial_x^3v^{I,0}\|_{L^2}\\&\quad+c_0\sum_{k=2}^{3}\|\partial_x^ku^{I,0}\|_{L^2}\|\partial_x^{3-k}v^{I,0}\|_{L^\infty}\|\partial_x^3v^{I,0}\|_{L^2}\\&\le c(T)\|\partial_x^3u^{I,0}\|_{L^2}^2+\|\partial_x^3v^{I,0}\|_{L^2}^2+c(T),\end{aligned}\tag{3.22}$$

where we have used (3.7), (3.15), (3.19) and the Cauchy–Schwarz inequality. On the other hand, by $(3.1)_1$, (3.7), (3.15), and (3.21), we get

$$\begin{aligned}\|\partial_x^3u^{I,0}\|_{L^2}^2&\le\|u_{tx}^{I,0}\|_{L^2}^2+\sum_{k=0}^{1}\|\partial_x^ku^{I,0}\|_{L^\infty}^2\|\partial_x^{3-k}v^{I,0}\|_{L^2}^2+\|(u_x^{I,0}v_x^{I,0})_x\|_{L^2}^2\\&\le c(T)\|\partial_x^3v^{I,0}\|_{L^2}^2+c(T).\end{aligned}\tag{3.23}$$

Therefore we update (3.22) as

$$\frac12\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{I}}|\partial_x^3v^{I,0}|^2\mathrm{d}x+\int_{\mathcal{I}}u^{I,0}|\partial_x^3v^{I,0}|^2\mathrm{d}x\le c(T)\|\partial_x^3v^{I,0}\|_{L^2}^2+c(T),\tag{3.24}$$

which, along with the Gronwall inequality, (3.23), and the fact $u^{I,0}\ge0$, entails that for any $t\in[0,T]$,

$$\|\partial_x^3v^{I,0}(\cdot,t)\|_{L^2}^2+\|\partial_x^3u^{I,0}(\cdot,t)\|_{L^2}^2\le c(T).\tag{3.25}$$

By the analogous arguments, one can also get

$$\|\partial_x^4v^{I,0}(\cdot,t)\|_{L^2}^2+\int_0^t\|\partial_x^4u^{I,0}\|_{L^2}^2\mathrm{d}\tau\le c(T)\tag{3.26}$$

for any $t\in[0,T]$. Now combining $(3.1)_2$, (3.7), (3.15), (3.16), (3.19), (3.21), (3.25), and (3.26) yields

$$\|v_t^{I,0}\|_{L_T^2H^4}+\|v_{tt}^{I,0}\|_{L_T^2H^2}\le c(T).$$

The rest of the estimates in (3.2) can be proved in a similar manner by applying ∂_t and ∂_t^2 to the equations in (3.17), and the details are omitted here for brevity. ∎

With the solution obtained in Lemma 3.1 for the problem (3.1), recalling the transformation (2.1), one can easily show the existence of unique classical solutions to (2.10). Precisely, we have the following lemma.

LEMMA 3.2. *Assume that* $(\varphi_0, v_0) \in H^7 \times H^7$ *and* $(\sqrt{v_0})_x \in L^2$ *satisfying* (2.20) *and* $\varphi_{0x} + M \geq 0$. *Then for any* $T > 0$, *there exists a unique solution* $(\varphi^{I,0}, v^{I,0})$ *to the problem* (2.10) *on* $[0, T]$ *satisfying*

$$\varphi_x^{I,0} + M \geq 0, \quad \|\partial_t^k \varphi^{I,0}\|_{L_T^2 H^{8-2k}} \leq c(T) \quad \text{for } k = 0,1,2,3,4, \tag{3.27a}$$

$$\|v^{I,0}\|_{L_T^\infty H^7} + \|\partial_t^k v^{I,0}\|_{L_T^2 H^{9-2k}} \leq c(T) \quad \text{for } k = 1,2,3,4. \tag{3.27b}$$

The next lemma gives the regularity of boundary layer profiles $v^{B,0}$ and $\varphi^{B,1}$.

LEMMA 3.3. *Let* $(\varphi^{I,0}, v^{I,0})$ *be the solution of* (2.10) *obtained in Lemma* 3.2. *Then for any* $T > 0$, *the problem* (2.12)–(2.13) *admits a unique solution* $v^{B,0}$ *on* $[0, T]$ *such that for any* $l \in \mathbb{N}$,

$$0 \leq v^{B,0} \leq v_*, \quad \langle z\rangle^l \partial_t^k v^{B,0} \in L_T^2 H_z^{6-2k}, \ \langle z\rangle^l \partial_t^k \varphi^{B,1} \in L_T^2 H_z^{7-2k} \ \text{for } k = 0,1,2,3. \tag{3.28}$$

Furthermore, it holds that

$$\|\langle z\rangle^l \partial_t^k v^{B,0}\|_{L_T^2 H_z^{6-2k}} \leq K_0(T, v_*) v_*, \ \|\langle z\rangle^l \partial_t^k \varphi^{B,1}\|_{L_T^2 H_z^{7-2k}} \leq c(v_*, T) v_*, \ k = 0,1,2,3, \tag{3.29}$$

$$\sum_{k=0}^{2} \|\langle z\rangle^l \partial_t^k v^{B,0}\|_{L_T^\infty H_z^{4-2k}} + \sum_{\lambda=0}^{1} \sum_{\ell=0}^{3-2\lambda} \|\langle z\rangle^l \partial_t^\lambda \partial_z^\ell v^{B,0}\|_{L_T^\infty L_z^\infty} \leq K_0(T, v_*) v_*, \tag{3.30}$$

where the constant $K_0(T, v_*) := C(T) \mathrm{e}^{c(v_*, T)} > 0$, *with* $c(v_*, T)$ *and* $C(T)$ *being as stated in section* 2. *Clearly,* $K_0(T, v_*)$ *is increasing in* T *and* v_* *with* $\lim_{T\to 0} K_0(T, v_*) = 0$ *and* $\lim_{T\to+\infty} K_0(T, v_*) = +\infty$.

Proof. The local existence and uniqueness of solutions to the problem (2.12) with regularity given in (3.28) can be proved by routine procedures: first, we study the linearized problem by the reflection method; second, we derive suitable estimates for solutions of the linearized problem and then prove the existence of solutions for the original nonlinear problem by the Banach's fixed point theorem. For completeness, we detail the proof in Appendix A. Below we derive the a priori estimates of solutions, which are used not only for the global existence of solutions but also for the convergence of boundary layers. We first prove that the solution of (2.12) is bounded and satisfies

$$0 \leq v^{B,0} \leq v_*. \tag{3.31}$$

To this end, we test the equation in (2.12) against $v^- := -\max\{0, -v^{B,0}\}$ to derive that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_+} |v^-|^2 \mathrm{d}z + \int_{\mathbb{R}_+} |\partial_z v^-|^2 \mathrm{d}z + \int_{\mathbb{R}_+} (\varphi_x^{I,0}(0,t) + M)\mathrm{e}^{v^{B,0}} |v^-|^2 \mathrm{d}z \\ + \int_{\{v^{B,0}<0\}} (\varphi_x^{I,0}(0,t) + M) v^{I,0}(0,t)(\mathrm{e}^{v^{B,0}} - 1) v^{B,0} \mathrm{d}z = 0,\end{aligned}$$

where, to ensure the validity of integration by parts, we have used the fact $v_* \geq v^{I,0}(0,t) \geq 0$ due to $\varphi_x^{I,0} + M \geq 0$ and

$$v^{I,0}(0,t) = v_* \exp\left(-\int_0^t (\varphi_x^{I,0}(0,\tau) + M)\mathrm{d}\tau\right). \tag{3.32}$$

This entails that

$$\int_{\mathbb{R}_+} |v^-|^2 \mathrm{d}z \leq 0,$$

which implies $v^- = 0$ and $v^{B,0} \geq 0$. Similarly, testing the equation (2.12) against $v^+ := \max\{v^{B,0} - v_*, 0\}$, we can show that $v^{B,0} \leq v_*$. Therefore (3.31) is proved.

Next, we shall derive some weighted estimates for $v^{B,0}$. Let $\eta(z) \in C^\infty([0,\infty))$ such that

$$\eta(0) = 1, \quad \eta(z) = 0 \quad \text{for } z \geq 1, \tag{3.33}$$

and denote $\overline{u^{I,0}} := \varphi_x^{I,0}(0,t) + M$. Then if we take $\vartheta = v^{B,0} - \eta(z)(v_* - v^{I,0}(0,t)) =: v^{B,0} - \phi(z,t)$, it follows that ϑ solves

$$\begin{cases} \vartheta_t = \vartheta_{zz} - \overline{u^{I,0}}\mathrm{e}^{\vartheta+\phi}(\vartheta + \phi) - \overline{u^{I,0}}\, v^{I,0}(0,t)(\mathrm{e}^{\vartheta+\phi} - 1) + \varrho, \\ \vartheta(0,t) = 0, \quad \vartheta(+\infty, t) = 0, \\ \vartheta(z,0) = 0, \end{cases} \tag{3.34}$$

where

$$\varrho = \eta_{zz}(z)(v_* - v^{I,0}(0,t)) - \eta(z)(v_* - v^{I,0}(0,t))_t.$$

By (3.27) and (C.2), we get that

$$\begin{cases} \|\partial_t^k \varphi_x^{I,0}(0,t)\|_{L^2(0,T)} \leq \|\partial_t^k \varphi_x^{I,0}\|_{L_T^2 H^1} \leq c(T) \quad \text{for} \quad 0 \leq k \leq 3, \\ \|\partial_t^k v^{I,0}(0,t)\|_{L^2(0,T)} \leq \|\partial_t^k v^{I,0}\|_{L_T^2 H^1} \leq c(T) \quad \text{for} \quad 0 \leq k \leq 4, \end{cases} \tag{3.35}$$

which gives rise to

$$\begin{aligned} &\|\partial_t^k \varphi_x^{I,0}(0,t)\|_{L^\infty(0,T)} \leq c(T) \text{ for } 0 \leq k \leq 2 \quad \text{and} \\ &\|\partial_t^k v^{I,0}(0,t)\|_{L^\infty(0,T)} \leq c(T) \text{ for } 0 \leq k \leq 3. \end{aligned} \tag{3.36}$$

Thanks to (2.20), (3.32), and (3.36), it holds for $l \in \mathbb{N}$ that $\langle z\rangle^l \partial_t^k \varrho \in L_T^2 H_z^{4-2k}$ $(k = 0,1,2)$ with

$$\|\langle z\rangle^l \partial_t^k \varrho\|_{L_T^2 H_z^{4-2k}} \leq C(T) v_*, \quad k = 0,1,2, \tag{3.37}$$

where the constant $C(T) > 0$ is as stated in section 2. Similarly, we get for $l \in \mathbb{N}$ that $\langle z\rangle^l \partial_t^k \phi \in L_T^2 H_z^{4-2k}$ $(k = 0,1,2)$ with

$$\|\langle z\rangle^l \partial_t^{k+\lambda} \phi\|_{L_T^2 H_z^{4-2k}} \leq C(T) v_*, \quad k = 0,1,2, \quad \lambda = 0,1. \tag{3.38}$$

Multiplying the equation $(3.34)_1$ by $\langle z\rangle^{2l}\vartheta$ followed by an integration over $\mathbb{R}_+$, we have

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_+} \langle z\rangle^{2l}\vartheta^2 \mathrm{d}z + \int_{\mathbb{R}_+} \langle z\rangle^{2l}\vartheta_z^2 \mathrm{d}z + \int_{\mathbb{R}_+} \langle z\rangle^{2l}\overline{u^{I,0}}\mathrm{e}^{\vartheta+\phi}\vartheta^2 \mathrm{d}z \\ &\quad = \int_{\mathbb{R}_+} \langle z\rangle^{2l}\vartheta\varrho \mathrm{d}z - 2l\int_{\mathbb{R}_+} \langle z\rangle^{2l-2} z\vartheta_z\vartheta \mathrm{d}z - \int_{\mathbb{R}_+} \langle z\rangle^{2l}\overline{u^{I,0}}\mathrm{e}^{\vartheta+\phi}\vartheta\phi \mathrm{d}z \\ &\qquad - \int_{\mathbb{R}_+} \langle z\rangle^{2l}\overline{u^{I,0}}\, v^{I,0}(0,t)\left(\mathrm{e}^{\vartheta+\phi} - 1\right)\vartheta \mathrm{d}z =: \mathcal{A}, \end{aligned} \tag{3.39}$$

where, due to $\overline{u^{I,0}} = \varphi_x^{I,0}(0,t) + M \geq 0$ and $0 \leq v^{B,0} \leq v_*$, it holds that

$$\int_{\mathbb{R}_+} \langle z\rangle^{2l} \overline{u^{I,0}} \mathrm{e}^{\vartheta+\phi} \vartheta^2 \mathrm{d}z \geq 0. \tag{3.40}$$

We now turn to estimating the terms on the right-hand side of (3.39). By (3.32), (3.36), $0 \leq v^{B,0} \leq v_*$, and the Cauchy–Schwarz inequality, we get

$$\begin{aligned} \mathcal{A} &\leq \|\langle z\rangle^l \vartheta\|_{L_z^2} \|\langle z\rangle^l \varrho\|_{L_z^2} + c_0 \|\langle z\rangle^l \vartheta\|_{L_z^2} \|\langle z\rangle^l \vartheta_z\|_{L_z^2} + c_0 \|\langle z\rangle^l \vartheta\|_{L_z^2} \|\langle z\rangle^l \phi\|_{L_z^2} \\ &\quad + c(v_*,T) \int_{\mathbb{R}_+} \langle z\rangle^{2l} \overline{u^{I,0}} v^{I,0}(0,t) \left(|\vartheta| + |\phi|\right) \vartheta \mathrm{d}z \\ &\leq \frac{1}{4} \int_{\mathbb{R}_+} \langle z\rangle^{2l} \vartheta_z^2 \mathrm{d}z + c(v_*,T) \int_{\mathbb{R}_+} \langle z\rangle^{2l} \vartheta^2 \mathrm{d}z + c_0 \|\langle z\rangle^l \varrho\|_{L_z^2}^2 + c_0 \|\langle z\rangle^l \phi\|_{L_z^2}^2, \end{aligned} \tag{3.41}$$

where the constant $c(v_*,T) > 0$ is as stated in section 2. Inserting (3.40)–(3.41) into (3.39) and integrating the result for any $t \in (0,T]$, we get

$$\int_{\mathbb{R}_+} \langle z\rangle^{2l} \vartheta^2(\cdot,t) \mathrm{d}z + \int_0^t \int_{\mathbb{R}_+} \langle z\rangle^{2l} \vartheta_z^2 \mathrm{d}z \mathrm{d}\tau \leq C(T) v_*^2 + c(v_*,T) \int_0^t \int_{\mathbb{R}_+} \langle z\rangle^{2l} \vartheta^2 \mathrm{d}z \mathrm{d}\tau, \tag{3.42}$$

where $C(T)$ and $c(v_*,T)$ are constants as stated in section 2. Applying the Gronwall inequality to (3.42), we get

$$\int_{\mathbb{R}_+} \langle z\rangle^{2l} \vartheta^2(\cdot,t) \mathrm{d}z + \int_0^t \int_{\mathbb{R}_+} \langle z\rangle^{2l} \vartheta_z^2 \mathrm{d}z \mathrm{d}\tau \leq C(T) \mathrm{e}^{c(v_*,T)} v_*^2. \tag{3.43}$$

Multiplying $(3.34)_1$ by $\langle z\rangle^{2l} \vartheta_t$ and integrating the resulting equation over $\mathbb{R}_+$, we have

$$\begin{aligned} &\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}_+} \langle z\rangle^{2l} \left(\vartheta_z^2 + \overline{u^{I,0}} \vartheta^2 \mathrm{e}^{\vartheta+\phi} \right) \mathrm{d}z + \int_{\mathbb{R}_+} \langle z\rangle^{2l} \vartheta_t^2 \mathrm{d}z \\ &= \frac{1}{2} \int_{\mathbb{R}_+} \langle z\rangle^{2l} \partial_t \overline{u^{I,0}} \vartheta^2 \mathrm{e}^{\vartheta+\phi} \mathrm{d}z + \frac{1}{2} \int_{\mathbb{R}_+} \langle z\rangle^{2l} \overline{u^{I,0}} \vartheta^2 \left(\vartheta_t + \phi_t\right) \mathrm{e}^{\vartheta+\phi} \mathrm{d}z - 2l \int_{\mathbb{R}_+} \langle z\rangle^{2l-2} z \vartheta_t \vartheta_z \mathrm{d}z \\ &\quad - \int_{\mathbb{R}_+} \langle z\rangle^{2l} \overline{u^{I,0}} \mathrm{e}^{\vartheta+\phi} \phi \vartheta_t \mathrm{d}z - \overline{u^{I,0}} v^{I,0}(0,t) \int_{\mathbb{R}_+} \langle z\rangle^{2l} (\mathrm{e}^{\vartheta+\phi} - 1) \vartheta_t \mathrm{d}z - \int_{\mathcal{I}} \langle z\rangle^{2l} \varrho \vartheta_t \mathrm{d}z \\ &\leq c(v_*,T) |\overline{\partial_t u^{I,0}}| \int_{\mathbb{R}_+} \langle z\rangle^{2l} \vartheta^2 \mathrm{d}z + c(v_*,T) \overline{u^{I,0}} \int_{\mathbb{R}_+} \langle z\rangle^{2l} \vartheta^2 (|\vartheta_t| + |\phi_t|) \mathrm{d}z \\ &\quad + c_0 \int_{\mathbb{R}_+} \langle z\rangle^{2l-1} |\vartheta_t| |\vartheta_z| \mathrm{d}z \\ &\quad + c(v_*,T) \int_{\mathbb{R}_+} \langle z\rangle^{2l} |\phi| |\vartheta_t| \mathrm{d}z + c(v_*,T) \overline{u^{I,0}} |v^{I,0}(0,t)| \int_{\mathbb{R}_+} \langle z\rangle^{2l} \left(|\vartheta| + |\phi|\right) |\vartheta_t| \mathrm{d}z \\ &\quad - \int_{\mathbb{R}_+} \langle z\rangle^{2l} \varrho \vartheta_t \mathrm{d}z \\ &\leq \frac{1}{8} \int_{\mathbb{R}_+} \langle z\rangle^{2l} \vartheta_t^2 \mathrm{d}z + c(v_*,T) \int_{\mathbb{R}_+} \langle z\rangle^{2l} \vartheta_z^2 \mathrm{d}z + c(v_*,T) \int_{\mathbb{R}_+} \langle z\rangle^{2l} (\phi^2 + \phi_t^2) \mathrm{d}z \\ &\quad + c_0 \int_{\mathbb{R}_+} \langle z\rangle^{2l} \varrho^2 \mathrm{d}z \\ &\quad + c(v_*,T) (|\overline{\partial_t u^{I,0}}| + |\overline{u^{I,0}}|^2 + |\overline{u^{I,0}}|^2 |v^{I,0}(0,t)|^2) \int_{\mathbb{R}_+} \langle z\rangle^{2l} \vartheta^2 \mathrm{d}z, \end{aligned} \tag{3.44}$$

where we have used (3.27), (3.43), $0 \le v^{B,0} \le v_*$, and the Cauchy–Schwarz inequality. By (3.36), we further update (3.44) as

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_+}\langle z\rangle^{2l}\left(\vartheta_z^2+\overline{u^{I,0}}\vartheta^2\mathrm{e}^{\vartheta+\phi}\right)\mathrm{d}z+\int_{\mathbb{R}_+}\langle z\rangle^{2l}\vartheta_t^2\mathrm{d}z$$
$$\le c(v_*,T)\int_{\mathbb{R}_+}\langle z\rangle^{2l}\left(\vartheta^2+\vartheta_z^2\right)\mathrm{d}z+c(v_*,T)\int_{\mathbb{R}_+}\langle z\rangle^{2l}(\phi^2+\phi_t^2)\mathrm{d}z+c_0\int_{\mathbb{R}_+}\langle z\rangle^{2l}\varrho^2\mathrm{d}z.$$

This, along with (3.27), (3.37), (3.38), (3.43), and the Gronwall inequality, yields for any $t\in(0,T]$ that

$$\int_{\mathbb{R}_+}\langle z\rangle^{2l}\vartheta_z^2(\cdot,t)\mathrm{d}z+\int_0^t\int_{\mathbb{R}_+}\langle z\rangle^{2l}\vartheta_\tau^2\mathrm{d}z\mathrm{d}\tau\le C(T)\mathrm{e}^{c(v_*,T)}v_*^2. \tag{3.45}$$

With (3.36), (3.37), and (3.45), we get from $(3.34)_1$ that

$$\int_0^T\int_{\mathbb{R}_+}\langle z\rangle^{2l}\vartheta_{zz}^2\mathrm{d}z\mathrm{d}t\le C(T)\mathrm{e}^{c(v_*,T)}v_*^2. \tag{3.46}$$

Denote $\tilde\vartheta=\vartheta_t$. Then by (3.34) and the compatibility condition (2.20), we find that $\tilde\vartheta$ satisfies

$$\begin{cases}\tilde\vartheta_t=\tilde\vartheta_{zz}-\overline{u^{I,0}}\mathrm{e}^{\vartheta+\phi}\tilde\vartheta-\overline{u^{I,0}}\mathrm{e}^{\vartheta+\phi}(\vartheta+\phi)\tilde\vartheta-\overline{u^{I,0}}\,v^{I,0}(0,t)\mathrm{e}^{\vartheta+\phi}\tilde\vartheta+\tilde\varrho,\\ \tilde\vartheta(0,t)=0,\quad\tilde\vartheta(+\infty,t)=0,\\ \tilde\vartheta(z,0)=0,\end{cases} \tag{3.47}$$

where $\tilde\varrho$ is given by

$$\tilde\varrho=-\partial_t\overline{u^{I,0}}\mathrm{e}^{\vartheta+\phi}(\vartheta+\phi)-\overline{u^{I,0}}\mathrm{e}^{\vartheta+\phi}\phi_t(1+\vartheta+\phi)-\overline{u^{I,0}}\,v^{I,0}(0,t)\mathrm{e}^{\vartheta+\phi}\phi_t$$
$$-\partial_t\left(\overline{u^{I,0}}\,v^{I,0}(0,t)\right)(\mathrm{e}^{\vartheta+\phi}-1)+\partial_t\varrho.$$

From (3.27), (3.37), (3.43), and (3.36)–(3.46), it follows for $l\in\mathbb{N}$ that $\langle z\rangle^l\partial_t^k\tilde\varrho\in L_T^2H_z^{2-2k}\,(k=0,1)$ with

$$\|\langle z\rangle^l\partial_t^k\tilde\varrho\|_{L_T^2H_z^{2-2k}}\le C(T)\mathrm{e}^{c(v_*,T)}v_*,\quad k=0,1. \tag{3.48}$$

With (3.48), by repeating the procedures in the proofs of (3.43), (3.45), and (3.46), we have

$$\int_{\mathbb{R}_+}\langle z\rangle^{2l}\left(\tilde\vartheta^2+\tilde\vartheta_z^2\right)(\cdot,t)\mathrm{d}z+\int_0^t\int_{\mathbb{R}_+}\langle z\rangle^{2l}\left(\tilde\vartheta_\tau^2+\tilde\vartheta_z^2+\tilde\vartheta_{zz}^2\right)\mathrm{d}z\mathrm{d}\tau\le C(T)\mathrm{e}^{c(v_*,T)}v_*^2$$

for any $t\in(0,T]$. This, along with (3.27), $(3.34)_1$, $(3.47)_1$, and the fact $\tilde v=v_t$, implies that

$$\int_0^T\int_{\mathbb{R}_+}\langle z\rangle^{2l}\left(\vartheta_{zzz}^2+\vartheta_{zzzz}^2+\vartheta_{tzz}^2+\vartheta_{tt}^2\right)\mathrm{d}z\mathrm{d}t\le C(T)\mathrm{e}^{c(v_*,T)}v_*^2,$$

where we have used $\|\langle z\rangle^l\partial_t^k\varrho\|_{L_T^2H_z^{2-2k}}\le C(T)v_*$ $(k=0,1)$ from (3.37) and the estimate $\|\langle z\rangle^l\tilde\varrho\|_{L_T^2L_z^2}\le C(T)\mathrm{e}^{c(v_*,T)}v_*$ from (3.48). Thus we conclude for the problem (3.34) that

$$\|\langle z\rangle^l\partial_t^k\vartheta\|_{L_T^2H_z^{4-2k}}\le C(T)\mathrm{e}^{c(v_*,T)}v_*,\quad k=0,1,2, \tag{3.49}$$

provided $\|\langle z\rangle^l\partial_t^k\varrho\|_{L_T^2H_z^{2-2k}}\le C(T)v_*$ with $k=0,1$. Notice that the initial value for the problem (3.47) is compatible up to order one, and that $\|\langle z\rangle^l\partial_t^k\tilde{\varrho}\|_{L_T^2H_z^{2-2k}}\le C(T)\mathrm{e}^{c(v_*,T)}v_*$ with $k=0,1$. Therefore, by the same arguments as those proving (3.49), we have for the problem (3.47) that

$$\|\langle z\rangle^l\partial_t^k\tilde{\vartheta}\|_{L_T^2H_z^{4-2k}}\le C(T)\mathrm{e}^{c(v_*,T)}v_*,\quad k=0,1,2. \tag{3.50}$$

This, along with (3.37) and (3.49),further gives

$$\int_0^T\|\partial_z^5\vartheta(\cdot,t)\|_{H^1}^2\mathrm{d}t\le C(T)\mathrm{e}^{c(v_*,T)}v_*^2. \tag{3.51}$$

Collecting (3.49)–(3.51), we have

$$\langle z\rangle^l\partial_t^kv^{B,0}\in L_T^2H_z^{6-2k},\quad k=0,1,2,3. \tag{3.52}$$

By (3.52) and Proposition C.4, we get for $k=0,1,2$, $\ell=0,1,\dots,4-2k$ that $\langle z\rangle^l\partial_t^kv^{B,0}\in C([0,T];H_z^{5-2k})$ and $\langle z\rangle^l\partial_t^k\partial_z^\ell v^{B,0}\in L_T^\infty L_z^\infty$ with

$$\begin{aligned}&\|\langle z\rangle^l\partial_t^kv^{B,0}\|_{L_T^\infty H_z^{4-2k}}+\sum_{\lambda=0}^{3}\|\langle z\rangle^l\partial_z^\lambda v^{B,0}\|_{L_T^\infty L_z^\infty}\\&\quad+\sum_{\lambda=0}^{1}\|\langle z\rangle^l\partial_z^\lambda\partial_tv^{B,0}\|_{L_T^\infty L_z^\infty}\le C(T)\mathrm{e}^{c(v_*,T)}v_*.\end{aligned} \tag{3.53}$$

Now let us derive estimates for $\varphi^{B,1}$. Since

$$\partial_z^\ell\mathrm{e}^{v^{B,0}}=\sum_{\substack{\ell_1+\cdots+\ell_r=\ell\\1\le\ell_1\le\cdots\le\ell_r,\,1\le r\le l}}C_r\mathrm{e}^{v^{B,0}}\partial_z^{\ell_1}v^{B,0}\cdots\partial_z^{\ell_r}v^{B,0},\quad\ell\ge1,$$

for some constant C_r independent of v_* and T, we get, thanks to (3.28) and (3.53),

(3.54)

$$\begin{aligned}&\left\|\langle z\rangle^l(\mathrm{e}^{v^{B,0}}-1)\right\|_{L_T^2H_z^6}^2\\&\quad\le\sum_{\substack{\ell_1+\cdots+\ell_r\le6\\1\le\ell_1\le\cdots\le\ell_r,\,1\le r\le6}}C_r\|\langle z\rangle^l\partial_z^{\ell_1}v^{B,0}\cdots\partial_z^{\ell_r}v^{B,0}\|_{L_T^2L_z^2}^2+c_0\|\langle z\rangle^l(\mathrm{e}^{v^{B,0}}-1)\|_{L_T^2L_z^2}^2\\&\quad\le\sum_{\substack{\ell_1+\cdots+\ell_r\le6\\1\le\ell_1\le\cdots\le\ell_r,\,1\le r\le6}}\int_0^T\|\partial_z^{\ell_1}v^{B,0}\|_{L_z^\infty}^2\cdots\|\partial_z^{\ell_{r-1}}v^{B,0}\|_{L_z^\infty}^2\|\langle z\rangle^l\partial_z^{\ell_r}v^{B,0}\|_{L_z^2}^2\mathrm{d}t\\&\qquad+c_0\|\langle z\rangle^lv^{B,0}\|_{L_T^2L_z^2}^2\\&\quad\le c(v_*,T)v_*\sum_{\ell=1}^{6}\|\langle z\rangle^l\partial_z^\ell v^{B,0}\|_{L_T^2L_z^2}^2+\|\langle z\rangle^lv^{B,0}\|_{L_T^2L_z^2}^2\le c(v_*,T)\|\langle z\rangle^lv^{B,0}\|_{L_T^2H_z^6}^2\\&\quad\le c(v_*,T)v_*^2.\end{aligned}$$

Similarly, we have for any $l\in\mathbb{N}$ that $\langle z\rangle^l(\mathrm{e}^{v^{B,0}}-1)\in L_T^\infty H^5$ with

$$\|\langle z\rangle^l(\mathrm{e}^{v^{B,0}}-1)\|_{L_T^\infty H_z^4}\le c(v_*,T)v_*. \tag{3.55}$$

Noting that

$$\partial_t^k \mathrm{e}^{v^{B,0}} = \sum_{\substack{\ell_1+\cdots\ell_r=k\\ 1\le\ell_1\le\cdots\le\ell_r,\,1\le r\le k}} C_r \mathrm{e}^{v^{B,0}} \partial_t^{\ell_1} v^{B,0}\cdots\partial_t^{\ell_r} v^{B,0} \quad \text{for } k\ge 1,$$

with C_r being a constant independent of v_* and ε, similar to (3.54), we get for $k = 1,2,3$ that

$$\begin{aligned}
\|\langle z\rangle^l \partial_t^k \mathrm{e}^{v^{B,0}}\|^2_{L^2_T H^{6-2k}} \le& \sum_{\substack{\ell_1+\cdots\ell_r=k\\ 1\le\ell_1\le\cdots\le\ell_r,\,1\le r\le k}} c(v_*,T)\|\langle z\rangle^l \partial_z(\mathrm{e}^{v^{B,0}})\partial_t^{\ell_1} v^{B,0}\cdots\partial_t^{\ell_r} v^{B,0}\|^2_{L^2_T H_z^{5-2k}}\\
&+ \sum_{\substack{\ell_1+\cdots\ell_r=k\\ 1\le\ell_1\le\cdots\le\ell_r,\,1\le r\le k}} c(v_*,T)\|\langle z\rangle^l \partial_t^{\ell_1} v^{B,0}\cdots\partial_t^{\ell_r} v^{B,0}\|^2_{L^2_T H_z^{6-2k}}\\
\le& \sum_{j=1}^k c(v_*,T)\|\langle z\rangle^l \partial_t^j v^{B,0}\|^2_{L^2_T H_z^{5-2k}} \|\langle z\rangle^l \partial_z(\mathrm{e}^{v^{B,0}})\|^2_{L^\infty_T H_z^{5-2k}}\\
&+ \sum_{j=1}^k c(v_*,T)\|\langle z\rangle^l \partial_t^j v^{B,0}\|^2_{L^2_T H_z^{6-2k}} \le \sum_{j=1}^k c(v_*,T)\|\langle z\rangle^l \partial_t^j v^{B,0}\|^2_{L^2_T H_z^{6-2j}}\\
\le& c(v_*,T)v_*^2,
\end{aligned}$$

where we have used (3.28), (3.53), (3.54), (3.55), and the fact

$$\|\langle x\rangle^l fg\|_{H^k(\mathbb{R}_+)} \le c_0 \|\langle x\rangle^l f\|_{H^k(\mathbb{R}_+)} \|\langle x\rangle^l g\|_{H^k(\mathbb{R}_+)} \tag{3.56}$$

for any $l\in\mathbb{N}$ and any integer $k\ge 1$, provided $\langle x\rangle^l f$, $\langle x\rangle^l g \in H^k(\mathbb{R}_+)$. Therefore we now have for $l\in\mathbb{N}$ that

(3.57)

$$\|\langle z\rangle^l \partial_t^k (\mathrm{e}^{v^{B,0}}-1)\|^2_{L^2_T H_z^{6-2k}} \le c(v_*,T)\sum_{j=0}^k \|\langle z\rangle^l \partial_t^j v^{B,0}\|^2_{L^2_T H_z^{6-2k}} \le c(v_*,T)v_*^2, \quad 0\le k\le 3,$$

where $c(v_*,T)>0$ is a constant as stated in section 2. With (2.13), (3.36), (3.57), and the Hölder inequality, we derive for $k=0,1,2$ that

(3.58)

$$\begin{aligned}
\|\langle z\rangle^l \partial_t^k \varphi^{B,1}\|^2_{L^2_T H_z^{7-2k}} \le& c_0 \sum_{i=0}^k \left\| \langle z\rangle^l \int_z^\infty \partial_t^{k-i}(\varphi_x^{I,0}(0,t)+M)\partial_t^i \left(\mathrm{e}^{v^{B,0}}-1\right)\mathrm{d}y\right\|^2_{L^2_T H_z^{7-2k}}\\
\le& c_0 \sum_{i=0}^k \|\partial_t^{k-i}\left(\varphi_x^{I,0}(0,t)+M\right)\|^2_{L^\infty(0,T)} \left\| \int_z^\infty \partial_t^i \left(\mathrm{e}^{v^{B,0}}-1\right)\mathrm{d}y\right\|^2_{L^2_T H_z^{7-2k}}\\
\le& c(v_*,T)\sum_{i=0}^k \left(1+\int_{\mathbb{R}_+}\int_z^\infty \langle y\rangle^{-4}\mathrm{d}y\mathrm{d}z\right) \|\langle z\rangle^{l+2}\partial_t^i(\mathrm{e}^{v^{B,0}}-1)\|^2_{L^2_T H_z^{6-2k}} \le c(v_*,T)v_*^2.
\end{aligned}$$

For $k=3$, we get

$$\begin{aligned}\|\langle z\rangle^l\partial_t^3\varphi^{B,1}\|^2_{L^2_TH^1_z} &\le c_0\sum_{i=0}^3\left\|\langle z\rangle^l\int_z^\infty\partial_t^{3-i}(\varphi_x^{I,0}(0,t)+M)\partial_t^i\left(\mathrm{e}^{v^{B,0}}-1\right)\mathrm{d}y\right\|^2_{L^2_TH^1_z}\\ &\le c_0\sum_{i=1}^3\|\partial_t^{3-i}\left(\varphi_x^{I,0}(0,t)+M\right)\|^2_{L^\infty(0,T)}\left\|\int_z^\infty\partial_t^i\left(\mathrm{e}^{v^{B,0}}-1\right)\mathrm{d}y\right\|^2_{L^2_TH^1_z}\\ &\quad+c_0\left\|\langle z\rangle^l\int_z^\infty\partial_t^3(\varphi_x^{I,0}(0,t)+M)\left(\mathrm{e}^{v^{B,0}}-1\right)\mathrm{d}y\right\|^2_{L^2_TH^1_z}=:A_1+A_2,\end{aligned}$$

where A_1 can be estimated by arguments similar to those proving (3.58):

$$A_1\le\sum_{i=1}^3c(v_*,T)\left(1+\int_{\mathbb{R}_+}\int_z^\infty\langle y\rangle^{-4}\mathrm{d}y\mathrm{d}z\right)\|\langle z\rangle^{l+2}\partial_t^i(\mathrm{e}^{v^{B,0}}-1)\|^2_{L^2_TL^2_z}\le c(v_*,T)v_*^2,$$

where (3.36) and (3.57) have been used. We proceed to estimate A_2. It follows from (3.35), (3.55), and the Hölder inequality that

$$\begin{aligned}A_2&\le c(v_*,T)\left(1+\int_{\mathbb{R}_+}\int_z^\infty\langle y\rangle^{-4}\mathrm{d}y\mathrm{d}z\right)\|\langle z\rangle^{l+2}(\mathrm{e}^{v^{B,0}}-1)\|^2_{L^\infty_TH^1_z}\|\partial_t^3\varphi_x^{I,0}(0,t)\|^2_{L^2(0,T)}\\ &\le c(v_*,T)v_*^2.\end{aligned}$$

Therefore we get for any $l\in\mathbb{N}$ that

$$\|\langle z\rangle^l\partial_t^k\varphi^{B,1}\|_{L^2_TH_z^{7-2k}}\le c(v_*,T)v_*,\quad k=0,1,2,3.$$

The proof is complete. ∎

The following lemma gives the regularity of $(\varphi^{b,1},v^{b,1})$ which can be proved by arguments similar to those proving Lemma 3.3.

LEMMA 3.4. *Assume the conditions in Lemma* 3.2 *hold. Then for any* $T>0$, *the problem* (2.14), (2.15) *admits a unique solution* $(v^{b,0},\varphi^{b,1})$ *on* $[0,T]$ *such that* $0\le v^{b,0}\le v_*$,

(3.59)

$$\|\langle z\rangle^l\partial_t^kv^{b,0}\|_{L^2_TH_\xi^{6-2k}}\le K_0(T,v_*)v_*,\ \|\langle z\rangle^l\partial_t^k\varphi^{b,1}\|_{L^2_TH_\xi^{7-2k}}\le c(v_*,T)v_*,\ k=0,1,2,3,$$

$$\|\langle\xi\rangle^l\partial_t^kv^{b,0}\|_{L^\infty_TH_\xi^{4-2k}}+\sum_{\lambda=0}^1\sum_{\ell=0}^{3-2\lambda}\|\langle\xi\rangle^l\partial_t^\lambda\partial_\xi^\ell v^{b,0}\|_{L^\infty_TL^\infty_\xi}\le K_0(v_*,T)v_*,\tag{3.60}$$

where $K_0(T,v_*)>0$ *is as in Lemma* 3.3, *and* $c(v_*,T)$ *is as stated in section* 2.

We next turn to the existence and regularity of the outer-layer profile $(\varphi^{I,1},v^{I,1})$.

LEMMA 3.5. *Assume the conditions in Lemma* 3.2 *hold, and let* $(v^{B,0},\varphi^{B,1})$ *and* $(v^{b,0},\varphi^{b,1})$ *be the solutions obtained in Lemmas* 3.3 *and* 3.4, *respectively. Then for any* $T>0$, *the problem* (2.11) *admits a unique classical solution* $(\varphi^{I,1},v^{I,1})$ *on* $[0,T]$ *satisfying*

$$\|\partial_t^k\varphi^{I,1}\|_{L^2_TH^{6-2k}}\le c(v_*,T)\quad\text{for } k=0,1,2,3,\tag{3.61a}$$

$$\|v^{I,1}\|_{L^\infty_TH^5}+\|\partial_t^kv^{I,1}\|_{L^2_TH^{7-2k}}\le c(v_*,T)\quad\text{for } k=1,2,3.\tag{3.61b}$$

Proof. The local existence and uniqueness of solutions to the problem (2.11) on $(\varphi^{I,1}, v^{I,1})$ can be proved by the classical PDE theory for linear parabolic equations (cf. [16, Section 7.1]) along with the Banach's fixed point theorem. In the following, we will devote ourselves to establishing some a priori estimates from which the global existence and the desired regularity of the solution follow.

Denote $b(x,t) := x\varphi^{b,1}(0,t) + (1-x)\varphi^{B,1}(0,t)$ and $\tilde{\varphi} := \varphi^{I,1} + b(x,t)$ with

$$\begin{cases} \varphi^{B,1}(0,t) = -\displaystyle\int_0^\infty (\varphi_x^{I,0}(0,t) + M)\left(\mathrm{e}^{v^{B,0}(y,t)} - 1\right)\mathrm{d}y, \\ \varphi^{b,1}(0,t) = \displaystyle\int_{-\infty}^0 (\varphi_x^{I,0}(1,t) + M)\left(\mathrm{e}^{v^{b,0}(y,t)} - 1\right)\mathrm{d}y. \end{cases}$$

Then we deduce from (2.11) that

$$\begin{cases} \tilde{\varphi}_t = \tilde{\varphi}_{xx} - (\varphi_x^{I,0} + M)v_x^{I,1} - \tilde{\varphi}_x v_x^{I,0} + f_1(x,t), \\ v_t^{I,1} = -\left(\varphi_x^{I,0} + M\right)v^{I,1} - \tilde{\varphi}_x v^{I,0} + f_2(x,t), \\ \tilde{\varphi}(0,t) = \tilde{\varphi}(1,t) = 0, \\ (\tilde{\varphi}, v^{I,1})(x,0) = (0,0), \end{cases} \tag{3.62}$$

where the fact $v^{B,0}(z,0) = v^{b,0}(\xi,0) = 0$ has been used, and $f_i(x,t)\ (i=1,2)$ are given by

$$f_1(x,t) := b_t + b_x v_x^{I,0}, \quad f_2(x,t) := b_x v^{I,0}, \quad k = 0,1. \tag{3.63}$$

To ensure the desired regularity of the solution, it is necessary to derive some estimates for the source terms involved. By (3.27), (3.28), (3.35), (3.36), and (3.57), we deduce for $k = 0,1,2$ that

$$\begin{aligned} &\|\partial_t^k \varphi^{B,1}(0,t)\|_{L^2(0,T)}^2 \\ &\quad \le c_0 \sum_{j=0}^k \int_0^T \left| \int_0^\infty \partial_t^{k-j}(\varphi_x^{I,0}(0,t) + M)\partial_t^j\left(\mathrm{e}^{v^{B,0}(y,t)} - 1\right)\mathrm{d}y\right|^2 \mathrm{d}t \\ &\quad \le c_0 \sum_{j=0}^k \|\partial_t^{k-j}(\varphi_x^{I,0}(0,t) + M)\|_{L^\infty(0,T)} \int_0^T \left|\int_0^\infty \partial_t^j\left(\mathrm{e}^{v^{B,0}(y,t)} - 1\right)\mathrm{d}y\right|^2 \mathrm{d}t \\ &\quad \le c(v_*, T)\sum_{j=0}^k \int_0^\infty \langle y\rangle^{-2}\mathrm{d}y \|\langle z\rangle \partial_t^j (\mathrm{e}^{v^{B,0}} - 1)\|_{L_T^2 L_z^2}^2 \\ &\quad \le c(v_*, T), \end{aligned}$$

and for $k = 3$ that

$$\begin{aligned} \|\partial_t^3 \varphi^{B,1}(0,t)\|_{L^2(0,T)}^2 &\le c_0 \sum_{j=1}^3 \int_0^T \left|\int_0^\infty \partial_t^{3-j}(\varphi_x^{I,0}(0,t) + M)\partial_t^j\left(\mathrm{e}^{v^{B,0}(y,t)} - 1\right)\mathrm{d}y\right|^2 \mathrm{d}t \\ &\quad + c_0 \int_0^T \left|\int_0^\infty \partial_t^3 \varphi_x^{I,0}(0,t)\left(\mathrm{e}^{v^{B,0}(y,t)} - 1\right)\mathrm{d}y\right|^2 \mathrm{d}t \\ &\le c(v_*, T)\int_{\mathbb{R}_+} \langle y\rangle^{-2}\mathrm{d}y \|\partial_t^3 \varphi_x^{I,0}(0,t)\|_{L^2(0,T)}^2 \|\langle z\rangle (\mathrm{e}^{v^{B,0}} - 1)\|_{L_T^\infty L_z^2}^2 \\ &\quad + c(v_*, T)\sum_{j=1}^3 \int_{\mathbb{R}_+} \langle y\rangle^{-2}\mathrm{d}y \|\langle z\rangle \partial_t^j(\mathrm{e}^{v^{B,0}} - 1)\|_{L_T^2 L_z^2}^2 \\ &\le c(v_*, T). \end{aligned}$$

Thus it holds for $k=0,1,2,3$ that

$$\|\partial_t^k \varphi^{B,1}(0,t)\|^2_{L^2(0,T)} \le c(v_*,T). \tag{3.64}$$

Similarly, by (3.27), (3.35), and (3.59), we have for $\varphi^{b,0}(0,t)$ that

$$\|\partial_t^k \varphi^{b,1}(0,t)\|^2_{L^2(0,T)} \le c(v_*,T) \quad \text{for } k=0,1,2,3. \tag{3.65}$$

With (3.27), (3.64), and (3.65), recalling the definitions of f_1 and f_2 in (3.63), we get for $k=0,1,2$ that

$$\begin{aligned} \|\partial_t^k f_1\|^2_{L^2_T H^{4-2k}} &\le c(v_*,T)\left(\|\partial_t^{k+1}\varphi^{B,1}(0,t)\|^2_{L^2(0,T)} + \|\partial_t^{k+1}\varphi^{b,1}(0,t)\|^2_{L^2(0,T)}\right) \\ &\quad + c_0\sum_{j=0}^{k}\left(\|\partial_t^j\varphi^{B,1}(0,t)\|^2_{L^2(0,T)} + \|\partial_t^j\varphi^{b,1}(0,t)\|^2_{L^2(0,T)}\right) \\ &\quad \times \|\partial_t^{k-j} v^{I,0}\|^2_{L^\infty_T H^{5-2k}} \le c(v_*,T), \end{aligned} \tag{3.66a}$$

$$\begin{aligned} \|\partial_t^k f_2\|^2_{L^2(0,T;H^{5-2k})} &\le c_0\sum_{j=0}^{k}\left(\|\partial_t^j\varphi^{B,1}(0,t)\|^2_{L^2(0,T)} + \|\partial_t^j\varphi^{b,1}(0,t)\|^2_{L^2(0,T)}\right) \\ &\quad \times \|\partial_t^{k-j} v^{I,0}\|^2_{L^\infty_T H^{5-2k}} \le c(v_*,T). \end{aligned} \tag{3.66b}$$

Now we are ready to establish estimates for the solution. Multiplying the first equation in (3.62) by $\tilde{\varphi}$ and integrating the resulting equation over $\mathcal{I}$, we have

(3.67)

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{I}}\tilde{\varphi}^2\mathrm{d}x + \int_{\mathcal{I}}\tilde{\varphi}_x^2\mathrm{d}x \\ &= -\int_{\mathcal{I}}(\varphi_x^{I,0}+M)v_x^{I,1}\tilde{\varphi}\mathrm{d}x - \int_{\mathcal{I}}\tilde{\varphi}_x v_x^{I,0}\tilde{\varphi}\mathrm{d}x + \int_{\mathcal{I}} f_1\tilde{\varphi}\mathrm{d}x \\ &= \int_{\mathcal{I}}\varphi_{xx}^{I,0}v^{I,1}\tilde{\varphi}\mathrm{d}x + \int_{\mathcal{I}}(\varphi_x^{I,0}+M)v^{I,1}\tilde{\varphi}_x\mathrm{d}x + \|v_x^{I,0}\|_{L^\infty}\|\tilde{\varphi}_x\|_{L^2}\|\tilde{\varphi}\|_{L^2} + \|f_1\|_{L^2}\|\tilde{\varphi}\|_{L^2} \\ &\le \|\varphi_{xx}^{I,0}\|_{L^\infty}\|v^{I,1}\|_{L^2}\|\tilde{\varphi}\|_{L^2} \\ &\quad + \frac{1}{16}\|\tilde{\varphi}_x\|^2_{L^2} + c(v_*,T)\left(\|\tilde{\varphi}_x\|_{L^2}\|v^{I,1}\|_{L^2} + \|f_1\|^2_{L^2} + \|\tilde{\varphi}\|^2_{L^2}\right) \\ &\le \frac{1}{8}\|\tilde{\varphi}_x\|^2_{L^2} + c(v_*,T)\left(\|\tilde{\varphi}\|^2_{L^2} + \|v^{I,1}\|^2_{L^2}\right) + c(v_*,T)\|f_1\|^2_{L^2}, \end{aligned}$$

where (3.27), integration by parts, and the Cauchy–Schwarz inequality have been used. On the other hand, testing the second equation in (3.62) against $v^{I,1}$, we have

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{I}}|v^{I,1}|^2\mathrm{d}x + \int_{\mathcal{I}}\left(\varphi_x^{I,0}+M\right)|v^{I,1}|^2\mathrm{d}x &= \int_{\mathcal{I}}\left(-\tilde{\varphi}_x v^{I,0} + f_2\right)v^{I,1}\mathrm{d}x \\ &\le c_0\int_{\mathcal{I}}|v^{I,1}|^2\mathrm{d}x + c_0\|v^{I,0}\|^2_{L^\infty}\|\tilde{\varphi}_x\|^2_{L^2} + c_0\|f_2\|^2_{L^2} \\ &\le c_0\int_{\mathcal{I}}|v^{I,1}|^2\mathrm{d}x + c(v_*,T)\|\tilde{\varphi}_x\|^2_{L^2} + c_0\|f_2\|^2_{L^2}, \end{aligned} \tag{3.68}$$

where we have used (3.27) and the Cauchy–Schwarz inequality. Combining (3.67) with (3.68) implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{I}}\left(\tilde{\varphi}^2+|v^{I,1}|^2\right)\mathrm{d}x \\ +\int_{\mathcal{I}}\tilde{\varphi}_x^2\mathrm{d}x \le c(v_*,T)\left(\|\tilde{\varphi}\|_{L^2}^2+\|v^{I,1}\|_{L^2}^2\right)+c(v_*,T)\left(\|f_1\|_{L^2}^2+\|f_2\|_{L^2}^2\right),$$

where we have used the fact $\varphi_x^{I,0}+M\ge 0$ from (3.27). This, along with (3.66) and the Gronwall inequality, immediately yields for any $t\in(0,T]$ that

$$\int_{\mathcal{I}}\left(\tilde{\varphi}^2+|v^{I,1}|^2\right)(\cdot,t)\mathrm{d}x+\int_0^t\int_{\mathcal{I}}\tilde{\varphi}_x^2\mathrm{d}x\mathrm{d}\tau \le c(v_*,T). \tag{3.69}$$

Multiplying the first equation in (3.62) by $\tilde{\varphi}_t$ followed by an integration over $\mathcal{I}$, it holds that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{I}}\tilde{\varphi}_x^2\mathrm{d}x+\int_{\mathcal{I}}\tilde{\varphi}_t^2\mathrm{d}x=-\int_{\mathcal{I}}\tilde{\varphi}_x v_x^{I,0}\tilde{\varphi}_t\mathrm{d}x+\int_{\mathcal{I}}f_1\tilde{\varphi}_t\mathrm{d}x-\int_{\mathcal{I}}(\varphi_x^{I,0}+M)v_x^{I,1}\tilde{\varphi}_t\mathrm{d}x. \tag{3.70}$$

By (3.27) and the Cauchy–Schwarz inequality, we have

$$\begin{aligned}-\int_{\mathcal{I}}\tilde{\varphi}_x v_x^{I,0}\tilde{\varphi}_t\mathrm{d}x+\int_{\mathcal{I}}f_1\tilde{\varphi}_t\mathrm{d}x &\le \frac{1}{4}\int_{\mathcal{I}}\tilde{\varphi}_t^2\mathrm{d}x+c_0\|v_x^{I,0}\|_{L^\infty}^2\|\tilde{\varphi}_x\|_{L^2}^2+c_0\|f_1\|_{L^2}^2\\ &\le \frac{1}{4}\|\tilde{\varphi}_t\|_{L^2}^2+c(v_*,T)\|\tilde{\varphi}_x\|_{L^2}^2+c_0\|f_1\|_{L^2}^2,\end{aligned} \tag{3.71}$$

where we have used the fact $\|v_x^{I,0}\|_{L_T^\infty L^\infty}\le c(v_*,T)$ due to (3.27). For the last term on the right-hand side of (3.70), we get by virtue of integration by parts and the Cauchy–Schwarz inequality that

$$\begin{aligned}&-\int_{\mathcal{I}}(\varphi_x^{I,0}+M)v_x^{I,1}\tilde{\varphi}_t\mathrm{d}x=\int_{\mathcal{I}}(\varphi_x^{I,0}+M)v^{I,1}\tilde{\varphi}_{tx}\mathrm{d}x+\int_{\mathcal{I}}\varphi_{xx}^{I,0}v^{I,1}\tilde{\varphi}_t\mathrm{d}x\\
&\quad=\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{I}}(\varphi_x^{I,0}+M)v^{I,1}\tilde{\varphi}_x\mathrm{d}x-\int_{\mathcal{I}}\varphi_{xt}^{I,0}v^{I,1}\tilde{\varphi}_x\mathrm{d}x-\int_{\mathcal{I}}(\varphi_x^{I,0}+M)v_t^{I,1}\tilde{\varphi}_x\mathrm{d}x\\
&\qquad+\int_{\mathcal{I}}\varphi_{xx}^{I,0}v^{I,1}\tilde{\varphi}_t\mathrm{d}x\\
&\quad\le\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{I}}(\varphi_x^{I,0}+M)v^{I,1}\tilde{\varphi}_x\mathrm{d}x+\|\varphi_{xt}^{I,0}\|_{L^\infty}\|v^{I,1}\|_{L^2}\|\tilde{\varphi}_x\|_{L^2}+c(v_*,T)\|v_t^{I,1}\|_{L^2}\|\tilde{\varphi}_x\|_{L^2}\\
&\qquad+\|\varphi_{xx}^{I,0}\|_{L^\infty}\|v^{I,1}\|_{L^2}\|\tilde{\varphi}_t\|_{L^2}\\
&\quad\le\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{I}}(\varphi_x^{I,0}+M)v^{I,1}\tilde{\varphi}_x\mathrm{d}x+\frac{1}{8}\|\tilde{\varphi}_t\|_{L^2}^2+c(v_*,T)\|\tilde{\varphi}_x\|_{L^2}^2\\
&\qquad+c(v_*,T)\left(\|v_t^{I,1}\|_{L^2}^2+\|v^{I,1}\|_{L^2}^2\right),\end{aligned} \tag{3.72}$$

where we have used $\|\varphi_{xx}^{I,0}\|_{L_T^\infty L^\infty}+\|\varphi_{xt}^{I,0}\|_{L_T^\infty L^\infty}\le c(v_*,T)$ due to (3.27) and Proposition C.4. Collecting (3.71) and (3.72), we thus have from (3.70) that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{I}}\tilde{\varphi}_x^2\mathrm{d}x-\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{I}}(\varphi_x^{I,0}+M)v^{I,1}\tilde{\varphi}_x\mathrm{d}x+\frac{1}{2}\int_{\mathcal{I}}\tilde{\varphi}_t^2\mathrm{d}x\\
&\quad\le c(v_*,T)(\|\tilde{\varphi}_x\|_{L^2}^2+\|v^{I,1}\|_{L^2}^2)+c(v_*,T)\|f_1\|_{L^2}^2+c(v_*,T)\|v_t^{I,1}\|_{L^2}^2.\end{aligned} \tag{3.73}$$

To control the term on $v_t^{I,1}$ on the right-hand side of (3.73), we test the second equation in (3.62) against $v_t^{I,1}$ and deduce that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{I}}\left(\varphi_x^{I,0}+M\right)|v^{I,1}|^2\mathrm{d}x+\int_{\mathcal{I}}|v_t^{I,1}|^2\mathrm{d}x\\
&\quad=\frac{1}{2}\int_{\mathcal{I}}\varphi_{xt}^{I,0}|v^{I,1}|^2\mathrm{d}x+\int_{\mathcal{I}}\left(-\tilde{\varphi}_x v^{I,0}+f_2\right)v_t^{I,1}\mathrm{d}x\\
&\quad\le\frac{1}{2}\int_{\mathcal{I}}|v_t^{I,1}|^2\mathrm{d}x+c_0\|\varphi_{xt}^{I,0}\|_{L^\infty}\|v^{I,1}\|_{L^2}^2+c_0\|v^{I,0}\|_{L^\infty}^2\|\tilde{\varphi}_x\|_{L^2}^2+c_0\|f_2\|_{L^2}^2\\
&\quad\le\frac{1}{2}\int_{\mathcal{I}}|v_t^{I,1}|^2\mathrm{d}x+c(v_*,T)\|v^{I,1}\|_{L^2}^2+c(v_*,T)\|\tilde{\varphi}_x\|_{L^2}^2+c_0\|f_2\|_{L^2}^2,
\end{aligned}\tag{3.74}$$

where we have used (3.36). Therefore we get from (3.73) and (3.74) that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{I}}\left(\tilde{\varphi}_x^2+(\varphi_x^{I,0}+M)|v^{I,1}|^2\right)(\cdot,t)\mathrm{d}x+\int_{\mathcal{I}}\left(\tilde{\varphi}_t^2+|v_t^{I,1}|^2\right)\mathrm{d}x\\
&\quad\le c(v_*,T)\left(\|\tilde{\varphi}_x\|_{L^2}^2+\|v^{I,1}\|_{L^2}^2\right)+c(v_*,T)\left(\|f_1\|_{L^2}^2+\|f_2\|_{L^2}^2\right)\\
&\qquad+c(v_*,T)\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{I}}(\varphi_x^{I,0}+M)v^{I,1}\tilde{\varphi}_x\mathrm{d}x.
\end{aligned}$$

Integrating the above inequality over $(0,t)$ for any $t\in(0,T]$ yields that

$$\begin{aligned}
&\int_{\mathcal{I}}\left(\tilde{\varphi}_x^2+(\varphi_x^{I,0}+M)|v^{I,1}|^2\right)(\cdot,t)\mathrm{d}x+\int_0^t\int_{\mathcal{I}}\left(\tilde{\varphi}_\tau^2+|v_\tau^{I,1}|^2\right)\mathrm{d}x\mathrm{d}\tau\\
&\quad\le c(v_*,T)+c(v_*,T)\int_{\mathcal{I}}(\varphi_x^{I,0}+M)v^{I,1}\tilde{\varphi}_x\mathrm{d}x+c(v_*,T)\int_0^t\left(\|f_1\|_{L^2}^2+\|f_2\|_{L^2}^2\right)\mathrm{d}\tau\\
&\qquad+c(v_*,T)\int_0^t\left(\|\tilde{\varphi}_x\|_{L^2}^2+\|v^{I,1}\|_{L^2}^2\right)\mathrm{d}\tau\\
&\quad\le\frac{1}{2}\int_{\mathcal{I}}\tilde{\varphi}_x^2\mathrm{d}x+c(v_*,T)+c(v_*,T)\int_{\mathcal{I}}|v^{I,1}|^2\mathrm{d}x+c(v_*,T)\int_0^t\left(\|\tilde{\varphi}_x\|_{L^2}^2+\|v^{I,1}\|_{L^2}^2\right)\mathrm{d}\tau\\
&\quad\le\frac{1}{2}\int_{\mathcal{I}}\tilde{\varphi}_x^2\mathrm{d}x+c(v_*,T)+c(v_*,T)\int_0^t\|\tilde{\varphi}_x\|_{L^2}^2\mathrm{d}\tau,
\end{aligned}$$

where we have used (3.66), (3.69), and the Cauchy–Schwarz inequality. We thus have

$$\int_{\mathcal{I}}\tilde{\varphi}_x^2(\cdot,t)\mathrm{d}x+\int_0^t\int_{\mathcal{I}}\left(\tilde{\varphi}_\tau^2+|v_\tau^{I,1}|^2\right)\mathrm{d}x\mathrm{d}\tau\le c(v_*,T)+c(v_*,T)\int_0^t\|\tilde{\varphi}_x\|_{L^2}^2\mathrm{d}\tau\le c(v_*,T),\tag{3.75}$$

where we have used (3.27) and (3.69). We proceed to derive estimates for $v_x^{I,1}$. Differentiating the second equation in (3.62) with respect to x leads to

$$v_{tx}^{I,1}=-(\varphi_x^{I,0}+M)v_x^{I,1}-\varphi_{xx}^{I,0}v^{I,1}-\tilde{\varphi}_{xx}v^{I,0}-\tilde{\varphi}_x v_x^{I,0}+\partial_x f_2.\tag{3.76}$$

Multiplying (3.76) by $v_x^{I,1}$ followed by an integration over $\mathcal{I}$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{I}}|v_x^{I,1}|^2\mathrm{d}x+\int_{\mathcal{I}}(\varphi_x^{I,0}+M)|v_x^{I,1}|^2\mathrm{d}x\\
&\quad=-\int_{\mathcal{I}}\varphi_{xx}^{I,0}v^{I,1}v_x^{I,1}\mathrm{d}x-\int_{\mathcal{I}}\tilde{\varphi}_{xx}v^{I,0}v_x^{I,1}\mathrm{d}x-\int_{\mathcal{I}}\tilde{\varphi}_x v_x^{I,0}v_x^{I,1}\mathrm{d}x+\int_{\mathcal{I}}\partial_x f_2 v_x^{I,1}\mathrm{d}x\\
&\quad\le\|\varphi_{xx}^{I,0}\|_{L^\infty}\|v^{I,1}\|_{L^2}\|v_x^{I,1}\|_{L^2}+\|v^{I,0}\|_{L^\infty}\|\tilde{\varphi}_{xx}\|_{L^2}\|v_x^{I,1}\|_{L^2}\\
&\qquad+\|v_x^{I,0}\|_{L^\infty}\|\tilde{\varphi}_x\|_{L^2}\|v_x^{I,1}\|_{L^2}+\|\partial_x f_2\|_{L^2}\|v_x^{I,1}\|_{L^2}\\
&\quad\le\frac{1}{8}\|\tilde{\varphi}_{xx}\|_{L^2}^2+c_0\left(\|\varphi_{xx}^{I,0}\|_{L^\infty}^2+\|v^{I,0}\|_{L^\infty}^2+\|v_x^{I,0}\|_{L^\infty}^2+\|v^{I,0}\|_{L^2}^2\right)\|v_x^{I,1}\|_{L^2}^2\\
&\qquad+c_0\left(\|v^{I,1}\|_{L^2}^2+\|\tilde{\varphi}_x\|_{L^2}^2+\|\partial_x f_2\|_{L^2}^2\right)\\
&\quad\le\frac{1}{8}\|\tilde{\varphi}_{xx}\|_{L^2}^2+c(v_*,T)\|v_x^{I,1}\|_{L^2}^2+c(v_*,T)+c(v_*,T)\|\partial_x f_2\|_{L^2}^2, \qquad (3.77)
\end{aligned}$$

where we have used (3.27), (3.75), Proposition C.4, and the Cauchy–Schwarz inequality. On the other hand, with (3.27), (3.69), and (3.75), we deduce from $(3.62)_1$ that

$$\begin{aligned}
\|\tilde{\varphi}_{xx}\|_{L^2}^2&\le c_0\|\tilde{\varphi}_t\|_{L^2}^2+c_0\|(\varphi_x^{I,0}+M)v_x^{I,1}\|_{L^2}^2+c_0\|v_x^{I,0}\|_{L^\infty}^2\|\tilde{\varphi}_x\|_{L^2}^2+c_0\|f_1\|_{L^2}^2\\
&\le c(v_*,T)\left(1+\|\tilde{\varphi}_t\|_{L^2}^2+\|v_x^{I,1}\|_{L^2}^2+\|f_1\|_{L^2}^2\right),
\end{aligned}$$

which, together with (3.27) and (3.77), yields that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{I}}|v_x^{I,1}|^2\mathrm{d}x+\|\tilde{\varphi}_{xx}\|_{L^2}^2\le c(v_*,T)\left(1+\|\tilde{\varphi}_t\|_{L^2}^2+\|v_x^{I,1}\|_{L^2}^2+\|f_1\|_{L^2}^2+\|\partial_x f_2\|_{L^2}^2\right). \qquad (3.78)$$

Applying the Gronwall inequality to (3.78), by virtue of (3.66) and (3.75), we then arrive at

$$\int_{\mathcal{I}}|v_x^{I,1}|^2(\cdot,t)\mathrm{d}x+\int_0^t\|\tilde{\varphi}_{xx}\|_{L^2}^2\mathrm{d}\tau\le c(v_*,T) \qquad (3.79)$$

for any $t\in(0,T]$. This, along with (3.27), (3.66b), and (3.76), further gives that $\|\partial_t v_x^{I,1}\|_{L_T^2L^2}\le c(v_*,T)$. Denote $\psi:=\tilde{\varphi}_t$ and $w:=v_t^{I,1}$. Then in view of (3.62) and the compatibility conditions of initial data, we have

$$\begin{cases}
\psi_t=\psi_{xx}-(\varphi_x^{I,0}+M)w_x^{I,1}-\psi_x v_x^{I,0}+\tilde{f}_1(x,t),\\
w_t=-\left(\varphi_x^{I,0}+M\right)w^{I,1}-\psi_x v^{I,0}+\tilde{f}_2(x,t),\\
\psi(0,t)=\psi(1,t)=0,\\
(\psi,w)(x,0)=(\tilde{\varphi}_t,v_t)|_{t=0}=(0,0),
\end{cases} \qquad (3.80)$$

where $\tilde{f}_i(x,t)\,(i=1,2)$ are given by

$$\tilde{f}_1(x,t)=-\varphi_{xt}^{I,0}v_x^{I,1}-\tilde{\varphi}_x v_{xt}^{I,0}+\partial_t f_1(x,t),\quad \tilde{f}_2(x,t)=-\varphi_{xt}^{I,0}v^{I,1}-\tilde{\varphi}_x v_t^{I,0}+\partial_t f_2(x,t).$$

Thanks to (3.27), (3.66), (3.75), (3.79), and Proposition C.4, we deduce that

$$\begin{aligned}
\|\tilde{f}_1\|_{L_T^2L^2}^2&\le\int_0^T\|\varphi_{xt}^{I,0}\|_{L^\infty}^2\|v_x^{I,1}\|_{L^2}^2\mathrm{d}t+\int_0^T\|v_{xt}^{I,0}\|_{L^\infty}^2\|\tilde{\varphi}_x\|_{L^2}^2\mathrm{d}t+\int_0^T\|\partial_t f_1(x,t)\|_{L^2}^2\mathrm{d}t\\
&\le c(v_*,T)\int_0^T\|\varphi_{xt}^{I,0}\|_{L^\infty}^2\mathrm{d}t+c(v_*,T)\int_0^T\|v_{xt}^{I,0}\|_{L^\infty}^2\mathrm{d}t+\int_0^T\|\partial_t f_1(x,t)\|_{L^2}^2\mathrm{d}t\\
&\le c(v_*,T), \qquad (3.81)
\end{aligned}$$

$$\begin{aligned}\|\tilde{f}_2\|^2_{L^2_T L^2} &\le \int_0^T \|\varphi^{I,0}_{xt}\|^2_{L^2}\|v^{I,1}\|^2_{L^\infty}\mathrm{d}t + \int_0^T \|v^{I,0}_t\|^2_{L^\infty}\|\tilde{\varphi}_x\|^2_{L^2}\mathrm{d}t + \int_0^T \|\partial_t f_2(x,t)\|^2_{L^2}\mathrm{d}t \\ &\le c(v_*,T)\int_0^T \|\varphi^{I,0}_{xt}\|^2_{L^2}\mathrm{d}t + c(v_*,T)\int_0^T \|v^{I,0}_t\|^2_{L^\infty}\mathrm{d}t + c(v_*,T) \le c(v_*,T) \qquad (3.82)\end{aligned}$$

and

$$\begin{aligned}&\|\partial_x\tilde{f}_2\|^2_{L^2_T L^2} \\ &\quad\le \int_0^T \left(\|\varphi^{I,0}_{xxt}\|^2_{L^2}\|v^{I,1}\|^2_{L^\infty} + \|\varphi^{I,0}_{xt}\|^2_{L^\infty}\|v^{I,1}_x\|^2_{L^2}\right)\mathrm{d}t \\ &\qquad + \int_0^T \left(\|\tilde{\varphi}_{xx}\|^2_{L^2}\|v^{I,0}_t\|^2_{L^\infty} + |\tilde{\varphi}_x\|^2_{L^\infty}\|v^{I,0}_{xt}\|^2_{L^2} + \|\partial_t\partial_x f_2\|^2_{L^2}\right)\mathrm{d}t \\ &\quad\le c(v_*,T)\int_0^T \left(\|\varphi^{I,0}_{xxt}\|^2_{L^2} + \|\varphi^{I,0}_x\|^2_{L^\infty} + \|\tilde{\varphi}_x\|^2_{H^1} + \|\partial_t\partial_x f_2\|^2_{L^2}\right)\mathrm{d}t \le c(v_*,T).\end{aligned}$$

Therefore by the above procedure for estimates on $(\tilde{\varphi}, v^{I,1})$, we conclude for any $t\in(0,T]$ that

$$\left(\|\psi(\cdot,t)\|^2_{H^1} + \|w(\cdot,t)\|^2_{H^1}\right) + \int_0^t \left(\|\psi_x\|^2_{H^1} + \|\psi_\tau\|^2_{L^2} + \|w_\tau\|^2_{H^1}\right)\mathrm{d}\tau \le c(v_*,T).$$

That is,

$$\left(\|\tilde{\varphi}_t(\cdot,t)\|^2_{H^1} + \|v^{I,1}_t(\cdot,t)\|^2_{H^1}\right) + \int_0^t \left(\|\tilde{\varphi}_{x\tau}\|^2_{H^1} + \|\tilde{\varphi}_{\tau\tau}\|^2_{L^2} + \|v^{I,1}_{\tau\tau}\|^2_{H^1}\right)\mathrm{d}\tau \le c(v_*,T). \qquad (3.83)$$

With (3.27), (3.66a), (3.69), (3.75), (3.79), and (3.83), we deduce from (3.62) that

$$\begin{aligned}&\int_0^t \|\partial_x^3\tilde{\varphi}\|^2_{L^2}\mathrm{d}\tau \\ &\quad\le c_0\int_0^t \|\tilde{\varphi}_{x\tau}\|^2_{L^2}\mathrm{d}\tau + c(v_*,T)\int_0^t \|v^{I,1}_{xx}\|^2_{L^2}\mathrm{d}\tau + c_0\int_0^t \|\varphi^{I,0}_{xx}\|^2_{L^\infty}\|v^{I,1}_x\|^2_{L^2}\mathrm{d}\tau \\ &\qquad + c_0\int_0^t \|v^{I,0}_x\|^2_{L^\infty}\|\tilde{\varphi}_{xx}\|^2_{L^2}\mathrm{d}\tau + c_0\int_0^t \|\tilde{\varphi}_x\|^2_{L^2}\|v^{I,0}_x\|^2_{L^\infty}\mathrm{d}\tau + c_0\int_0^t \|\partial_x f_1\|^2_{L^2}\mathrm{d}\tau \\ &\quad\le c(v_*,T) + c(v_*,T)\int_0^t \|v^{I,1}_{xx}\|^2_{L^2}\mathrm{d}\tau\end{aligned} \qquad (3.84)$$

for any $t\in(0,T]$. Differentiating (3.76) with respect to x gives

$$\begin{aligned}v^{I,1}_{txx} =& -(\varphi^{I,0}_x + M)v^{I,1}_{xx} - 2\varphi^{I,0}_{xx}v^{I,1}_x - \partial_x^3\varphi^{I,0}v^{I,1} - \partial_x^3\tilde{\varphi}v^{I,0} - 2\tilde{\varphi}_{xx}v^{I,0}_x \\ &- \tilde{\varphi}_x v^{I,0}_{xx} + \partial_x^2 f_2.\end{aligned} \qquad (3.85)$$

Testing (3.85) against $v_{xx}^{I,1}$, we have

$$
\begin{aligned}
(3.86)\qquad &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{I}}|v_{xx}^{I,1}|^2\mathrm{d}x+\int_{\mathcal{I}}(\varphi_x^{I,0}+M)|v_{xx}^{I,1}|^2\mathrm{d}x\\
&=-\int_{\mathcal{I}}2\varphi_{xx}^{I,0}v_x^{I,1}v_{xx}^{I,1}\mathrm{d}x-\int_{\mathcal{I}}\varphi_{xxx}^{I,0}v^{I,1}v_{xx}^{I,1}\mathrm{d}x-\int_{\mathcal{I}}\tilde{\varphi}_{xxx}v^{I,0}v_{xx}^{I,1}\mathrm{d}x\\
&\quad-\int_{\mathcal{I}}2\tilde{\varphi}_{xx}v_x^{I,0}v_{xx}^{I,1}\mathrm{d}x-\int_{\mathcal{I}}\tilde{\varphi}_x v_{xx}^{I,0}v_{xx}^{I,1}\mathrm{d}x+\int_{\mathcal{I}}\partial_x^2 f_2 v_{xx}^{I,1}\mathrm{d}x\\
&\le 2\|\varphi_{xx}^{I,0}\|_{L^\infty}\|v_x^{I,1}\|_{L^2}\|v_{xx}^{I,1}\|_{L^2}+\|v^{I,1}\|_{L^\infty}\|\partial_x^3\varphi^{I,0}\|_{L^2}^2\|v_{xx}^{I,1}\|_{L^2}\\
&\quad+\|\partial_x^3\tilde{\varphi}\|_{L^2}\|v^{I,0}\|_{L^\infty}\|v_{xx}^{I,1}\|_{L^2}+2\|v_x^{I,0}\|_{L^\infty}\|\tilde{\varphi}_{xx}\|_{L^2}\|v_{xx}^{I,1}\|_{L^2}\\
&\quad+\|v_x^{I,0}\|_{L^\infty}\|\tilde{\varphi}_x\|_{L^2}\|v_x^{I,1}\|_{L^2}+\|\partial_x^2 f_2\|_{L^2}\|v_{xx}^{I,1}\|_{L^2}\\
&\le c(v_*,T)\|v_{xx}^{I,1}\|_{L^2}^2+c(v_*,T)\left(1+\|\tilde{\varphi}_{xx}\|_{L^2}^2+\|\partial_x^3\tilde{\varphi}\|_{L^2}^2+\|\partial_x^2 f_2\|_{L^2}^2\right),
\end{aligned}
$$

where we have used (3.27), (3.75), (3.83), Proposition C.4, and the Cauchy–Schwarz inequality. Integrating (3.86) over $(0,t)$ gives

$$
\begin{aligned}
&\int_{\mathcal{I}}|v_{xx}^{I,1}|^2(\cdot,t)\mathrm{d}x+\int_0^t\int_{\mathcal{I}}(\varphi_x^{I,0}+M)|v_{xx}^{I,1}|^2\mathrm{d}x\mathrm{d}\tau\\
&\quad\le c(v_*,T)\int_0^t\int_{\mathcal{I}}\left(|v_{xx}^{I,1}|^2+|\partial_x^3\tilde{\varphi}|^2\right)\mathrm{d}x\mathrm{d}\tau+c(v_*,T),
\end{aligned}
$$

where we have used (3.66b) and (3.79). This, combined with (3.84), yields that

$$
\begin{aligned}
&\int_{\mathcal{I}}|v_{xx}^{I,1}|^2(\cdot,t)\mathrm{d}x+\int_0^t\int_{\mathcal{I}}\left((\varphi_x^{I,0}+M)|v_{xx}^{I,1}|^2+|\partial_x^3\tilde{\varphi}|^2\right)\mathrm{d}x\mathrm{d}\tau\\
&\quad\le c(v_*,T)\int_0^t\int_{\mathcal{I}}|v_{xx}^{I,1}|^2\mathrm{d}x\mathrm{d}\tau+c(v_*,T).
\end{aligned}
$$

Applying the Gronwall inequality to the above inequality, we have

$$
(3.87)\qquad \int_{\mathcal{I}}|v_{xx}^{I,1}|^2(\cdot,t)\mathrm{d}x+\int_0^t\int_{\mathcal{I}}|\partial_x^3\tilde{\varphi}|^2\mathrm{d}x\mathrm{d}\tau\le c(v_*,T),
$$

where $\varphi_x^{I,0}+M\ge 0$ from (3.27) has been used. Similar to the proof of (3.87), we can derive that

$$
(3.88)\qquad \int_{\mathcal{I}}|\partial_x^3 v^{I,1}|^2(\cdot,t)\mathrm{d}x+\int_0^t\int_{\mathcal{I}}\left|\partial_x^4\tilde{\varphi}\right|^2\mathrm{d}x\mathrm{d}\tau\le c(v_*,T),
$$

where we have used $\|f_2\|_{L_T^2H^3}\le c(v_*,T)$ by (3.66b). Furthermore, by (3.27), (3.66b), (3.69), (3.75), (3.79), (3.83), (3.87), and (3.88), we deduce from $(3.62)_2$ that $\|\partial_t^k v^{I,1}\|_{L_T^2H^{5-2k}}\le c(v_*,T)$ for $k=1,2$. Hence, we conclude for the problem (3.62) that

$$
(3.89)\qquad \|\partial_t^k\varphi^{I,1}\|_{L_T^2H^{4-2k}}\le c(v_*,T)\quad\text{for } k=0,1,2,
$$

$$
(3.90)\qquad \|v^{I,1}\|_{L_T^\infty H^3}\le c(v_*,T),\quad \|\partial_t^k v^{I,1}\|_{L_T^2H^{5-2k}}\le c(v_*,T)\quad\text{for } k=1,2,
$$

provided $\|\partial_t^k f_1\|_{L_T^2H^{2-2k}}\le c(v_*,T)$, $\|\partial_t^k f_2\|_{L_T^2H^{3-2k}}\le c(v_*,T)$ for $k=0,1$. With (3.27), (3.66), (3.89), (3.90), and Proposition C.4, we can update the estimates in (3.81) and (3.82), respectively, for $\tilde{f}_1$ and $\tilde{f}_2$ as

$$
\|\partial_t^k\tilde{f}_1\|_{L_T^2H^{2-2k}}\le c(v_*,T),\ \ \|\partial_t^k\tilde{f}_2\|_{L_T^2H^{3-2k}}\le c(v_*,T),\quad k=0,1.
$$

Indeed, it holds for $\tilde{f}_1$ that

$$\begin{aligned}\|\tilde{f}_1\|^2_{L^2_T H^2} &\le c_0\int_0^T \|\varphi^{I,0}_{xt}\|^2_{H^2}\|v^{I,1}_x\|^2_{H^2}\mathrm{d}t + c_0\int_0^T \|v^{I,0}_{xt}\|^2_{H^2}\|\tilde{\varphi}_x\|^2_{H^2}\mathrm{d}t\\ &\quad + c_0\int_0^T \|\partial_t f_1(x,t)\|^2_{H^2}\mathrm{d}t\\ &\le c_0\|\varphi^{I,0}_t\|^2_{L^\infty_T H^3}\int_0^T \|v^{I,1}\|^2_{H^3}\mathrm{d}t + c_0\|v^{I,0}_t\|^2_{L^\infty_T H^3}\int_0^T \|\tilde{\varphi}\|^2_{H^3}\mathrm{d}t + c(v_*,T)\\ &\le c(v_*,T)\end{aligned}$$

and

$$\begin{aligned}&\|\partial_t\tilde{f}_1\|^2_{L^2_T L^2}\\ &\quad\le c_0\int_0^T \left(\|\varphi^{I,0}_{ttx}\|^2_{L^\infty}\|v^{I,1}_x\|^2_{L^2}\|\varphi^{I,0}_{xt}\|^2_{L^\infty}\|v^{I,1}_{xt}\|^2_{L^2}\right)\mathrm{d}t c_0\int_0^T \|\partial_t f_1\|^2_{L^2}\mathrm{d}t\\ &\quad\quad c_0\int_0^T \left(\|v^{I,0}_{ttx}\|^2_{L^\infty}\|\tilde{\varphi}_x\|^2_{L^2}\|v^{I,0}_{xt}\|^2_{L^\infty}\|\tilde{\varphi}_{xt}\|^2_{L^2}\right)\mathrm{d}t\\ &\quad\le c_0\int_0^T \left(\|v^{I,1}_x\|^2_{L^2}\|v^{I,1}_{xt}\|^2_{L^2}\|\tilde{\varphi}_x\|^2_{L^2}\|\tilde{\varphi}_{xt}\|^2_{L^2}\right)\mathrm{d}t c(v_*,T) \le c(v_*,T).\end{aligned}$$

For $\tilde{f}_2$, it follows that

$$\begin{aligned}&\|\tilde{f}_2\|^2_{L^2_T H^3}\\ &\quad\le c_0\int_0^T \|\varphi^{I,0}_x\|^2_{H^3}\|v^{I,1}\|^2_{H^3}\mathrm{d}t + c_0\int_0^T \|\tilde{\varphi}_x\|^2_{H^3}\|v^{I,0}_t\|^2_{H^3}\mathrm{d}t + c_0\int_0^T \|\partial_t f_2\|^2_{H^3}\mathrm{d}t\\ &\quad\le c_0\|\varphi^{I,0}\|^2_{L^\infty_T H^4}\int_0^T \|v^{I,1}\|^2_{H^3}\mathrm{d}t + c_0\int_0^T \|\partial_t f_2\|^2_{H^3}\mathrm{d}t\\ &\quad\quad + c_0\|v^{I,0}_t\|^2_{L^\infty_T H^3}\int_0^T \|\tilde{\varphi}\|^2_{H^4}\mathrm{d}t \le c(v_*,T)\end{aligned}$$

and

$$\begin{aligned}&\|\partial_t\tilde{f}_2\|^2_{L^2_T H^1}\\ &\quad\le c_0\int_0^T \left(\|\varphi^{I,0}_{xtt}\|^2_{H^1}\|v^{I,1}\|^2_{H^1} + \|\varphi^{I,0}_{xt}\|^2_{H^1}\|v^{I,1}_t\|^2_{H^1}\right)\mathrm{d}t + c_0\int_0^T \|\partial_t^2 f_2\|^2_{H^1}\mathrm{d}t\\ &\quad\quad + c_0\int_0^T \left(\|\tilde{\varphi}_{xt}\|^2_{H^1}\|v^{I,0}_t\|^2_{H^1} + \|\tilde{\varphi}_x\|^2_{H^1}\|v^{I,0}_{tt}\|^2_{H^1}\right)\mathrm{d}t\\ &\quad\le c_0\int_0^T \left(\|v^{I,1}\|^2_{H^1} + \|v^{I,1}_t\|^2_{H^1} + \|\tilde{\varphi}_{xt}\|^2_{H^1} + \|\tilde{\varphi}_x\|^2_{H^1}\right)\mathrm{d}t + c(v_*,T) \le c(v_*,T).\end{aligned}$$

Here we have used the Sobolev inequality $\|fg\|_{H^k(\mathbb{R}_+)} \le C_k\|f\|_{H^k(\mathbb{R}_+)}\|g\|_{H^k(\mathbb{R}_+)}$ for any integer $k \ge 1$. On the other hand, it can be verified that the initial value of the problem (3.80) is compatible up to order one. Therefore, by arguments similar to those proving (3.89) and (3.90), we have for the problem (3.80) that

$$\|\partial_t^k\psi\|_{L^2_T H^{4-2k}} \le c(v_*,T) \quad \text{for } k = 0,1,2, \tag{3.91}$$

$$\|w\|_{L^\infty_T H^3} + \|\partial_t^k w\|_{L^2_T H^{5-2k}} \le c(v_*,T) \quad \text{for } k = 1,2. \tag{3.92}$$

Collecting estimates (3.89), (3.90), (3.91), and (3.92), and also making use of (3.27) and (3.66), one can deduce

$$\|\varphi^{I,1}\|_{L_T^2H^6}+\|v^{I,1}\|_{L_T^\infty H^5}+\|\partial_t^k v^{I,1}\|_{L_T^2H^{7-2k}}\le c(v_*,T)\quad\text{for}\quad k=1,2,3$$

and ultimately obtain (3.61). The proof of Lemma 3.5 is complete. ∎

With Lemmas 3.2–3.5 at hand, we proceed to study the problems (2.16) and (2.17).

LEMMA 3.6. *Assume the conditions in Lemmas* 3.2, 3.3, *and* 3.5 *hold. Then the problem* (2.16) *admits a unique solution* $(v^{B,1},\varphi^{B,2})$ *on* $[0,T]$ *for any* $T\in(0,\infty)$ *which satisfies, for any* $l\in\mathbb{N}$,

$$\|\langle z\rangle^l\partial_t^k v^{B,1}\|_{L_T^2H_z^{6-2k}}+\|\langle z\rangle^l\partial_t^j\varphi^{B,2}\|_{L_T^2H_z^{6-2j}}\le c(v_*,T),\tag{3.93}$$

where $k=0,1,2,3,$ *and* $j=0,1,2.$

Proof. From $(2.16)_1$, we have

$$\begin{aligned}\varphi_z^{B,2}&=-\mathrm{e}^{v^{B,0}}\int_z^\infty v_y^{B,1}(\varphi_x^{I,0}(0,t)+M+\varphi_y^{B,1})\mathrm{e}^{-v^{B,0}}\mathrm{d}y\\&\quad-\mathrm{e}^{v^{B,0}}\int_z^\infty\left[v_y^{B,0}(\varphi_{xx}^{I,0}(0,t)y+\varphi_x^{I,1}(0,t))+\varphi_y^{B,1}v_x^{I,0}(0,t)\right]\mathrm{e}^{-v^{B,0}}\mathrm{d}y\\&=-\mathrm{e}^{v^{B,0}}\int_z^\infty\left[v_y^{B,0}(\varphi_{xx}^{I,0}(0,t)y+\varphi_x^{I,1}(0,t))+\varphi_y^{B,1}v_x^{I,0}(0,t)\right]\mathrm{e}^{-v^{B,0}}\mathrm{d}y\\&\quad+\mathrm{e}^{v^{B,0}}\int_z^\infty v^{B,1}\partial_y\left[(\varphi_x^{I,0}(0,t)+M+\varphi_y^{B,1})\mathrm{e}^{-v^{B,0}}\right]\mathrm{d}y\\&\quad+v^{B,1}(\varphi_x^{I,0}(0,t)+M+\varphi_z^{B,1}).\end{aligned}\tag{3.94}$$

This, together with $(2.16)_2$, gives

$$\begin{aligned}v_t^{B,1}&=v_{zz}^{B,1}-(\varphi_x^{I,0}(0,t)+M)v^{B,1}-v^{B,1}(\varphi_x^{I,0}(0,t)+M+\varphi_z^{B,1})(v^{I,0}(0,t)+v^{B,0})\\&\quad-\mathrm{e}^{v^{B,0}}\int_z^\infty v^{B,1}\partial_y\left[(\varphi_x^{I,0}(0,t)+M+\varphi_y^{B,1})\mathrm{e}^{-v^{B,0}}\right]\mathrm{d}y(v^{I,0}(0,t)+v^{B,0})\\&\quad+\mathrm{e}^{v^{B,0}}\int_z^\infty\left[v_y^{B,0}(\varphi_{xx}^{I,0}(0,t)y+\varphi_x^{I,1}(0,t))+\varphi_y^{B,1}v_x^{I,0}(0,t)\right]\mathrm{e}^{-v^{B,0}}\\&\qquad\times\ \mathrm{d}y(v^{I,0}(0,t)+v^{B,0})\\&\quad-\varphi_z^{B,1}(v_x^{I,0}(0,t)z+v^{I,1}(0,t)+v^{B,1})-(\varphi_{xx}^{I,0}(0,t)z+\varphi_x^{I,1}(0,t))v^{B,0}.\end{aligned}\tag{3.95}$$

Take

$$\tilde v=v^{B,1}+\eta(z)v^{I,1}(0,t)$$

with $\eta(z)$ as in (3.33). Then we deduce from $(2.16)_3$, $(2.16)_4$, and (3.95) that $\tilde v$ solves

$$\begin{cases}\tilde v_t=\tilde v_{zz}-(\varphi_x^{I,0}(0,t)+M)\tilde v-\tilde v(\varphi_x^{I,0}(0,t)+M+\varphi_z^{B,1})(v^{I,0}(0,t)+v^{B,0})-\tilde v\varphi_z^{B,1}\\\qquad-\mathrm{e}^{v^{B,0}}\displaystyle\int_z^\infty\tilde v\partial_y\left[(\varphi_x^{I,0}(0,t)+M+\varphi_y^{B,1})\mathrm{e}^{-v^{B,0}}\right]\mathrm{d}y(v^{I,0}(0,t)+v^{B,0})+g,\\\tilde v(0,t)=0,\quad\tilde v(+\infty,t)=0,\\\tilde v(z,0)=0,\end{cases}\tag{3.96}$$

where g is given by

(3.97)
$$
\begin{aligned}
g = & \, \mathrm{e}^{v^{B,0}} \int_z^\infty \eta v^{I,1}(0,t)\partial_y \left[(\varphi_x^{I,0}(0,t) + M + \varphi_y^{B,1})\mathrm{e}^{-v^{B,0}} \right] \mathrm{d}y (v^{I,0}(0,t) + v^{B,0}) \\
& + \mathrm{e}^{v^{B,0}} \int_z^\infty \left[v_y^{B,0}(\varphi_{xx}^{I,0}(0,t)y + \varphi_x^{I,1}(0,t)) + \varphi_y^{B,1} v_x^{I,0}(0,t) \right] \mathrm{e}^{-v^{B,0}} \\
& \qquad \times \mathrm{d}y (v^{I,0}(0,t) + v^{B,0}) \\
& + (\varphi_x^{I,0}(0,t) + M)\eta(z) v^{I,1}(0,t) + \eta(z) v_t^{I,1}(0,t) - \varphi_z^{B,1}(v_x^{I,0}(0,t)z + v^{I,1}(0,t)) \\
& - (\varphi_{xx}^{I,0}(0,t)z + \varphi_x^{I,1}(0,t)) v^{B,0} \\
& + \eta(z) v^{I,1}(0,t)(\varphi_x^{I,0}(0,t) + M + \varphi_z^{B,1})(v^{I,0}(0,t) + v^{B,0}) \\
& - \eta''(z) v^{I,1}(0,t) + \eta(z) v^{I,1}(0,t)\varphi_z^{B,1}.
\end{aligned}
$$

The existence of solutions to the problem (3.96) can be proved by using the reflection method along with the Banach's fixed point theorem. Since the argument is similar to that in Appendix A for the linearized problem of (2.12), we omit the details here. In the following, we are devoted to deriving some weighted estimates for the solution. It can be verified that the initial datum for the problem (3.96) is compatible up to order two. That is, if we define $\partial_t^k \tilde{v}|_{t=0}$ $(k = 1,2)$ through the first equation in (3.96), then $\partial_t^k \tilde{v}|_{t=0}$ $(k = 0,1,2)$ vanish at the boundary. Furthermore, we have for $k = 0,1,2$ and $l \in \mathbb{N}$ that

$$
\|\langle z\rangle^l \partial_t^k g\|_{L_T^2 H_z^{4-2k}} \le c(v_*, T). \tag{3.98}
$$

The proof of (3.98) will be detailed in Appendix B. We proceed to prove for $m = 1,2,3$ and $l \in \mathbb{N}$ that

$$
\|\langle z\rangle^l \partial_t^k \tilde{v}\|_{L_T^2 H_z^{2m-2k}} \le c(v_*, T) \quad \text{for } k = 0,1,\ldots,m. \tag{3.99}
$$

Indeed, for the case $m = 1$, multiplying the first equation in (3.96) by $\langle z\rangle^{2l}\tilde{v}$ followed by an integration over $\mathbb{R}_+$, we have

(3.100)
$$
\begin{aligned}
& \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_+} \langle z\rangle^{2l}\tilde{v}^2 \mathrm{d}z + \int_{\mathbb{R}_+} \langle z\rangle^{2l}\tilde{v}_z^2 \mathrm{d}z + \int_{\mathbb{R}_+} \langle z\rangle^{2l}(\varphi_x^{I,0}(0,t) + M)(v^{I,0}(0,t) + v^{B,0})\tilde{v}^2 \mathrm{d}z \\
& = -\int_{\mathbb{R}_+} \varphi_z^{B,1}(v^{I,0}(0,t) + v^{B,0} + 1)\langle z\rangle^{2l}\tilde{v}^2 \mathrm{d}z + \int_{\mathbb{R}_+} g\langle z\rangle^{2l}\tilde{v}\mathrm{d}z - 2l\int_{\mathbb{R}_+} \langle z\rangle^{2l-2} z\tilde{v}\tilde{v}_z \mathrm{d}z \\
& \quad - \int_{\mathbb{R}_+} \mathrm{e}^{v^{B,0}} \int_z^\infty \tilde{v}\partial_y \left[(\varphi_x^{I,0}(0,t) + M + \varphi_y^{B,1})\mathrm{e}^{-v^{B,0}} \right] \mathrm{d}y (v^{I,0}(0,t) + v^{B,0})\langle z\rangle^{2l}\tilde{v}\mathrm{d}z,
\end{aligned}
$$

where by (3.27)–(3.29), the Sobolev inequality $\|f\|_{L_z^\infty} \le c_0\|f\|_{H_z^1}$, and the Cauchy–Schwarz inequality, it holds that

$$
\begin{aligned}
&\int_{\mathbb{R}_+} \varphi_z^{B,1}(v^{I,0}(0,t)+v^{B,0}+1)\langle z\rangle^{2l}\tilde v^2 \mathrm{d}z\\
&\quad \le c_0 \left\|\varphi_z^{B,1}\right\|_{L_z^\infty}\left(\|v^{I,0}\|_{L^\infty}+\|v^{B,0}\|_{L_z^\infty}+1\right)\int_{\mathbb{R}_+}\langle z\rangle^{2l}\tilde v^2\mathrm{d}z\\
&\quad \le c(v_*,T)\left\|\varphi_z^{B,1}\right\|_{L_z^\infty}\int_{\mathbb{R}_+}\langle z\rangle^{2l}\tilde v^2\mathrm{d}z \le c(v_*,T)\int_{\mathbb{R}_+}\langle z\rangle^{2l}\tilde v^2\mathrm{d}z,\\
&\int_{\mathbb{R}_+} g\langle z\rangle^{2l}\tilde v\mathrm{d}z - 2l\int_{\mathbb{R}_+}\langle z\rangle^{2l-2}z\tilde v\tilde v_z\mathrm{d}z\\
&\quad \le \frac{1}{8}\int_{\mathbb{R}_+}\langle z\rangle^{2l}\tilde v_z^2\mathrm{d}z + c_0\int_{\mathbb{R}_+}\langle z\rangle^{2l}\tilde v^2\mathrm{d}z + c_0\int_{\mathbb{R}_+}\langle z\rangle^{2l}g^2\mathrm{d}z
\end{aligned}
$$

and

$$
\begin{aligned}
&-\int_{\mathbb{R}_+}\mathrm{e}^{v^{B,0}}\int_z^\infty \tilde v\partial_y\left[(\varphi_x^{I,0}(0,t)+M+\varphi_y^{B,1})\mathrm{e}^{-v^{B,0}}\right]\mathrm{d}y(v^{I,0}(0,t)+v^{B,0})\langle z\rangle^{2l}\tilde v\mathrm{d}z\\
&\le c(v_*,T)\left(|v^{I,0}(0,t)|+\|v^{B,0}\|_{L_z^\infty}\right)\\
&\quad\times\int_{\mathbb{R}_+}\langle z\rangle^{2l}\left|\int_z^\infty|\tilde v|\Big(|v_y^{B,0}|+|\varphi_{yy}^{B,1}|+|\varphi_y^{B,1}||v_y^{B,0}|\Big)\mathrm{d}y\right|^2\mathrm{d}z\\
&\qquad + c(v_*,T)\|\langle z\rangle^l\tilde v\|_{L^2}^2\\
&\le c(v_*,T)\|\langle z\rangle^l\tilde v\|_{L_z^2}^2\\
&\quad + c(v_*,T)\|\langle z\rangle^l\tilde v\|_{L_z^2}^2\int_{\mathbb{R}_+}\langle z\rangle^{2l}\int_z^\infty\langle y\rangle^{-2l}\left(|v_y^{B,0}|+|\varphi_{yy}^{B,1}|+|\varphi_y^{B,1}||v_y^{B,0}|\right)^2\mathrm{d}y\mathrm{d}z\\
&\le c(v_*,T)\|\langle z\rangle^l\tilde v\|_{L_z^2}^2\\
&\quad + c(v_*,T)\|\langle z\rangle^l\tilde v\|_{L_z^2}^2\left(\|\langle z\rangle^l v_z^{B,0}\|_{L_z^2}^2+\|\langle z\rangle^l\varphi_{zz}^{B,1}\|_{L_z^2}^2+\|\varphi_z^{B,1}\|_{L_z^\infty}^2\|\langle z\rangle^l v_z^{B,0}\|_{L_z^2}^2\right)\\
&\le c(v_*,T)\|\langle z\rangle^l\tilde v\|_{L_z^2}^2.
\end{aligned}
$$

Therefore we update (3.100) as

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_+}\langle z\rangle^{2l}\tilde v^2\mathrm{d}z+\frac{1}{2}\int_{\mathbb{R}_+}\langle z\rangle^{2l}\tilde v_z^2\mathrm{d}z\le c(v_*,T)\|\langle z\rangle^l\tilde v\|_{L_z^2}^2+c(v_*,T)\int_{\mathbb{R}_+}\langle z\rangle^{2l}g^2\mathrm{d}z,
$$

where we have used (3.27) and (3.32). This, along with (3.98) and the Gronwall inequality, yields that

$$
\|\langle z\rangle^l\tilde v(\cdot,t)\|_{L_z^2}^2+\int_0^t\|\langle z\rangle^l\tilde v_z\|_{L_z^2}^2\mathrm{d}\tau\le c(v_*,T) \tag{3.101}
$$

for any $t\in(0,T]$. Multiplying the first equation in (3.96) by $\langle z\rangle^{2l}\tilde v_t$ and then integrating the resulting equation over $\mathbb{R}_+$, we get

(3.102)

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\int_{\mathbb{R}_+}\langle z\rangle^{2l}\tilde{v}_z^2\mathrm{d}z+\int_{\mathbb{R}_+}\langle z\rangle^{2l}(\varphi_x^{I,0}(0,t)+M)(v^{I,0}(0,t)+v^{B,0})\tilde{v}^2\mathrm{d}z\right\}+\int_{\mathbb{R}_+}\langle z\rangle^{2l}\tilde{v}_t^2\mathrm{d}z\\
&=\frac{1}{2}\int_{\mathbb{R}_+}\langle z\rangle^{2l}\left[(\varphi_x^{I,0}(0,t)+M)(v^{I,0}(0,t)+v^{B,0})\right]_t\tilde{v}^2\mathrm{d}z\\
&\quad-\int_{\mathbb{R}_+}\varphi_z^{B,1}(v^{I,0}(0,t)+v^{B,0}+1)\langle z\rangle^{2l}\tilde{v}\tilde{v}_t\mathrm{d}z+\int_{\mathbb{R}_+}g\langle z\rangle^{2l}\tilde{v}_t\mathrm{d}z-2l\int_{\mathbb{R}_+}\langle z\rangle^{2l-2}z\tilde{v}_t\tilde{v}_z\mathrm{d}z\\
&\quad-\int_{\mathbb{R}_+}\mathrm{e}^{v^{B,0}}\int_z^\infty\tilde{v}\partial_y\left[(\varphi_x^{I,0}(0,t)+M+\varphi_y^{B,1})\mathrm{e}^{-v^{B,0}}\right]\mathrm{d}y(v^{I,0}(0,t)+v^{B,0})\langle z\rangle^{2l}\tilde{v}_t\mathrm{d}z\\
&=\sum_{i=1}^{5}\mathcal{I}_i.
\end{aligned}$$

We now estimate $\mathcal{I}_i$ $(1\le i\le 5)$ term by term. By (3.27), (3.28), and Proposition C.4, we have

$$\mathcal{I}_1\le c(v_*,T)\left(\|\varphi_{xt}^{I,0}\|_{L^\infty}+\|v_t^{I,0}\|_{L^\infty}+\|v_t^{B,0}\|_{L_z^\infty}\right)\|\langle z\rangle^l\tilde{v}\|_{L_z^2}^2\le c(v_*,T)\|\langle z\rangle^l\tilde{v}\|_{L_z^2}^2.$$

Similarly, for $\mathcal{I}_2$, we get

$$\begin{aligned}
\mathcal{I}_2&\le\|\varphi_z^{B,1}\|_{L_z^\infty}\left(\|v^{I,0}\|_{L^\infty}+\|v^{B,0}\|_{L_z^\infty}+1\right)\|\langle z\rangle^l\tilde{v}\|_{L_z^2}\|\langle z\rangle^l\tilde{v}_t\|_{L_z^2}\\
&\le c(v_*,T)\|\langle z\rangle^l\tilde{v}\|_{L_z^2}\|\langle z\rangle^l\tilde{v}_t\|_{L_z^2}\\
&\le\frac{1}{8}\|\langle z\rangle^l\tilde{v}_t\|_{L_z^2}^2+c(v_*,T)\|\langle z\rangle^l\tilde{v}\|_{L_z^2}^2.
\end{aligned}$$

By the Cauchy–Schwarz inequality, we have

$$\mathcal{I}_3+\mathcal{I}_4\le\frac{1}{8}\int_{\mathbb{R}_+}\langle z\rangle^{2l}\tilde{v}_t^2\mathrm{d}z+c(v_*,T)\int_{\mathbb{R}_+}\langle z\rangle^{2l}g^2\mathrm{d}z+c(v_*,T)\int_{\mathbb{R}_+}\langle z\rangle^{2l}\tilde{v}_z^2\mathrm{d}z.$$

Finally, in view of (3.27), (3.28), (3.29), and the Cauchy–Schwarz inequality, we get for $\mathcal{I}_5$ that

$$\begin{aligned}
&\mathcal{I}_5\\
&\le\frac{1}{8}\int_{\mathbb{R}_+}\langle z\rangle^{2l}\tilde{v}_t^2\mathrm{d}z+c(v_*,T)\int_{\mathbb{R}_+}\langle z\rangle^{2l}\left|\int_z^\infty\tilde{v}\partial_y\left[(\varphi_x^{I,0}(0,t)+M+\varphi_y^{B,1})\mathrm{e}^{-v^{B,0}}\right]\mathrm{d}y\right|^2\mathrm{d}z\\
&\le\frac{1}{8}\int_{\mathbb{R}_+}\langle z\rangle^{2l}\tilde{v}_t^2\mathrm{d}z+c(v_*,T)\|\langle z\rangle^l\tilde{v}\|_{L_z^2}^2\left(\|\langle z\rangle^{l+2}\varphi_{zz}^{B,1}\|_{L_z^2}^2+\|\langle z\rangle^{l+2}v_z^{B,0}\|_{L_z^2}^2\right)\int_{\mathbb{R}_+}\langle z\rangle^{-2}\mathrm{d}z\\
&\le\frac{1}{8}\int_{\mathbb{R}_+}\langle z\rangle^{2l}\tilde{v}_t^2\mathrm{d}z+c(v_*,T)\|\langle z\rangle^l\tilde{v}\|_{L_z^2}^2.
\end{aligned}$$

Collecting estimates for $\mathcal{I}_i$ $(1\le i\le 5)$, we have from (3.100) that

$$\int_{\mathbb{R}_+}\langle z\rangle^{2l}\tilde{v}_z^2(\cdot,t)\mathrm{d}z+\int_0^t\int_{\mathbb{R}_+}\langle z\rangle^{2l}\tilde{v}_\tau^2\mathrm{d}z\mathrm{d}\tau\le c(v_*,T)\int_0^t\int_{\mathbb{R}_+}\langle z\rangle^{2l}\left(\tilde{v}_z^2+\tilde{v}^2\right)\mathrm{d}z\mathrm{d}\tau$$

for any $t\in(0,T]$, where we have used the facts $\varphi_x^{I,0}(0,t)+M\ge 0$ and $v^{B,0}\ge 0$. Therefore we utilize (3.101) and the Gronwall inequality to deduce that

$$\int_{\mathbb{R}_+}\langle z\rangle^{2l}\tilde{v}_z^2(\cdot,t)\mathrm{d}z+\int_0^t\int_{\mathbb{R}_+}\langle z\rangle^{2l}\tilde{v}_\tau^2\mathrm{d}z\mathrm{d}\tau\le c(v_*,T).\tag{3.103}$$

This along with $(3.96)_1$, (3.27), (3.28), (3.29), and (3.98) leads to

$$\int_0^T \|\langle z\rangle^l \tilde{v}_{zz}\|_{L_z^2}^2 \mathrm{d}t \le c(v_*,T).$$

Then we finish the proof of (3.99) for $m=1$. To proceed, set $\hat{v}=\tilde{v}_t$. Then $\hat{v}$ satisfies

(3.104)
$$\begin{cases} \hat{v}_t = \hat{v}_{zz} - (\varphi_x^{I,0}(0,t)+M)\hat{v} - \hat{v}(\varphi_x^{I,0}(0,t)+M+\varphi_z^{B,1})(v^{I,0}(0,t)+v^{B,0}) - \hat{v}\varphi_z^{B,1} \\ \qquad -\mathrm{e}^{v^{B,0}}\displaystyle\int_z^\infty \hat{v}\partial_y\left[(\varphi_x^{I,0}(0,t)+M+\varphi_y^{B,1})\mathrm{e}^{-v^{B,0}}\right]\mathrm{d}y(v^{I,0}(0,t)+v^{B,0})+\tilde{g}, \\ \hat{v}(0,t)=0, \quad \hat{v}(+\infty,t)=0, \\ \hat{v}(z,0)=\tilde{v}_t|_{t=0}, \end{cases}$$

where $\tilde{v}_t|_{t=0}$ is defined through the equation $(3.96)_1$, and $\tilde{g}$ is given by

$$\begin{aligned} \tilde{g} = & -\partial_t[(\varphi_x^{I,0}(0,t)+M)]\tilde{v} - \tilde{v}(\partial_t\varphi_x^{I,0}(0,t)+\varphi_{zt}^{B,1})(v^{I,0}(0,t)+v^{B,0}) \\ & - \tilde{v}(\varphi_x^{I,0}(0,t)+M+\varphi_z^{B,1})(v_t^{I,0}(0,t)+v_t^{B,0}) - \tilde{v}\varphi_{zt}^{B,1} + g_t \\ & - \int_z^\infty \tilde{v}\partial_y\left[(\varphi_x^{I,0}(0,t)+M+\varphi_y^{B,1})\mathrm{e}^{-v^{B,0}}\right]\mathrm{d}y\left[\mathrm{e}^{v^{B,0}}(v^{I,0}(0,t)+v^{B,0})\right]_t \\ & - \mathrm{e}^{v^{B,0}}(v^{I,0}(0,t)+v^{B,0})\int_z^\infty \tilde{v}\partial_{yt}\left[(\varphi_x^{I,0}(0,t)+M+\varphi_y^{B,1})\mathrm{e}^{-v^{B,0}}\right]\mathrm{d}y. \end{aligned}$$

By virtue of (3.27), (3.28), (3.29), (3.98), (3.101), (3.103), and arguments similar to those proving (3.98), it holds for $k=0,1$ that

$$\|\langle z\rangle^l \partial_t^k \tilde{g}\|_{L_T^2 H_z^{2-2k}} \le c(v_*,T). \tag{3.105}$$

Repeating the argument in the proofs of (3.101) and (3.103), we then arrive at

$$\int_{\mathbb{R}_+} \langle z\rangle^{2l}\hat{v}_z^2(\cdot,t)\mathrm{d}z + \int_0^t\int_{\mathbb{R}_+}\langle z\rangle^{2l}\left(\hat{v}_z^2+\hat{v}_\tau^2\right)\mathrm{d}z\mathrm{d}\tau \le c(v_*,T) \tag{3.106}$$

for any $t\in(0,T]$. Furthermore, from $(3.104)_1$ and (3.106), we get

$$\int_0^T\int_{\mathbb{R}_+}\langle z\rangle^{2l}\tilde{v}_{tzz}^2\mathrm{d}z\mathrm{d}t \le c(v_*,T). \tag{3.107}$$

In view of $(3.96)_1$, (3.101), (3.103), $(3.104)_1$, and (3.107), we also have that

$$\int_0^T\int_{\mathbb{R}_+}\langle z\rangle^{2l}\left(|\partial_z^3\tilde{v}|^2+|\partial_z^4\tilde{v}|^2\right)\mathrm{d}z\mathrm{d}t \le c(v_*,T).$$

Thus we finish the proof of (3.99) for $m=2$. Now let us turn to the proof of (3.99) for the case $m=3$. Based on (3.105) and the fact that the initial datum of the problem (3.104) is compatible up to order one, we apply the procedure in the proof of the cases $m=1,2$ to the problem (3.104) and get that $\|\langle z\rangle^l\partial_t^k\hat{v}\|_{L_T^2H_z^{4-2k}} \le C$ for any $l\in\mathbb{N}$ and $k=0,1,2$. That is,

$$\|\langle z\rangle^l\partial_t^k\tilde{v}\|_{L_T^2H_z^{6-2k}} \le c(v_*,T) \quad \text{for } k=1,2,3.$$

This, along with (3.27), (3.28), (3.29), $(3.96)_1$, and (3.98), further gives that

$$\int_0^T \left(\|\langle z\rangle^l \partial_z^5 \tilde{v}\|_{L_z^2}^2 + \|\langle z\rangle^l \partial_z^6 \tilde{v}\|_{L_z^2}^2\right) \mathrm{d}t \le c(v_*, T).$$

Then (3.99) is proved. With the definition of $\tilde{v}$ and (3.99), one can immediately obtain the estimate for $v^{B,1}$ in (3.93). The estimates for $\varphi^{B,2}$ follow from (3.27), (3.28), (3.29), (3.61a), (3.61b), $(3.93)_1$, and (3.94) along with arguments similar to those proving (3.98). We thus finish the proof of Lemma 3.6. ∎

By arguments analogous to those proving Lemma 3.6, we have the following existence and regularity result on $(\varphi^{b,2}, v^{b,1})$.

LEMMA 3.7. *Assume the conditions in Lemmas* 3.2, 3.3, *and* 3.5 *hold. Then there exists a unique solution* $(\varphi^{b,2}, v^{b,1})$ *to the problem* (2.17) *on* $[0,T]$ *for any* $T \in (0,\infty)$ *such that for any* $l \in \mathbb{N}$,

$$\|\langle \xi\rangle^l \partial_t^k v^{b,1}\|_{L_T^2 H_\xi^{6-2k}} + \|\langle \xi\rangle^l \partial_t^j \varphi^{b,2}\|_{L_T^2 H_\xi^{6-2j}} \le c(v_*, T), \tag{3.108}$$

where $k = 0,1,2,3,$ *and* $j = 0,1,2$.

4. Convergence of boundary layers.

4.1. Reformulation of the problem. Denote by $(\varphi^\varepsilon, v^\varepsilon)$ the solution to problem (2.2)–(2.3). To prove Theorem 2.1, normally we shall construct a perturbation as

$$\begin{cases} \varphi^\varepsilon = \varphi^{I,0} + \varepsilon^{1/2}\left(\varphi^{I,1}(x,t) + \varphi^{B,1}(z,t) + \varphi^{b,1}(\xi,t)\right) + \mathcal{E}_1^\varepsilon, \\ v^\varepsilon = v^{I,0} + v^{B,0} + v^{b,0} + \mathcal{E}_2^\varepsilon \end{cases} \tag{4.1}$$

and estimate the remainder $(\mathcal{E}_1^\varepsilon, \mathcal{E}_2^\varepsilon)$ to show that

$$\|\mathcal{E}_1^\varepsilon\|_{L_T^\infty L^\infty} = O(\varepsilon^{5/8}), \quad \|\mathcal{E}_2^\varepsilon\|_{L_T^\infty L^\infty} = O(\varepsilon^{1/2}), \quad \|\partial_x \mathcal{E}_1^\varepsilon\|_{L_T^\infty L^\infty} = O(\varepsilon^{1/4}) \tag{4.2}$$

for some $T > 0$. However, if we substitute (4.1) into (2.2), we shall find that the equations for $(\mathcal{E}_1^\varepsilon, \mathcal{E}_2^\varepsilon)$ involves terms that converge to nonzero constants as $\varepsilon \to 0$, but we need estimates in (4.2), where $\mathcal{E}_1^\varepsilon$ behaves like $o(\varepsilon^{1/2})$. This gap causes trouble in the analysis. To circumvent this difficulty, we resort to higher-order outer- and boundary layer profiles by introducing an approximate solution to the problem (2.2)–(2.3) as follows:

$$\begin{aligned} \Phi^A(x,t) := {} & \varphi^{I,0} + \varepsilon^{1/2}\left(\varphi^{I,1}(x,t) + \varphi^{B,1}(z,t) + \varphi^{b,1}(\xi,t)\right) \\ & + \varepsilon\left(\varphi^{B,2}(z,t) + \varphi^{b,2}(\xi,t)\right) + b_\varphi^\varepsilon(x,t), \end{aligned} \tag{4.3a}$$

$$V^A(x,t) := v^{I,0} + v^{B,0} + v^{b,0} + \varepsilon^{1/2}\left(v^{I,1}(x,t) + v^{B,1}(z,t) + v^{b,1}(\xi,t)\right) + b_v^\varepsilon(x,t), \tag{4.3b}$$

where the functions $b_\varphi^\varepsilon(x,t)$ and $b_v^\varepsilon(x,t)$ are constructed below to homogenize the boundary values of (Φ^A, V^A):

$$\begin{aligned} b_\varphi^\varepsilon(x,t) = {} & -(1-x)\left[\varepsilon^{1/2}\varphi^{b,1}\left(-\frac{1}{\varepsilon^{1/2}}, t\right) + \varepsilon\varphi^{b,2}\left(-\frac{1}{\varepsilon^{1/2}}, t\right) + \varepsilon\varphi^{B,2}(0,t)\right] \\ & - x\left[\varepsilon^{1/2}\varphi^{B,1}\left(\frac{1}{\varepsilon^{1/2}}, t\right) + \varepsilon\varphi^{B,2}\left(\frac{1}{\varepsilon^{1/2}}, t\right) + \varepsilon\varphi^{b,2}(0,t)\right], \end{aligned} \tag{4.4a}$$

$$\begin{aligned} b_v^\varepsilon(x,t) = (x-1)\left[v^{b,0}\left(-\frac{1}{\varepsilon^{1/2}},t\right)+\varepsilon^{1/2}v^{b,1}\left(-\frac{1}{\varepsilon^{1/2}},t\right)\right] \\ -x\left[v^{B,0}\left(\frac{1}{\varepsilon^{1/2}},t\right)+\varepsilon^{1/2}v^{B,1}\left(\frac{1}{\varepsilon^{1/2}},t\right)\right]. \end{aligned} \tag{4.4b}$$

Then we can write $(\varphi^\varepsilon, v^\varepsilon)$ as

$$\varphi^\varepsilon = \Phi^A + \varepsilon^{1/2}\Phi^\varepsilon, \quad v^\varepsilon = V^A + \varepsilon^{1/2}V^\varepsilon \tag{4.5}$$

with $(\Phi^\varepsilon, V^\varepsilon)$ being the perturbation functions, which, along with (4.1), imply that

$$\mathcal{E}_1^\varepsilon = \varepsilon^{1/2}\Phi^\varepsilon + \varepsilon\left(\varphi^{B,2}(z,t)+\varphi^{b,2}(\xi,t)\right)+b_\varphi^\varepsilon(x,t), \tag{4.6}$$

$$\mathcal{E}_2^\varepsilon = \varepsilon^{1/2}V^\varepsilon + +\varepsilon^{1/2}\left(v^{I,1}(x,t)+v^{B,1}(z,t)+v^{b,1}(\xi,t)\right)+b_v^\varepsilon(x,t). \tag{4.7}$$

We remark that we have omitted the term $\varepsilon\varphi^{I,2}$ in the above construction of Φ^A. Indeed, this term is of order ε and is unnecessary for our analysis. On the other hand, if this term is included, then the upper bound on $\|\partial_t\varphi_x^{I,2}\|_{L_T^2L^2}$ is needed for the estimate of $\mathcal{R}_1^\varepsilon$ in the subsequent analysis. This will require higher-order regularities on the initial data (φ_0, v_0). Substituting (4.5) into (2.2)–(2.3), we see that the perturbation functions $(\Phi^\varepsilon, V^\varepsilon)$ satisfy

$$\begin{cases} \Phi_t^\varepsilon = \Phi_{xx}^\varepsilon - \varepsilon^{1/2}\Phi_x^\varepsilon V_x^\varepsilon - \Phi_x^\varepsilon V_x^A - V_x^\varepsilon(\Phi_x^A+M)+\varepsilon^{-1/2}\mathcal{R}_1^\varepsilon, \\ V_t^\varepsilon = \varepsilon V_{xx}^\varepsilon - \varepsilon^{1/2}\Phi_x^\varepsilon V^\varepsilon - \Phi_x^\varepsilon V^A - (\Phi_x^A+M)V^\varepsilon + \varepsilon^{-1/2}\mathcal{R}_2^\varepsilon, \\ (\Phi^\varepsilon, V^\varepsilon)(x,0) = (0,0), \\ (\Phi^\varepsilon, V^\varepsilon)(0,t) = (\Phi^\varepsilon, V^\varepsilon)(1,t) = (0,0), \end{cases} \tag{4.8}$$

where

$$\mathcal{R}_1^\varepsilon = \Phi_{xx}^A - (\Phi_x^A+M)V_x^A - \Phi_t^A, \quad \mathcal{R}_2^\varepsilon = \varepsilon V_{xx}^A - (\Phi_x^A+M)V^A - V_t^A. \tag{4.9}$$

Notice that the coefficients and source terms in (4.8) involve only the outer- and boundary layer profiles studied in the previous section. By standard arguments (e.g., [37, 42]), one can prove the local-in-time existence and uniqueness of solutions to the problem (4.8) with $\varepsilon > 0$ in the time interval $[0, T_\varepsilon]$ for some $T_\varepsilon > 0$ which may be small. Now the key is to establish some uniform-in-ε estimates for $(\Phi^\varepsilon, V^\varepsilon)$ so that the ε-independent lifespan of the solution and the convergence of boundary layers can be extracted. To this end, we present the following results for the problem (4.8), which will be proved in the next subsection.

PROPOSITION 4.1. *Assume the conditions in Theorem* 2.1 *hold. Then for any* $v_* > 0$, *there exist constants* $T > 0$ *and* $\varepsilon_0 > 0$ *such that for any* $\varepsilon \in (0, \varepsilon_0)$, *the problem* (4.8) *admits a unique solution* $(\Phi^\varepsilon, V^\varepsilon) \in L^\infty(0,T;H^2\times H^2)$ *which satisfies, for any* $t \in [0,T]$,

$$\|\Phi^\varepsilon(\cdot,t)\|_{L^2}^2 + \varepsilon^{1/2}\|\Phi_x^\varepsilon(\cdot,t)\|_{L^2}^2 + \varepsilon^{3/2}\|\Phi_{xx}^\varepsilon\|_{L^2}^2 + \varepsilon^\ell\|\partial_x^\ell V^\varepsilon(\cdot,t)\|_{L^2}^2 \le c(v_*,T)\varepsilon^{1/2}$$

and

$$\begin{aligned} &\int_0^t \left(\|\Phi_x^\varepsilon\|_{L^2}^2 + \varepsilon^{1/2}\|\Phi_\tau^\varepsilon\|_{L^2}^2 + \varepsilon\|\Phi_{x\tau}^\varepsilon\|_{L^2}^2 + \varepsilon\|V_x^2\|_{L^2}^2 + \|V_\tau^\varepsilon\|_{L^2}^2 + \varepsilon^{5/2}\|V_{x\tau}\|_{L^2}^2\right)\mathrm{d}\tau \\ &\quad \le c(v_*,T)\varepsilon^{1/2}, \end{aligned}$$

where $\ell = 0,1,2$, *and* $c(v_*,T) > 0$ *is a constant depending on* T *but independent of* ε.

4.2. ***A priori*** **estimates.**

4.2.1. Preliminaries. We introduce some basic facts for later use. By (3.28), (3.29), (3.93), (3.108), and Proposition C.4, we have for $l\in\mathbb{N}$ that

$$\|\langle z\rangle^l\partial_t^k\varphi^{B,1}\|_{L_T^\infty H_z^{5-2k}}+\|\langle z\rangle^l\partial_t^k\partial_z^j\varphi^{B,1}\|_{L_T^\infty L_z^\infty}\le K_0(T,v_*)v_*, \tag{4.10a}$$

$$\|\langle \xi\rangle^l\partial_t^k\varphi^{b,1}\|_{L_T^\infty H_\xi^{5-2k}}+\|\langle \xi\rangle^l\partial_t^k\partial_\xi^j\varphi^{b,1}\|_{L_T^\infty L_\xi^\infty}\le K_0(T,v_*)v_* \tag{4.10b}$$

for $k=0,1,2,\ j=0,1,\ldots,4-2k$, and that

$$\|\langle z\rangle^l\partial_t^k\varphi^{B,2}\|_{L_T^\infty H_z^{4-2k}}+\|\langle z\rangle^l\partial_t^k\partial_z^j\varphi^{B,2}\|_{L_T^\infty L_z^\infty}\le c(v_*,T), \tag{4.11a}$$

$$\|\langle \xi\rangle^l\partial_t^k\varphi^{b,2}\|_{L_T^\infty H_\xi^{4-2k}}+\|\langle \xi\rangle^l\partial_t^k\partial_\xi^j\varphi^{b,2}\|_{L_T^\infty L_\xi^\infty}\le c(v_*,T) \tag{4.11b}$$

for $k=0,1,\ j=0,1,\ldots,4-2k$. Hereafter the constant $c(v_*,T)>0$ is as stated in section 2, and $K_0(T,v_*)>0$ is as in Lemma 3.3. Also, we collect some basic estimates on the boundary layer profiles of v as follows:

$$\|\langle z\rangle^l\partial_t^k v^{B,1}\|_{L_T^\infty H_z^{4-2k}}+\sum_{\lambda=0}^{1}\sum_{\ell=0}^{3-2\lambda}\|\langle z\rangle^l\partial_t^\lambda\partial_z^\ell v^{B,1}\|_{L_T^\infty L_z^\infty}\le c(v_*,T), \tag{4.12a}$$

$$\|\langle \xi\rangle^l\partial_t^k v^{b,1}\|_{L_T^\infty H_\xi^{4-2k}}+\sum_{\lambda=0}^{1}\sum_{\ell=0}^{3-2\lambda}\|\langle \xi\rangle^l\partial_t^\lambda\partial_z^\ell v^{b,1}\|_{L_T^\infty L_\xi^\infty}\le c(v_*,T), \tag{4.12b}$$

and

$$\|\langle z\rangle^l\partial_t^k v^{B,0}\|_{L_T^\infty H_z^{4-2k}}+\sum_{\lambda=0}^{1}\sum_{\ell=0}^{3-2\lambda}\|\langle z\rangle^l\partial_t^\lambda\partial_z^\ell v^{B,0}\|_{L_T^\infty L_z^\infty}\le K_0(T,v_*)v_*, \tag{4.13a}$$

$$\|\langle \xi\rangle^l\partial_t^k v^{b,0}\|_{L_T^\infty H_\xi^{4-2k}}+\sum_{\lambda=0}^{1}\sum_{\ell=0}^{3-2\lambda}\|\langle \xi\rangle^l\partial_t^\lambda\partial_\xi^\ell v^{b,0}\|_{L_T^\infty L_\xi^\infty}\le K_0(T,v_*)v_* \tag{4.13b}$$

for $k=0,1,2$, due to Lemmas 3.3, 3.4, 3.6, 3.7, and Proposition C.4. From (3.27) and (4.10)–(4.12), one can deduce some estimates on the approximate solution (Φ^A,V^A):

$$\|\partial_x^l\Phi^A\|_{L_T^\infty L^\infty}+\|\partial_t\partial_x^l\Phi^A\|_{L_T^\infty L^\infty}\le c(v_*,T),\quad l=0,1, \tag{4.14a}$$

$$\|V^A\|_{L_T^\infty L^\infty}+\|V_t^A\|_{L_T^\infty L^\infty}+\varepsilon^{1/2}\|\partial_t^l\partial_x V^A\|_{L_T^\infty L^\infty}\le c(v_*,T),\quad l=0,1. \tag{4.14b}$$

4.2.2. Estimates on the error terms. Now let us turn to estimates on the error terms $\mathcal{R}_1^\varepsilon$ and $\mathcal{R}_2^\varepsilon$.

LEMMA 4.2. *Let $0<\varepsilon<1$. It holds for any $T>0$ that*

$$\|\mathcal{R}_1^\varepsilon\|_{L_T^\infty L^\infty}\le c(v_*,T)\varepsilon^{1/2},\quad \|\mathcal{R}_1^\varepsilon\|_{L_T^\infty L^2}+\|\partial_t\mathcal{R}_1^\varepsilon\|_{L_T^2 L^2}\le c(v_*,T)\varepsilon^{3/4}. \tag{4.15}$$

Proof. First, recalling the definitions of G_i $(i=-1,0)$ and $\tilde{G}_i$ $(i=-1,0)$ in (2.8) and (2.9), respectively, using (4.9) and the first equation in (2.10) and in (2.11), we get from a direct computation that

$$
\begin{aligned}
\mathcal{R}_1^\varepsilon = & -v_x^{B,0}\left[\varphi_x^{I,0} - \varphi_x^{I,0}(0,t) - x\varphi_{xx}^{I,0}(0,t)\right]\\
& + v_x^{b,0}\left[\varphi_x^{I,0} - \varphi_x^{I,0}(1,t) - (x-1)\varphi_{xx}^{I,0}(1,t)\right]\\
& - \left[\varepsilon^{1/2}v_x^{B,1}(\varphi_x^{I,0}(x,t) - \varphi_x^{I,0}(0,t)) + \varepsilon^{1/2}v_x^{b,1}(\varphi_x^{I,0}(x,t) - \varphi_x^{I,0}(1,t))\right]\\
& - \left[\varepsilon^{1/2}v_x^{B,0}(\varphi_x^{I,1}(x,t) - \varphi_x^{I,1}(0,t)) + \varepsilon^{1/2}v_x^{b,0}(\varphi_x^{I,1}(x,t) - \varphi_x^{I,1}(1,t))\right]\\
& - \varepsilon^{1/2}\left[\varphi_x^{B,1}(v_x^{I,0}(x,t) - v_x^{I,0}(0,t)) + \varphi_x^{b,1}(v_x^{I,0}(x,t) - v_x^{I,0}(1,t))\right]\\
& - \varepsilon^{1/2}(v_x^{B,0}\varphi_x^{b,1} + v_x^{b,0}\varphi_x^{B,1})\\
& - \varepsilon(v_x^{B,1}\varphi_x^{b,1} + v_x^{b,1}\varphi_x^{B,1} + \varphi_x^{B,2}v_x^{b,0} + \varphi_x^{b,2}v_x^{B,0})\\
& - \varepsilon\left[\varphi_x^{I,1}(v_x^{B,1} + v_x^{b,1}) + v_x^{I,1}(\varphi_x^{B,1} + \varphi_x^{b,1})\right]\\
& - \varepsilon\varphi_x^{I,1}v_x^{I,1} - \varepsilon\varphi_x^{B,2}\left[v^{I,0} + \varepsilon(v^{I,1} + v^{B,1} + v^{b,1})\right]_x\\
& - \varepsilon\varphi_x^{b,2}\left[v^{I,0} + \varepsilon(v^{I,1} + v^{B,1} + v^{b,1})\right]_x\\
& - \left[\varepsilon^{1/2}(\varphi_t^{B,1} + \varphi_t^{b,1}) + \varepsilon(\varphi_t^{B,2} + \varphi_t^{b,2})\right] + F^\varepsilon =: \sum_{i=1}^{12}\mathcal{P}_i + F^\varepsilon,
\end{aligned}
$$

where

(4.16)

$$
\begin{aligned}
F^\varepsilon = & -\partial_x b_v^\varepsilon\left(\varphi_x^{I,0} + M + \varepsilon^{1/2}(\varphi_x^{I,1} + \varphi_x^{B,1} + \varphi_x^{b,1}) + \varepsilon(\varphi_x^{B,2} + \varphi_x^{b,2})\right)\\
& - \partial_x b_\varphi^\varepsilon\left(v^{I,0} + v^{B,0} + v^{b,0} + \varepsilon^{1/2}(v^{I,1} + v^{B,1} + v^{b,1})\right)_x - (\partial_x b_\varphi^\varepsilon \partial_x b_v^\varepsilon) - \partial_t b_\varphi^\varepsilon.
\end{aligned}
$$

Now we are ready to estimate $\|\mathcal{R}_1^\varepsilon\|_{L_T^2 L_z^2}$. By (3.27), (4.12a), (C.6a), and Taylor's formula, we have

$$
\begin{aligned}
\|\mathcal{P}_1\|_{L_T^\infty L^2} &= \left\|\frac{\varphi_x^{I,0}(x,t) - \varphi_x^{I,0} - x\varphi_{xx}^{I,0}(0,t)}{x^2}x^2 v_x^{B,0}\right\|_{L_T^\infty L^2}\\
&\le \frac{1}{2}\|\partial_x^3\varphi^{I,0}\|_{L_T^\infty L^\infty}\|x^2 v_x^{B,0}\|_{L_T^\infty L^2} \le c_0\varepsilon\|\varphi^{I,0}\|_{L_T^\infty H^4}\|z^2 v_x^{B,0}\|_{L_T^\infty L^2}\\
&\le c_0\varepsilon^{3/4}\|\varphi^{I,0}\|_{L_T^\infty H^4}\|z^2 v_z^{B,0}\|_{L_T^\infty L_z^2} \le c(v_*,T)\varepsilon^{3/4}.
\end{aligned} \tag{4.17}
$$

The same argument as above yields

$$
\|\mathcal{P}_2\|_{L_T^\infty L^2} \le \frac{1}{2}\|\partial_x^3\varphi^{I,0}\|_{L_T^\infty L^\infty}\|(x-1)^2 v_x^{b,0}\|_{L_T^\infty L^2} \le c(v_*,T)\varepsilon^{3/4}.
$$

Furthermore, in the same manner, we get from (3.27) and (4.10)–(4.12) that

$$
\|\mathcal{P}_i\|_{L_T^\infty L^2} \le c(v_*,T)\varepsilon^{3/4}, \quad i = 3,4,5.
$$

Notice that $\frac{1}{2\varepsilon^{1/2}} < z = \frac{x}{\varepsilon^{1/2}} < \frac{1}{\varepsilon^{1/2}}$ for $1/2 \le x \le 1$, and that $-\frac{1}{\varepsilon^{1/2}} \le \xi = \frac{x-1}{\varepsilon^{1/2}} \le -\frac{1}{2\varepsilon^{1/2}}$ for $0 \le x \le 1/2$. This, along with (3.59), (3.93), (3.108), (4.12), and (C.6), implies for $m \in \mathbb{N}$ and $k = 0,1,2$ that

(4.18)

$$
\begin{aligned}
& \varepsilon^{-\frac{m}{2}}\|\partial_t^k\partial_x^i v^{B,j}\|_{L^\infty((1/2,1)\times(0,T))} + \varepsilon^{-\frac{m}{2}}\|\partial_t^k\partial_x^i v^{b,j}\|_{L^\infty((0,\frac{1}{2})\times(0,T))}\\
&\quad \le c_0\|z^{m+i}\partial_t^k\partial_z^i v^{B,j}\|_{L^\infty(0,T;L_z^\infty(0,\varepsilon^{-1/2}))} + c_0\|\xi^{m+i}\partial_t^k\partial_\xi^i v^{b,j}\|_{L^\infty(0,T;L_\xi^\infty(-\varepsilon^{-1/2},0))}\\
&\quad \le c_0\|\langle z\rangle^{m+i}\partial_t^k\partial_z^i v^{B,j}\|_{L_z^\infty} + c_0\|\langle\xi\rangle^{m+i}\partial_t^k\partial_\xi^i v^{b,j}\|_{L_\xi^\infty} \le c(v_*,T),
\end{aligned}
$$

where $j=0,1,\ i=0,1,\dots,4-2k$. Similarly, we have

$$\varepsilon^{-m/2}\left(\|\partial_t^k\partial_x^i\varphi^{B,1}\|_{L^\infty((1/2,1)\times(0,T))}+\|\partial_t^k\partial_x^i\varphi^{b,1}\|_{L^\infty((0,1/2)\times(0,T))}\right)\le c(v_*,T) \tag{4.19}$$

for $k=0,1,2,\ i=0,1,\dots,4-2k$, and

$$\varepsilon^{-m/2}\left(\|\partial_t^k\partial_x^i\varphi^{B,2}\|_{L^\infty((1/2,1)\times(0,T))}+\|\partial_t^k\partial_x^i\varphi^{b,2}\|_{L^\infty((0,1/2)\times(0,T))}\right)\le c(v_*,T) \tag{4.20}$$

if $k=0,1,\ i=0,1,\dots,2-2k$. Therefore, we deduce for $m\in\mathbb{N}_+$ that

$$\begin{aligned}\varepsilon^{1/2}\|v_x^{B,0}\varphi_x^{b,1}\|_{L_T^\infty L^2}&\le\varepsilon^{1/2}\|v_x^{B,0}\varphi_x^{b,1}\|_{L^\infty(0,T;L^2(0,1/2))}+\varepsilon^{1/2}\|v_x^{B,0}\varphi_x^{b,1}\|_{L^\infty(0,T;L^2(1/2,1))}\\&\le\varepsilon^{\frac{m+1}{2}}\Big(\|\varepsilon^{-m/2}\varphi_x^{b,1}\|_{L^\infty((0,1/2)\times(0,T))}\|v_x^{B,0}\|_{L_T^\infty L^2}\\&\qquad+\|\varepsilon^{-m/2}v_x^{B,0}\|_{L^\infty((1/2,1)\times(0,T))}\|\varphi_x^{b,1}\|_{L_T^\infty L^2}\Big)\\&\le c_0\varepsilon^{\frac{2m+1}{4}}\left(\|v_z^{B,0}\|_{L_T^\infty L_z^2}+\|v_\xi^{b,1}\|_{L_T^\infty L_\xi^2}\right)\\&\le c(v_*,T)\varepsilon^{\frac{2m+1}{4}}\end{aligned} \tag{4.21}$$

and

$$\begin{aligned}\varepsilon^{1/2}\|v_x^{b,0}\varphi_x^{B,1}\|_{L_T^\infty L^2}&\le\varepsilon^{1/2}\|v_x^{b,0}\varphi_x^{B,1}\|_{L^\infty(0,T;L^2(0,1/2))}+\varepsilon^{1/2}\|v_x^{b,0}\varphi_x^{B,1}\|_{L^\infty(0,T;L^2(1/2,1))}\\&\le\varepsilon^{\frac{m+1}{2}}\Big(\|\varepsilon^{-m/2}v_x^{b,0}\|_{L^\infty((0,1/2)\times(0,T))}\|\varphi_x^{B,1}\|_{L_T^\infty L^2}\\&\qquad+\|\varepsilon^{-m/2}\varphi_x^{B,1}\|_{L^\infty((1/2,1)\times(0,T))}\|v_x^{b,0}\|_{L_T^\infty L^2}\Big)\\&\le c_0\varepsilon^{\frac{2m+1}{4}}\left(\|\varphi_z^{B,1}\|_{L_T^\infty L_z^2}+\|v_\xi^{b,0}\|_{L_T^\infty L_\xi^2}\right)\\&\le c(v_*,T)\varepsilon^{\frac{2m+1}{4}},\end{aligned} \tag{4.22}$$

where we have used (4.10), (4.12), and (C.6). Thus $\|\mathcal{P}_6\|_{L_T^\infty L^2}\le c(v_*,T)\varepsilon^{3/4}$. By the same argument as that proving estimates for $\mathcal{P}_6$, one can infer that

$$\|\mathcal{P}_7\|_{L_T^\infty L_z^2}\le c(v_*,T)\varepsilon^{3/4}.$$

With (3.61), (4.10), (4.12), and (C.6), we obtain

$$\begin{aligned}\|\mathcal{P}_8\|_{L_T^\infty L^2}&\le c(v_*,T)\varepsilon^{3/4}\|\varphi_x^{I,1}\|_{L_T^\infty L^\infty}\left(\|v_z^{B,1}\|_{L_T^\infty L_z^2}+\|v_\xi^{b,1}\|_{L_T^\infty L_\xi^2}\right)\\&\quad+c(v_*,T)\varepsilon^{3/4}\|v_x^{I,1}\|_{L_T^\infty L^\infty}\left(\|\varphi_z^{B,1}\|_{L_T^\infty L_z^2}+\|\varphi_\xi^{b,1}\|_{L_T^\infty L_\xi^2}\right)\le c(v_*,T)\varepsilon^{3/4}.\end{aligned}$$

Similarly, we also have

$$\|\mathcal{P}_i\|_{L_T^\infty L^2}\le c(v_*,T)\varepsilon^{3/4},\quad i=9,10,11,12.$$

For the last term F^ε, we first deduce from (4.4a) and (4.10)–(4.12) that

$$\begin{aligned}&\|\partial_t^k b_\varphi^\varepsilon\|_{L_T^\infty H^1}\\&\quad\le c(v_*,T)\varepsilon\Big(\|\varepsilon^{-1/2}\partial_t^k\varphi^{b,1}(-\varepsilon^{-1/2},t)\|_{L^\infty(0,T)}+\|\partial_t^k\varphi^{b,2}(-\varepsilon^{-1/2},t)\|_{L^\infty(0,T)}\\&\qquad+\|\partial_t^k\varphi^{B,2}(0,t)\|_{L^\infty(0,T)}\Big)\\&\quad\le c(v_*,T)\varepsilon\left(\|\langle\xi\rangle\partial_t^k\varphi^{b,1}\|_{L_T^\infty L_\xi^\infty}+\|\partial_t^k\varphi^{b,2}\|_{L_T^\infty L_\xi^\infty}+\|\partial_t^k\varphi^{B,2}\|_{L_T^\infty L_z^\infty}\right)\\&\quad\le c(v_*,T)\varepsilon\end{aligned} \tag{4.23}$$

for $k=0,1$. By similar arguments, we have from (4.4b) that

$$\|\partial_t^k b_v^\varepsilon\|_{L_T^\infty H^1} \le c(v_*,T)\varepsilon, \quad k=0,1. \tag{4.24}$$

Similar arguments, along with (3.28), (3.59), (3.93), and (3.108), further imply that

$$\|\partial_t^2 b_\varphi^\varepsilon\|_{L_T^2 H^1} + \left\|\partial_t^2 b_v^\varepsilon\right\|_{L_T^2 H^1} \le c(v_*,T)\varepsilon. \tag{4.25}$$

Notice also that $\partial_x b_\varphi^\varepsilon$ and $\partial_x b_v^\varepsilon$ are independent of x. Thus it holds that

$$\|\partial_t^2 \partial_x b_\varphi^\varepsilon\|_{L^2(0,T)} + \|\partial_t^2 \partial_x b_v^\varepsilon\|_{L^2(0,T)} + \|\partial_t^k \partial_x b_\varphi^\varepsilon\|_{L^\infty(0,T)} + \|\partial_t^k \partial_x b_v^\varepsilon\|_{L^\infty(0,T)} \le c(v_*,T)\varepsilon, \tag{4.26}$$

where $k=0,1$. With (3.27), (4.10)–(4.12), and (4.23)–(4.26), recalling the definition of F^ε in (4.16), we have

$$\begin{aligned}
&\|F^\varepsilon\|_{L_T^\infty L^2} \\
&\le c(v_*,T)\|\partial_x b_v^\varepsilon\|_{L^\infty}\Big(1+\left\|\varphi^{I,0}\right\|_{L_T^\infty H^1} + \varepsilon^{1/2}\|\varphi^{I,1}\|_{L_T^\infty H^1} + \varepsilon^{1/4}\|\varphi^{B,1}\|_{L_T^\infty H_z^1} \\
&\quad + \varepsilon^{1/4}\|\varphi^{b,1}\|_{L_T^\infty H_\xi^1} + \varepsilon^{3/4}\|\varphi^{B,2}\|_{L_T^\infty H_z^1} + \varepsilon^{3/4}\|\varphi^{b,2}\|_{L_T^\infty H_\xi^1}\Big) \\
&\quad + \|\partial_x b_\varphi^\varepsilon\|_{L^\infty(0,T)}\Big(\|v^{I,0}\|_{L_T^\infty H^1} + \varepsilon^{1/4}\|v^{B,1}\|_{L_T^\infty H_z^1} + \varepsilon^{1/4}\|v^{b,1}\|_{L_T^\infty H_\xi^1}\Big) \\
&\quad + \|\partial_x b_\varphi^\varepsilon\|_{L^\infty(0,T)}\|\partial_x b_v^\varepsilon\|_{L^\infty(0,T)} + \|\partial_t b_\varphi^\varepsilon\|_{L_T^\infty L^2} \\
&\le c(v_*,T)\varepsilon^{5/4}.
\end{aligned} \tag{4.27}$$

In summary, we now have for $0<\varepsilon<1$ that

$$\|\mathcal{R}_1^\varepsilon\|_{L_T^\infty L^2} \le \sum_{i=1}^{12}\|\mathcal{P}_i\|_{L_T^\infty L^2} + \|F^\varepsilon\|_{L_T^\infty L^2} \le c(v_*,T)\varepsilon^{3/4}. \tag{4.28}$$

Repeating the above procedure with the L^2-norm replaced by the L^∞-norm, we have that

$$\|\mathcal{R}_1^\varepsilon\|_{L_T^\infty L^\infty} \le c(v_*,T)\varepsilon^{1/2}. \tag{4.29}$$

We proceed to estimate $\|\partial_t \mathcal{R}_1^\varepsilon\|_{L_T^2 L^2}$. Notice that if $\|h\chi\|_Z \le c_0\|h\|_X\|\chi\|_Y$ for $h\in X$ and $\chi \in Y$, with X, Y, and Z being Banach spaces, then

$$\|\partial_t(h\chi)\|_Z \le c_0\|\partial_t h\|_X\|\chi\|_Y + c_0\|h\|_X\|\partial_t\chi\|_Y, \tag{4.30}$$

provided that $\partial_t f \in X$ and $\partial_t \chi \in Y$. Therefore, by (3.27), (4.12a), and arguments similar to those proving (4.17), we have

$$\begin{aligned}
&\|\partial_t \mathcal{P}_1\|_{L_T^\infty L^2} \\
&\le \left\|\frac{\partial_t\varphi_x^{I,0}(x,t) - \partial_t\varphi_x^{I,0}(0,t) - x\partial_t\varphi_{xx}^{I,0}(0,t)}{x^2}x^2 v_x^{B,0}\right\|_{L_T^\infty L^2} \\
&\quad + \left\|\frac{\varphi_x^{I,0}(x,t) - \varphi_x^{I,0}(0,t) - x\varphi_{xx}^{I,0}(0,t)}{x^2}x^2 v_{xt}^{B,0}\right\|_{L_T^\infty L^2} \\
&\le \|\partial_t\partial_x^3\varphi^{I,0}\|_{L_T^\infty L^\infty}\|x^2 v_x^{B,0}\|_{L_T^\infty L^2} + \|\partial_x^3\varphi^{I,0}\|_{L_T^\infty L^\infty}\|x^2 v_{xt}^{B,0}\|_{L_T^\infty L^2} \\
&\le c(v_*,T)\varepsilon^{3/4}\|\varphi_t^{I,0}\|_{L_T^\infty H^4}\|z^2 v_z^{B,0}\|_{L_T^\infty L_z^2} + c(v_*,T)\varepsilon^{3/4}\|\varphi^{I,0}\|_{L_T^\infty H^4}\|z^2 v_{zt}^{B,0}\|_{L_T^\infty L_z^2} \\
&\le c(v_*,T)\varepsilon^{3/4}.
\end{aligned} \tag{4.31}$$

Similar arguments further yield that

$$\|\partial_t \mathcal{P}_i\|_{L^\infty_T L^2} \le c(v_*, T)\varepsilon^{3/4}, \quad i = 2, 3, \ldots, 11. \tag{4.32}$$

Now it remains to prove $\|\partial_t \mathcal{P}_{12}\|_{L^2_T L^2} \le c(v_*, T)\varepsilon^{3/4}$ and $\|\partial_t F^\varepsilon\|_{L^2_T L^2} \le c(v_*, T)\varepsilon^{3/4}$. For the former, it follows from (3.29), (3.59), (3.93), (3.108), and $0 < \varepsilon < 1$ that

$$\begin{aligned} \|\partial_t \mathcal{P}_{12}\|_{L^2_T L^2} &\le c_0 \varepsilon^{\frac{3}{4}} (\|\partial_t^2 \varphi^{B,1}\|_{L^2_T L^2_z} + \|\partial_t^2 \varphi^{b,1}\|_{L^2_T L^2_\xi}) \\ &\quad + c_0 \varepsilon^{5/4} (\|\partial_t^2 \varphi^{B,2}\|_{L^2_T L^2_z} + \|\partial_t^2 \varphi^{b,2}\|_{L^2_T L^2_\xi}) \le c(v_*, T)\varepsilon^{3/4}. \end{aligned} \tag{4.33}$$

For the latter, we split $\partial_t F^\varepsilon$ into two parts:

$$\begin{aligned} &\partial_t F^\varepsilon \\ &\quad = -\partial_t \Big[\partial_x b_v^\varepsilon \left(\partial_x \varphi^{I,0} + M + \varepsilon^{1/2} (\partial_x \varphi^{I,1} + \partial_x \varphi^{B,1} + \partial_x \varphi^{b,1}) + \varepsilon (\partial_x \varphi^{B,2} + \partial_x \varphi^{b,2}) \right) \\ &\qquad - \partial_x b_\varphi^\varepsilon \left(v^{I,0} + v^{B,0} + v^{b,0} + \varepsilon^{1/2} (v^{I,1} + v^{B,1} + v^{b,1}) \right)_x + (\partial_x b_\varphi^\varepsilon \partial_x b_v^\varepsilon) \Big] - \partial_t^2 b_\varphi^\varepsilon \\ &\quad =: \tilde{F} - \partial_t^2 b_\varphi^\varepsilon, \end{aligned}$$

where $\|\partial_t^2 b_\varphi^\varepsilon\|_{L^2_T L^2} \le c(v_*, T)\varepsilon$ due to (4.25). In view of (4.30) along with a modification of the arguments in (4.27), it holds that $\|\tilde{F}\|_{L^2_T L^2} \le c(v_*, T)\varepsilon^{3/4}$. Therefore we have

$$\|\partial_t F^\varepsilon\|_{L^2_T L^2} \le c(v_*, T)\varepsilon^{3/4}.$$

This, alongside (4.28), (4.29), and (4.31)–(4.33), gives rise to (4.15) and thus complete the proof of Lemma 4.2. ∎

LEMMA 4.3. *For any $0 < T < \infty$ and $0 < \varepsilon < 1$, it holds that*

$$\|\mathcal{R}_2^\varepsilon\|_{L^\infty_T L^2} + \|\partial_t \mathcal{R}_2^\varepsilon\|_{L^2_T L^2} \le c(v_*, T)\varepsilon^{3/4}, \quad \|\mathcal{R}_2^\varepsilon\|_{L^\infty_T L^\infty} \le c(v_*, T)\varepsilon^{1/2}. \tag{4.34}$$

Proof. From (2.12)–(2.15), we know that

$$\begin{cases} v_{zz}^{B,0} = v_t^{B,0} + (\partial_x \varphi^{I,0}(0,t) + M) v^{B,0} + \varphi_z^{B,1}(v^{B,0} + v^{I,0}(0,t)), \\ v_{\xi\xi}^{b,0} = v_t^{b,0} + (\partial_x \varphi^{I,0}(1,t) + M) v^{b,0} + \varphi_z^{b,1}(v^{b,0} + v^{I,0}(1,t)). \end{cases} \tag{4.35}$$

Plugging (4.35) into $\mathcal{R}_2^\varepsilon$ in (4.9) and recalling the definition of Φ^A and V^A, we have

$$\begin{aligned} \mathcal{R}_2^\varepsilon = &-\big[v^{B,0} (\varphi_x^{I,0}(x,t) - \varphi_x^{I,0}(0,t) - x \varphi_{xx}^{I,0}(0,t)) \\ &+ v^{b,0} (\varphi_x^{I,0}(x,t) - \varphi_x^{I,0}(1,t) - (x-1) \varphi_{xx}^{I,0}(1,t)) \big] \end{aligned}$$

$$
\begin{aligned}
&- \varepsilon^{1/2} \left[v^{B,0}(\varphi_x^{I,1}(x,t) - \varphi_x^{I,1}(0,t)) + v^{b,0}(\varphi_x^{I,1}(x,t) - \varphi_x^{I,1}(1,t))\right] \\
&- \varepsilon^{1/2} \left[v^{B,1}(\varphi_x^{I,0}(x,t) - \varphi_x^{I,0}(0,t)) + v^{b,1}(\varphi_x^{I,0}(x,t) - \varphi_x^{I,0}(1,t))\right] \\
&- \varepsilon^{1/2} \big[\varphi_x^{B,1}(v^{I,0} - v^{I,0}(0,t) - x v_x^{I,0}(0,t)) \\
&+ \varphi_x^{b,1}\left(v^{I,0} - v^{I,0}(1,t) - (x-1) v_x^{I,0}(1,t)\right)\big] \\
&- \varepsilon \left[\varphi_x^{B,1}(v^{I,1}(x,t) - v^{I,1}(0,t)) + \varphi_x^{b,1}(v^{I,1}(x,t) - v^{I,1}(1,t))\right] \\
&- \varepsilon \left[\varphi_x^{B,2}(v^{I,0}(x,t) - v^{I,0}(0,t)) + \varphi_x^{b,2}(v^{I,0}(x,t) - v^{I,0}(1,t))\right] \\
&- \varepsilon \varphi_x^{I,1}(v^{I,1} + v^{B,1} + v^{b,1}) - \varepsilon^{1/2}(\varphi_x^{B,1} v^{b,0} + \varphi_x^{b,1} v^{B,0}) \\
&- \varepsilon(\varphi_x^{B,1} v^{b,1} + \varphi_x^{b,1} v^{B,1}) \\
&- \varepsilon(\varphi_x^{B,2} v^{b,0} + \varphi_x^{b,2} v^{\varepsilon B,0}) - \varepsilon^{3/2} \varphi_x^{B,2}(v^{I,1} + v^{B,1} + v^{b,1}) \\
&- \varepsilon^{3/2} \varphi_x^{b,2}(v^{I,1} + v^{B,1} + v^{b,1}) \\
&- b_v^\varepsilon[\varphi_x^{I,0} + M + \varepsilon^{1/2}(\varphi_x^{I,1} + \varphi_x^{B,1} + \varphi_x^{b,1}) + \partial_x b_\varphi^\varepsilon] \\
&- \partial_x b_\varphi^\varepsilon (v^{I,0} + v^{B,0} + v^{b,0} + \varepsilon^{\frac12}(v^{I,1} + v^{B,1} + v^{b,1})) \\
&+ \left[\varepsilon v_{xx}^{I,0} + \varepsilon^{3/2} v_{xx}^{I,1}\right] - \partial_t b_v^\varepsilon =: \sum_{i=1}^{16} \mathcal{K}_i.
\end{aligned}
$$

To prove (4.34), it suffices to establish estimates for $\mathcal{K}_i$ $(1 \le i \le 15)$. The proof is quite similar to the one for Lemma 4.2. We first prove $\|\mathcal{R}_2^\varepsilon\|_{L_T^\infty L^2} \le c(v_*, T)\varepsilon^{3/4}$. By (4.18), (C.6), and Taylor's formula, we get

$$
\|\mathcal{K}_1\|_{L_T^\infty L^2} \le c_0 \varepsilon \|\partial_x^3 \varphi^{I,0}\|_{L_T^\infty L^\infty} \left(\|v^{B,0}\|_{L_T^\infty L_z^2} + \|v^{b,0}\|_{L_T^\infty L_\xi^2}\right) \le c(v_*, T)\varepsilon^{3/4}.
$$

Similar arguments imply that $\|\mathcal{K}_i\|_{L_T^\infty L^2} \le c(v_*, T)\varepsilon^{3/4}$ for $i = 2,3,4,5,6$. From (3.27), (4.10)–(4.12), and (C.6), we get

$$
\|\mathcal{K}_7\|_{L_T^\infty L^2} \le \varepsilon \|\varphi_x^{I,1}\|_{L_T^\infty L^\infty} \left(\|v^{I,1}\|_{L_T^\infty L^2} + \|v^{B,1}\|_{L_T L^2} + \|v^{b,1}\|_{L_T^\infty L^2}\right) \le c(v_*, T)\varepsilon,
$$

where the constraint $0 < \varepsilon < 1$ has been used. Analogously, we further have that

$$
\|\mathcal{K}_i\|_{L_T^\infty L^2} \le c(v_*, T)\varepsilon^{5/4}, \quad i = 11, 12,
$$

and

$$
\begin{aligned}
\|\mathcal{K}_{13}\|_{L_T^\infty L^2} &\le c_0 \Big(1 + \|\varphi_x^{I,0}\|_{L_T^\infty L^2} + \varepsilon^{\frac12}\|\varphi_x^{I,1}\|_{L_T^\infty L^2} + \varepsilon^{\frac14}\|\varphi_z^{B,1}\|_{L_T^\infty L_z^2} \\
&\qquad + \varepsilon^{\frac14}\|\varphi_\xi^{b,1}\|_{L_T^\infty L_\xi^2} + \|\partial_x b_\varphi^\varepsilon\|_{L_T^\infty L^2}\Big) \|b_v^\varepsilon\|_{L_T^\infty L^\infty} \le c(v_*, T)\varepsilon, \\
\|\mathcal{K}_{14}\|_{L_T^\infty L^2} &\le c(v_*, T)\|\partial_x b_\varphi^\varepsilon\|_{L_T^\infty L^2}\Big(1 + \varepsilon^{\frac12}(\|v^{I,1}\|_{L_T^\infty L^\infty} + \|v^{B,1}\|_{L_T^\infty L^\infty} + \|v^{b,1}\|_{L_T^\infty L^\infty})\Big) \\
&\le c(v_*, T)\varepsilon.
\end{aligned}
$$

Recalling the arguments in (4.21) and (4.22), we proceed to estimate $\|\mathcal{K}_8\|_{L_T^\infty L^2}$ as follows:

$$\begin{aligned}
\|\mathcal{K}_8\|_{L^\infty_T L^2} &\le \varepsilon^{1/2}\|v^{b,0}\varphi_x^{B,1}\|_{L^\infty_T L^2} + \varepsilon^{1/2}\|v^{B,0}\varphi_x^{b,1}\|_{L^\infty_T L^2}\\
&\le c_0\varepsilon^{\frac{2m+1}{4}}\Big(\|\varepsilon^{-m/2}v^{b,0}\|_{L^\infty((0,1/2)\times(0,T))}\|\varphi_z^{B,1}\|_{L^\infty_T L^2_z}\\
&\quad +\|\varepsilon^{-m/2}\varphi_x^{B,1}\|_{L^\infty((1/2,1)\times(0,T))}\|v^{b,0}\|_{L^\infty_T L^2_\xi}\Big)\\
&\quad + c_0\varepsilon^{\frac{2m+1}{4}}\Big(\|\varepsilon^{-m/2}\varphi_x^{b,1}\|_{L^\infty((0,1/2)\times(0,T))}\|v^{B,0}\|_{L^\infty_T L^2_z}\\
&\quad +\|\varepsilon^{-m/2}v^{B,0}\|_{L^\infty((1/2,1)\times(0,T))}\|\varphi_\xi^{b,1}\|_{L^\infty_T L^2_\xi}\Big)\\
&\le c(v_*,T)\varepsilon^{\frac{2m+1}{4}}
\end{aligned}$$

for any integer $m \ge 1$, where we have used (4.18), (4.19), (C.6), and $0 < \varepsilon < 1$. Thus, $\|\mathcal{K}_8\|_{L^\infty_T L^2} \le c(v_*,T)\varepsilon^{3/4}$. Similarly, we have also $\|\mathcal{K}_i\|_{L^\infty_T L^2} \le c(v_*,T)\varepsilon^{3/4}$ for $i = 9, 10$. For $\mathcal{K}_{15}$ and $\mathcal{K}_{16}$, it follows from (3.27), (3.61), and (4.24) that

$$\|\mathcal{K}_{15}\|_{L^\infty_T L^2} \le \varepsilon\|v_{xx}^{I,0}\|_{L^\infty_T L^2} + \varepsilon^{\frac{3}{2}}\|v_{xx}^{I,1}\|_{L^\infty_T L^2} \le c(v_*,T)\varepsilon, \quad \|\mathcal{K}_{16}\|_{L^\infty_T L^2} \le c(v_*,T)\varepsilon,$$

where we have used $0<\varepsilon<1$. Therefore we conclude that

$$\|\mathcal{R}_2^\varepsilon\|_{L^\infty_T L^2} \le c_0\sum_{i=1}^{16}\|\mathcal{K}_i\|_{L^\infty_T L^2} \le c(v_*,T)\varepsilon^{3/4}.$$

Repeating the above arguments with $\|\cdot\|_{L^\infty_T L^2}$ replaced by $\|\cdot\|_{L^\infty_T L^\infty}$, from (3.27), (4.10)–(4.12), (4.18), (4.23)–(4.26), and (C.6), one can deduce that

$$\|\mathcal{R}_2^\varepsilon\|_{L^\infty_T L^\infty} \le c(v_*,T)\varepsilon^{1/2}.$$

Finally, in view of (4.30), the above estimates for $\mathcal{K}_i$ $(1 \le i \le 16)$, and Lemmas 3.2–3.7, we have that $\|\partial_t\mathcal{R}_2^\varepsilon\|_{L^\infty_T L^2} \le c(v_*,T)\varepsilon^{3/4}$. This ends the proof of Lemma 4.3. ◻

4.2.3. Lower-order estimates. From now on, we shall establish some uniform-in-ε estimates for $(\Phi^\varepsilon, V^\varepsilon)$. Throughout this section, we assume that $(\Phi^\varepsilon, V^\varepsilon)$ satisfies, for any $T > 0$,

$$\sup_{t\in[0,T]}\|\Phi^\varepsilon(\cdot,t)\|_{L^\infty}^2 \le \delta, \tag{4.36}$$

where $\delta > 0$ is a small constant to be determined later, and may depend on T. The results in Lemmas 3.2–3.7, 4.2, and 4.3 will be frequently used in the subsequent analysis without further clarification. We emphasize that these estimates are all independent of δ. We begin with the L^2 estimates of $(\Phi^\varepsilon, V^\varepsilon)$.

LEMMA 4.4. *Let the conditions in Proposition* 4.1 *hold. Assume* $0 < \varepsilon < 1$ *and that the solution* $(\Phi^\varepsilon, V^\varepsilon)$ *to* (4.8) *on* $[0,T]$ *satisfies* (4.36)*. Then there exists a positive constant* $\delta_1 > 0$ *independent of* ε *and* δ *such that for any* $t \in [0,T]$,

(4.37)
$$\|V^\varepsilon(\cdot,t)\|_{L^2}^2 + \|\Phi^\varepsilon(\cdot,t)\|_{L^2}^2 + \int_0^t \left(\|V^\varepsilon\|_{L^2}^2 + \varepsilon\|V_x^\varepsilon\|_{L^2}^2 + \|\Phi_x^\varepsilon\|_{L^2}^2\right)\mathrm{d}\tau \le c(v_*,T)\varepsilon^{1/2},$$

provided $\delta \le \delta_1$ *and* $K_1(T,v_*)v_* \le 1/16$, *where* $K_1(T,v_*)$ *is given as in* (4.40), *and* $c(v_*,T) > 0$ *is a constant depending on* T *but independent of* ε *and* δ.

Proof. Multiplying the first equation in (4.8) by Φ^ε, followed by an integration over $\mathcal{I}$ and integration by parts, one deduces that

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{I}}|\Phi^\varepsilon|^2\mathrm{d}x+\int_{\mathcal{I}}|\Phi_x^\varepsilon|^2\mathrm{d}x\\
&\quad=-\varepsilon^{1/2}\int_{\mathcal{I}}\Phi^\varepsilon\Phi_x^\varepsilon V_x^\varepsilon\mathrm{d}x+\int_{\mathcal{I}}\varepsilon^{-1/2}\mathcal{R}_1^\varepsilon\Phi^\varepsilon\mathrm{d}x-\int_{\mathcal{I}}(\Phi_x^A+M)V_x^\varepsilon\Phi^\varepsilon\mathrm{d}x-\int_{\mathcal{I}}\Phi^\varepsilon\Phi_x^\varepsilon V_x^A\mathrm{d}x.
\end{aligned}
\tag{4.38}
$$

The terms on the right-hand side of (4.38) can be treated as follows. Thanks to (4.36), Lemma 4.2, and the Cauchy–Schwarz inequality, it holds that

$$
\begin{aligned}
-\varepsilon^{1/2}\int_{\mathcal{I}}\Phi^\varepsilon\Phi_x^\varepsilon V_x^\varepsilon\mathrm{d}x&\le\frac{\varepsilon}{8}\int_{\mathcal{I}}|V_x^\varepsilon|^2\,\mathrm{d}x+c_0\|\Phi^\varepsilon\|_{L^\infty}^2\int_{\mathcal{I}}|\Phi_x^\varepsilon|^2\mathrm{d}x\\
&\le\frac{\varepsilon}{8}\int_{\mathcal{I}}|V_x^\varepsilon|^2\,\mathrm{d}x+c_0\delta\int_{\mathcal{I}}|\Phi_x^\varepsilon|^2\mathrm{d}x
\end{aligned}
\tag{4.39}
$$

and that

$$
\int_{\mathcal{I}}\varepsilon^{-1/2}\mathcal{R}_1^\varepsilon\Phi^\varepsilon\mathrm{d}x\le c_0\|\Phi^\varepsilon\|_{L^2}^2+c_0\varepsilon^{-1}\|\mathcal{R}_1^\varepsilon\|_{L^2}^2\le c_0\|\Phi^\varepsilon\|_{L^2}^2+c(v_*,T)\varepsilon^{1/2}.
$$

Hereafter the constant $c(v_*,T)>0$ is independent of ε and δ. The integration by parts, along with (3.27), (4.10), (4.11), (4.14a), the fact $\partial_x^2 b_\varphi^\varepsilon=0$, the Hardy inequality (C.5), and the Cauchy–Schwarz inequality gives

$$
\begin{aligned}
&-\int_{\mathcal{I}}(\Phi_x^A+M)V_x^\varepsilon\Phi^\varepsilon\mathrm{d}x=\int_{\mathcal{I}}V^\varepsilon\Phi^\varepsilon\Phi_{xx}^A\mathrm{d}x+\int_{\mathcal{I}}(\Phi_x^A+M)V^\varepsilon\Phi_x^\varepsilon\mathrm{d}x\\
&\quad\le\varepsilon^{1/2}\int_{\mathcal{I}}V^\varepsilon\Phi^\varepsilon(\varphi_{xx}^{B,1}+\varphi_{xx}^{b,1})\mathrm{d}x\\
&\qquad+\int_{\mathcal{I}}V^\varepsilon\Phi^\varepsilon\left(\varphi_{xx}^{I,0}+\varepsilon^{1/2}\varphi_{xx}^{I,1}+\varepsilon\varphi_{xx}^{B,2}+\varepsilon\varphi_{xx}^{b,2}+\partial_x^2 b_\varphi^\varepsilon\right)\mathrm{d}x\\
&\qquad+c_0\|\Phi_x^A+M\|_{L^\infty}\|V^\varepsilon\|_{L^2}\|\Phi_x^\varepsilon\|_{L^2}\\
&\quad\le\|V^\varepsilon\|_{L^2}\left\|\frac{\Phi^\varepsilon}{x(1-x)}\right\|_{L^2}\left(\left\|\frac{x(1-x)}{\varepsilon^{1/2}}\varphi_{zz}^{B,1}\right\|_{L^\infty}+\left\|\frac{x(1-x)}{\varepsilon^{1/2}}\varphi_{\xi\xi}^{b,1}\right\|_{L^\infty}\right)\\
&\qquad+c(v_*,T)\|V^\varepsilon\|_{L^2}\|\Phi_x^\varepsilon\|_{L^2}+c(v_*,T)\|V^\varepsilon\|_{L^2}\|\Phi^\varepsilon\|_{L^2}\\
&\qquad\times\left(\|\varphi_{xx}^{I,0}\|_{L^\infty}+\varepsilon^{1/2}\|\varphi_{xx}^{I,1}\|_{L^\infty}+\|\varphi_{zz}^{B,2}\|_{L_z^\infty}+\|\varphi_{\xi\xi}^{b,2}\|_{L_\xi^\infty}+\|\partial_x^2 b_\varphi^\varepsilon\|_{L^\infty}\right)\\
&\quad\le c(v_*,T)\|V^\varepsilon\|_{L^2}\|\Phi_x^\varepsilon\|_{L^2}\left(\|\langle z\rangle\varphi_{zz}^{B,1}\|_{L_z^\infty}+\|\langle\xi\rangle\varphi_{\xi\xi}^{b,1}\|_{L_\xi^\infty}\right)\\
&\qquad+c(v_*,T)\|V^\varepsilon\|_{L^2}\left(\|\Phi_x^\varepsilon\|_{L^2}+\|\Phi^\varepsilon\|_{L^2}\right)\\
&\quad\le\frac{1}{8}\|\Phi_x\|_{L^2}^2+c(v_*,T)(\|\Phi^\varepsilon\|_{L^2}^2+\|V^\varepsilon\|_{L^2}^2).
\end{aligned}
$$

For the last term on the right-hand side of (4.38), from (3.27), (3.61), (4.3b), (4.12), (4.13), (C.4), the Hardy inequality (C.5), and the Cauchy–Schwarz inequality, we get

(4.40)
$$
\begin{aligned}
-\int_{\mathcal{I}}\Phi^\varepsilon\Phi^\varepsilon_x V^A_x \mathrm{d}x &= -\int_{\mathcal{I}}\Phi^\varepsilon\Phi^\varepsilon_x\left(v_x^{B,0}+v_x^{b,0}\right)\mathrm{d}x\\
&\quad -\int_{\mathcal{I}}\Phi^\varepsilon\Phi^\varepsilon_x\left[v_x^{I,0}+\varepsilon^{1/2}(v_x^{I,1}+v_x^{B,1}+v_x^{b,1})+\partial_x b_v^\varepsilon\right]\\
&\le \left\|\frac{\Phi^\varepsilon}{x(1-x)}\right\|_{L^2}\|\Phi^\varepsilon_x\|_{L^2}\left(\left\|\frac{x(1-x)}{\varepsilon^{1/2}}v_z^{B,0}\right\|_{L^\infty}+\left\|\frac{x(1-x)}{\varepsilon^{1/2}}v_\xi^{b,0}\right\|_{L^\infty}\right)\\
&\quad +\|\Phi^\varepsilon\|_{L^2}\|\Phi^\varepsilon_x\|_{L^2}\left(\|v_x^{I,0}\|_{L^\infty}+\|v_z^{B,1}\|_{L^\infty_z}+\|v_\xi^{b,1}\|_{L^\infty_\xi}+\varepsilon^{1/2}\|v_x^{I,1}\|_{L^\infty}+\|\partial_x b_v^\varepsilon\|_{L^\infty}\right)\\
&\le c_0\|\Phi^\varepsilon_x\|^2_{L^2}\left(\|\langle z\rangle v_z^{B,0}\|_{L^\infty_z}+\|\langle\xi\rangle v_\xi^{b,0}\|_{L^\infty_\xi}\right)+c(v_*,T)\|\Phi^\varepsilon_x\|_{L^2}\|\Phi^\varepsilon\|_{L^2}\\
&\le\left(\frac18+K_1(T,v_*)v_*\right)\|\Phi^\varepsilon_x\|^2_{L^2}+c(v_*,T)\|\Phi^\varepsilon\|^2_{L^2},
\end{aligned}
$$

where $K_1(T,v_*):=K_0(T,v_*)c_0>0$ is constant with $K_0(v_*,T)$ as in (4.13). Thus, plugging (4.39)–(4.40) into (4.38), it follows that

(4.41)
$$
\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{I}}|\Phi^\varepsilon|^2\mathrm{d}x+\int_{\mathcal{I}}|\Phi^\varepsilon_x|^2\mathrm{d}x\le\frac{\varepsilon}{4}\int_{\mathcal{I}}|V^\varepsilon_x|^2\mathrm{d}x+c(v_*,T)(\|\Phi^\varepsilon\|^2_{L^2}+\|V^\varepsilon\|^2_{L^2})+c(v_*,T)\varepsilon^{1/2},
$$

provided $c_0\delta\le 1/8$ in (4.39) and $K_1(T,v_*)v_*\le 1/8$ in (4.40). To proceed, multiplying the second equation in (4.8) by V^ε and integrating the resulting equation over $\mathcal{I}$, we get

$$
\begin{aligned}
\frac12\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{I}}|V^\varepsilon|^2\mathrm{d}x+\varepsilon\int_{\mathcal{I}}|V^\varepsilon_x|^2\mathrm{d}x &= -\int_{\mathcal{I}}\varepsilon^{1/2}\Phi^\varepsilon_x V^\varepsilon V^\varepsilon\mathrm{d}x-\int_{\mathcal{I}}\Phi^\varepsilon_x V^A V^\varepsilon\mathrm{d}x\\
&\quad -\int_{\mathcal{I}}(\Phi^A_x+M)|V^\varepsilon|^2\mathrm{d}x+\int_{\mathcal{I}}V^\varepsilon\varepsilon^{-1/2}\mathcal{R}^\varepsilon_2\mathrm{d}x=\sum_{i=1}^4\mathcal{N}_i,
\end{aligned}\tag{4.42}
$$

where, due to (4.36), integration by parts, and the Cauchy–Schwarz inequality,

$$
\begin{aligned}
\mathcal{N}_1=\varepsilon^{1/2}\int_{\mathcal{I}}\Phi^\varepsilon V^\varepsilon_x V^\varepsilon\mathrm{d}x &\le\frac{\varepsilon}{8}\int_{\mathcal{I}}|V^\varepsilon_x|^2\mathrm{d}x+c(v_*,T)\|\Phi^\varepsilon\|^2_{L^\infty}\int_{\mathcal{I}}|V^\varepsilon|^2\mathrm{d}x\\
&\le\frac{\varepsilon}{8}\int_{\mathcal{I}}|V^\varepsilon_x|^2\mathrm{d}x+c(v_*,T)\|V^\varepsilon\|^2_{L^2},
\end{aligned}\tag{4.43}
$$

provided $\delta<1$. For $\mathcal{N}_2$ and $\mathcal{N}_3$, the estimates in (4.14) along with the Cauchy–Schwarz inequality yield that

$$
\mathcal{N}_2\le\|V^A\|_{L^\infty}\|\Phi^\varepsilon_x\|_{L^2}\|V^\varepsilon\|_{L^2}\le\frac18\|\Phi^\varepsilon_x\|^2_{L^2}+c(v_*,T)\|V^\varepsilon\|^2_{L^2},\quad \mathcal{N}_3\le c(v_*,T)\|V^\varepsilon\|^2_{L^2}.
$$

For the last term $\mathcal{N}_4$, by the Cauchy–Schwarz inequality, one has $\mathcal{N}_4\le\|V^\varepsilon\|^2_{L^2}+c(v_*,T)\varepsilon^{-1}\|\mathcal{R}^\varepsilon_2\|^2_{L^2}$. Inserting the estimates on $\mathcal{N}_i$ $(1\le i\le 4)$ into (4.42), by virtue of Lemma 4.3 we get that

(4.44)
$$
\frac12\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{I}}|V^\varepsilon|^2\mathrm{d}x+\varepsilon\int_{\mathcal{I}}|V^\varepsilon_x|^2\mathrm{d}x\le\frac14\int_{\mathcal{I}}|\Phi^\varepsilon_x|^2\mathrm{d}x+c(v_*,T)\|V^\varepsilon\|^2_{L^2}+c(v_*,T)\varepsilon^{1/2}.
$$

Combining (4.44) with (4.41), we obtain that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{I}}\left(|V^\varepsilon|^2+|\Phi^\varepsilon|^2\right)\mathrm{d}x+\int_{\mathcal{I}}\left(|\Phi^\varepsilon_x|^2+\varepsilon|V^\varepsilon_x|^2\right)\mathrm{d}x\\
&\quad\le c(v_*,T)\int_{\mathcal{I}}\left(|V^\varepsilon|^2+|\Phi^\varepsilon|^2\right)\mathrm{d}x+c(v_*,T)\varepsilon^{1/2},
\end{aligned}
$$

which, along with the Gronwall inequality, entails for any $t \in [0,T]$ that

$$\|V^\varepsilon(\cdot,t)\|_{L^2}^2 + \|\Phi^\varepsilon(\cdot,t)\|_{L^2}^2 + \int_0^t \left(\varepsilon\|V_x^\varepsilon\|_{L^2}^2 + \|\Phi_x^\varepsilon\|_{L^2}^2\right) \mathrm{d}\tau \le c(v_*,T)\varepsilon^{1/2}.$$

The proof of Lemma 4.4 is complete. ∎

We proceed to establish the H^1 estimate for $(\Phi^\varepsilon, V^\varepsilon)$.

LEMMA 4.5. *Under the conditions of Lemma* 4.4, *it holds that*

$$\varepsilon\|\Phi_x^\varepsilon(\cdot,t)\|_{L^2}^2 + \varepsilon\|V_x^\varepsilon(\cdot,t)\|_{L^2}^2 + \int_0^t \left(\varepsilon\|\Phi_\tau^\varepsilon\|_{L^2}^2 + \|V_\tau^\varepsilon\|_{L^2}^2\right) \mathrm{d}\tau \le c(v_*,T)\varepsilon^{1/2} \quad \forall t \in (0,T], \tag{4.45}$$

where the constant $c(v_*,T) > 0$ is independent of ε and δ.

Proof. Multiplying the second equation in (4.8) by V_t^ε and integrating the resulting equation over $\mathcal{I}$, we have

$$\begin{aligned}
&\frac{\varepsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{I}} |V_x^\varepsilon|^2\,\mathrm{d}x + \int_{\mathcal{I}} |V_t^\varepsilon|^2\,\mathrm{d}x \\
&= -\int_{\mathcal{I}} (\Phi_x^A + M)V^\varepsilon V_t^\varepsilon \mathrm{d}x - \int_{\mathcal{I}} \Phi_x^\varepsilon V^A V_t^\varepsilon \mathrm{d}x - \varepsilon^{1/2}\int_{\mathcal{I}} \Phi_x^\varepsilon V^\varepsilon V_t^\varepsilon \mathrm{d}x + \varepsilon^{-1/2}\int_{\mathcal{I}} \mathcal{R}_2^\varepsilon V_t^\varepsilon \mathrm{d}x,
\end{aligned} \tag{4.46}$$

where, due to $\|\Phi_x^A\|_{L_T^\infty L^\infty} \le c(v_*,T)$ from (4.14a) and the Cauchy–Schwarz inequality,

$$-\int_{\mathcal{I}} (\Phi_x^A + M)V^\varepsilon V_t^\varepsilon \mathrm{d}x \le \|\Phi_x^A + M\|_{L^\infty}\|V^\varepsilon\|_{L^2}\|V_t^\varepsilon\|_{L^2} \le \frac{1}{4}\|V_t^\varepsilon\|_{L^2}^2 + c(v_*,T)\|V^\varepsilon\|_{L^2}^2. \tag{4.47}$$

By (4.14b) and the Cauchy–Schwarz inequality, we get

$$-\int_{\mathcal{I}} \Phi_x^\varepsilon V^A V_t^\varepsilon \mathrm{d}x \le \|V^A\|_{L^\infty}\|\Phi_x^\varepsilon\|_{L^2}\|V_t^\varepsilon\|_{L^2} \le \frac{1}{4}\|V_t^\varepsilon\|_{L^2}^2 + c(v_*,T)\|\Phi_x^\varepsilon\|_{L^2}^2. \tag{4.48}$$

Thanks to (4.37), the Cauchy–Schwarz inequality, the Sobolev inequality (C.2), and Lemma 4.3, we deduce that

$$\begin{aligned}
-\varepsilon^{1/2}\int_{\mathcal{I}} \Phi_x^\varepsilon V^\varepsilon V_t^\varepsilon \mathrm{d}x &\le \frac{1}{8}\int_{\mathcal{I}} |V_t^\varepsilon|^2 \mathrm{d}x + c(v_*,T)\varepsilon\|V^\varepsilon\|_{L^\infty}^2 \int_{\mathcal{I}} |\Phi_x^\varepsilon|^2 \mathrm{d}x \\
&\le \frac{1}{8}\int_{\mathcal{I}} |V_t^\varepsilon|^2 \mathrm{d}x + c(v_*,T)\varepsilon\|V^\varepsilon\|_{L^2}\|V_x^\varepsilon\|_{L^2} \int_{\mathcal{I}} |\Phi_x^\varepsilon|^2 \mathrm{d}x \\
&\le \frac{1}{8}\int_{\mathcal{I}} |V_t^\varepsilon|^2 \mathrm{d}x + c(v_*,T)\varepsilon^{5/4}\|V_x^\varepsilon\|_{L^2} \int_{\mathcal{I}} |\Phi_x^\varepsilon|^2 \mathrm{d}x \\
&\le \frac{1}{8}\int_{\mathcal{I}} |V_t^\varepsilon|^2 \mathrm{d}x + c(v_*,T)\varepsilon\|V_x^\varepsilon\|_{L^2}^2\|\Phi_x^\varepsilon\|_{L^2}^2 + c(v_*,T)\varepsilon^{3/2}\|\Phi_x^\varepsilon\|_{L^2}^2
\end{aligned}$$

and that

$$\varepsilon^{-1/2}\int_{\mathcal{I}} \mathcal{R}_2^\varepsilon V_t^\varepsilon \mathrm{d}x \le \frac{1}{8}\int_{\mathcal{I}} |V_t^\varepsilon|^2 \mathrm{d}x + c(v_*,T)\varepsilon^{-1}\|\mathcal{R}_2^\varepsilon\|_{L^2}^2 \le \frac{1}{8}\int_{\mathcal{I}} |V_t^\varepsilon|^2\,\mathrm{d}x + c(v_*,T)\varepsilon^{1/2}. \tag{4.49}$$

With (4.47)–(4.49) and the fact $0<\varepsilon<1$, we thus update (4.46) as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{I}}\varepsilon|V_x^\varepsilon|^2\mathrm{d}x+\|V_t^\varepsilon\|_{L^2}^2\le c(v_*,T)\|\Phi_x^\varepsilon\|_{L^2}^2(\varepsilon\|V_x^\varepsilon\|_{L^2}^2)+C\|\Phi_x^\varepsilon\|_{L^2}^2+c(v_*,T)\varepsilon^{1/2},$$

which, followed by an integration over $[0,t]$ for any $t\in(0,T]$, gives

$$\varepsilon\|V_x^\varepsilon(\cdot,t)\|_{L^2}^2+\int_0^t\|V_\tau^\varepsilon\|_{L^2}^2\mathrm{d}\tau\le c(v_*,T)\int_0^t\|\Phi_x^\varepsilon\|_{L^2}^2(\varepsilon\|V_x^\varepsilon\|_{L^2}^2)\mathrm{d}\tau+c(v_*,T)\varepsilon^{1/2},\tag{4.50}$$

where we have used Lemma 4.4. Applying the Gronwall inequality to (4.50), we then obtain that

$$\varepsilon\|V_x^\varepsilon(\cdot,t)\|_{L^2}^2+\int_0^t\|V_\tau^\varepsilon\|_{L^2}^2\mathrm{d}\tau\le c(v_*,T)\varepsilon^{1/2},\tag{4.51}$$

where we have used (4.37). Now let us turn to the estimate on Φ_x^ε. Taking the L^2 inner product of the first equation in (4.8) with Φ_t^ε, followed by integration by parts, we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{I}}|\Phi_x^\varepsilon|^2\mathrm{d}x+\int_{\mathcal{I}}|\Phi_t^\varepsilon|^2\mathrm{d}x&=-\int_{\mathcal{I}}\Phi_t^\varepsilon\Phi_x^\varepsilon V_x^A\mathrm{d}x-\int_{\mathcal{I}}\varepsilon^{1/2}\Phi_t^\varepsilon\Phi_x^\varepsilon V_x^\varepsilon\mathrm{d}x\\&\quad-\int_{\mathcal{I}}V_x^\varepsilon\Phi_t^\varepsilon(\Phi_x^A+M)\mathrm{d}x+\int_{\mathcal{I}}\varepsilon^{-1/2}\mathcal{R}_1^\varepsilon\Phi_t^\varepsilon\mathrm{d}x=:\sum_{i=1}^4\mathcal{Q}_i.\end{aligned}\tag{4.52}$$

Next, we estimate $\mathcal{Q}_i$ $(1\le i\le 4)$. From (4.14) and the Cauchy–Schwarz inequality, we deduce that

$$\begin{aligned}\mathcal{Q}_1&\le\|V_x^A\|_{L^\infty}\|\Phi_t^\varepsilon\|_{L^2}\|\Phi_x^\varepsilon\|_{L^2}\le\frac{1}{8}\|\Phi_t^\varepsilon\|_{L^2}^2+c(v_*,T)\varepsilon^{-1}\|\Phi_x^\varepsilon\|_{L^2}^2,\\\mathcal{Q}_3&\le\|\Phi_x^A+M\|_{L^\infty}\|V_x^\varepsilon\|_{L^2}\|\Phi_t\|_{L^2}\le\frac{1}{8}\|\Phi_t^\varepsilon\|_{L^2}^2+c(v_*,T)\|V_x^\varepsilon\|_{L^2}^2.\end{aligned}$$

By the Cauchy–Schwarz inequality, we get

$$\mathcal{Q}_2\le\frac{1}{8}\int_{\mathcal{I}}|\Phi_t^\varepsilon|^2\mathrm{d}x+C\varepsilon\int_{\mathcal{I}}|\Phi_x|^2|V_x^\varepsilon|^2\mathrm{d}x\le\frac{1}{8}\int_{\mathcal{I}}|\Phi_t^\varepsilon|^2\mathrm{d}x+c_0\varepsilon\|V_x^\varepsilon\|_{L^\infty}^2\|\Phi_x^\varepsilon\|_{L^2}^2.\tag{4.53}$$

Similarly, we have

$$\mathcal{Q}_4\le\frac{1}{8}\int_{\mathcal{I}}|\Phi_t^\varepsilon|^2\mathrm{d}x+c_0\varepsilon^{-1}\|\mathcal{R}_1^\varepsilon\|_{L^2}^2\le\frac{1}{8}\int_{\mathcal{I}}|\Phi_t^\varepsilon|^2\mathrm{d}x+c(v_*,T)\varepsilon^{1/2},\tag{4.54}$$

where we have used Lemma 4.2. Therefore, inserting the estimates on $\mathcal{Q}_i$ $(0\le i\le 4)$ into (4.52), we get

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{I}}|\Phi_x^\varepsilon|^2\mathrm{d}x+\frac{1}{2}\int_{\mathcal{I}}|\Phi_t^\varepsilon|^2\mathrm{d}x\\&\quad\le c(v_*,T)(\varepsilon\|V_x^\varepsilon\|_{L^\infty}^2)\|\Phi_x^\varepsilon\|_{L^2}^2+c(v_*,T)(\varepsilon^{-1}\|\Phi_x^\varepsilon\|_{L^2}^2+\|V_x^\varepsilon\|_{L^2}^2)+c(v_*,T)\varepsilon^{1/2}.\end{aligned}$$

Integrating the above inequality over $(0,t)$ for any $t\in[0,T]$, we arrive at

$$\|\Phi_x^\varepsilon(\cdot,t)\|_{L^2}+\int_0^t\|\Phi_\tau^\varepsilon\|_{L^2}^2\mathrm{d}\tau\le c(v_*,T)\int_0^t\varepsilon\|V_x^\varepsilon\|_{L^\infty}^2\|\Phi_x^\varepsilon\|_{L^2}^2\mathrm{d}\tau+c(v_*,T)\varepsilon^{-1/2},\tag{4.55}$$

where we have used (4.37) and $0<\varepsilon<1$. To close the estimate, it now suffices to show that

$$\varepsilon\int_0^T \|V_x^\varepsilon\|_{L^\infty}^2 \mathrm{d}t \le c(v_*,T) \tag{4.56}$$

for some constant $c(v_*,T)>0$ independent of ε and δ. Indeed, if (4.56) holds true, then by the Gronwall inequality, we get from (4.55) that for any $t\in[0,T]$,

$$\|\Phi_x^\varepsilon(\cdot,t)\|_{L^2}+\int_0^t \|\Phi_\tau^\varepsilon\|_{L^2}^2 \mathrm{d}\tau \le c(v_*,T)\varepsilon^{-1/2}.$$

This, along with (4.51), yields (4.45). To prove (4.56), we first derive from the second equation in (4.8) that

$$\begin{aligned}\varepsilon^2\|V_{xx}^\varepsilon\|_{L^2}^2 &\le \|V_t^\varepsilon\|_{L^2}^2+\varepsilon\|V^\varepsilon\|_{L^\infty}^2\|\Phi_x^\varepsilon\|_{L^2}^2+c(v_*,T)\|\Phi_x^\varepsilon\|_{L^2}^2\\ &\quad + c(v_*,T)\|V^\varepsilon\|_{L^2}^2+\varepsilon^{-1}\|\mathcal{R}_2^\varepsilon\|^2,\end{aligned} \tag{4.57}$$

where (4.14) has been used. Therefore,

$$\begin{aligned}\varepsilon^2\int_0^T\|V_{xx}^\varepsilon\|_{L^2}^2\mathrm{d}t &\le \int_0^T\|V_t^\varepsilon\|_{L^2}^2\mathrm{d}t+\int_0^T\|\Phi_x^\varepsilon\|_{L^2}^2\mathrm{d}t+c(v_*,T)\int_0^T\|\Phi_x^\varepsilon\|_{L^2}^2\mathrm{d}t\\ &\quad +c(v_*,T)\int_0^T\|V^\varepsilon\|_{L^2}^2\mathrm{d}t+\varepsilon^{-1}\int_0^T\|\mathcal{R}_2^\varepsilon\|_{L^2}^2\mathrm{d}\tau\\ &\le c(v_*,T)\varepsilon^{1/2},\end{aligned} \tag{4.58}$$

where we have used (4.14b), (4.37), (4.51), Lemma 4.3, and the following fact:

$$\|V^\varepsilon\|_{L^\infty}^2\le c(v_*,T)\left(\|V^\varepsilon\|_{L^2}^2+\|V^\varepsilon\|_{L^2}\|V_x^\varepsilon\|_{L^2}\right)\le c(v_*,T)\varepsilon^{1/2}+c(v_*,T)\le c(v_*,T), \tag{4.59}$$

due to (4.37), (4.51), $0<\varepsilon<1$, and the Sobolev inequality (C.2). Therefore, we utilize the Sobolev inequality (C.2) again, along with (4.37), to derive that

$$\begin{aligned}&\int_0^T\|V_x^\varepsilon\|_{L^\infty}^2\mathrm{d}t\\ &\quad\le c(v_*,T)\int_0^T\|V_x^\varepsilon\|_{L^2}\|V_{xx}^\varepsilon\|_{L^2}\mathrm{d}t+c(v_*,T)\int_0^T\|V_x^\varepsilon\|_{L^2}^2\mathrm{d}t\\ &\quad\le c(v_*,T)\varepsilon^{1/2}\int_0^T\|V_{xx}^\varepsilon\|_{L^2}^2\mathrm{d}t+c(v_*,T)\varepsilon^{-1/2}\int_0^T\|V_x^\varepsilon\|_{L^2}^2\mathrm{d}t\le c(v_*,T)\varepsilon^{-1},\end{aligned}$$

where the constant $c(v_*,T)>0$ is independent of ε and δ. This gives (4.56). Thus we finish the proof of Lemma 4.5. ∎

As a direct consequence of Lemmas 4.4 and 4.5, we have the following corollary.

Corollary 4.6. *Assume the conditions of Lemmas 4.4 and 4.5 hold. Then for any solution $(\Phi^\varepsilon,V^\varepsilon)$ to (4.8) on $[0,T]$ satisfying (4.36), we have*

$$\int_0^T\left(\varepsilon^{1/2}\|\Phi_{xx}^\varepsilon\|_{L^2}^2+\|\Phi_x^\varepsilon\|_{L^\infty}^2+\varepsilon^{3/2}\|V_{xx}^\varepsilon\|_{L^2}^2\right)\mathrm{d}t\le c(v_*,T), \tag{4.60}$$

where $c(v_*,T)>0$ is a constant depending on T but independent of ε and δ.

Proof. The estimate on V^ε_{xx} follows from (4.58) directly. We now show estimates on Φ^ε_{xx} and Φ^ε_x. By the first equation in (4.8), we have

$$\begin{aligned}\|\Phi^\varepsilon_{xx}\|^2_{L^2} &\le \|\Phi^\varepsilon_t\|^2_{L^2} + \varepsilon\|\Phi^\varepsilon_x\|^2_{L^\infty}\|V^\varepsilon_x\|^2_{L^2} + \|V^A_x\|^2_{L^\infty}\|\Phi^\varepsilon_x\|^2_{L^2} + c(v_*,T)\|V^\varepsilon_x\|^2_{L^2}\\ &\quad + \varepsilon^{-1}\|\mathcal{R}^\varepsilon_1\|^2_{L^2},\end{aligned} \tag{4.61}$$

where we have used (4.14a). Therefore we derive that

$$\begin{aligned}&\int_0^T \|\Phi^\varepsilon_{xx}\|^2_{L^2}\mathrm{d}t\\ &\quad\le \int_0^T \|\Phi^\varepsilon_t\|^2_{L^2}\mathrm{d}t + \varepsilon \sup_{t\in[0,T]}\|\Phi^\varepsilon_x\|^2_{L^2}\int_0^T\|V^\varepsilon_x\|^2_{L^\infty}\mathrm{d}t + c(v_*,T)\varepsilon^{-1}\int_0^T\|\Phi^\varepsilon_x\|^2_{L^2}\mathrm{d}t\\ &\qquad + c(v_*,T)\int_0^T\|V^\varepsilon_x\|^2_{L^2}\mathrm{d}t + \varepsilon^{-1}\int_0^T\|\mathcal{R}^\varepsilon_1\|^2_{L^2}\mathrm{d}t\\ &\quad\le c(v_*,T)\varepsilon^{-1/2} + c(v_*,T)\varepsilon^{1/2} \le c(v_*,T)\varepsilon^{-1/2}\end{aligned} \tag{4.62}$$

for some constant $c(v_*,T)>0$ depending on T but independent of ε and δ, where we have used (4.14), (4.37), (4.51), (4.56), $0<\varepsilon<1$, and Lemma 4.2. This, along with (4.37) and the Sobolev inequality (C.2), further entails for $0<\varepsilon<1$ that

$$\begin{aligned}\int_0^T \|\Phi^\varepsilon_x\|^2_{L^\infty}\mathrm{d}t &\le c(v_*,T)\int_0^T\left(\|\Phi^\varepsilon_x\|^2_{L^2} + \|\Phi^\varepsilon_x\|_{L^2}\|\Phi^\varepsilon_{xx}\|_{L^2}\right)\mathrm{d}t\\ &\le c(v_*,T)(1+\varepsilon^{-1/2})\int_0^T\|\Phi^\varepsilon_x\|^2_{L^2}\mathrm{d}t + c(v_*,T)\varepsilon^{1/2}\int_0^T\|\Phi^\varepsilon_{xx}\|^2_{L^2}\mathrm{d}t\\ &\le c(v_*,T).\end{aligned}$$

The proof is complete. ∎

4.2.4. Higher-order estimates. To prove the convergence result in Theorem 2.1, we derive some higher-order estimates for $(\Phi^\varepsilon, V^\varepsilon)$ in this subsection.

LEMMA 4.7. *Assume that the conditions of Lemmas* 4.4 *and* 4.5 *hold. Then it holds for any* $t\in(0,T]$ *that*

$$\begin{aligned}&\|\Phi^\varepsilon_t(\cdot,t)\|^2_{L^2} + \|V^\varepsilon_t(\cdot,t)\|^2_{L^2} + \varepsilon^{1/2}\|\Phi^\varepsilon_{xx}(\cdot,t)\|^2_{L^2} + \varepsilon\|V^\varepsilon_{xx}(\cdot,t)\|^2_{L^2}\\ &\quad + \int_0^t\left(\|\Phi^\varepsilon_{x\tau}\|^2_{L^2} + \varepsilon\|V_{x\tau}\|^2_{L^2}\right)\mathrm{d}\tau \le c(v_*,T)\varepsilon^{-1/2}, \quad t\in(0,T],\end{aligned} \tag{4.63}$$

where $c(v_*,T)>0$ *is a constant independent of* ε *and* δ.

Proof. Differentiating the equations in (4.8) with respect to t, we end up with

$$\begin{cases}\Phi^\varepsilon_{tt} = \Phi^\varepsilon_{xxt} - \varepsilon^{\frac12}\Phi^\varepsilon_{xt}V^\varepsilon_x - \varepsilon^{\frac12}\Phi^\varepsilon_x V^\varepsilon_{xt} - \Phi^\varepsilon_{xt}V^A_x - \Phi^\varepsilon_x V^A_{xt} - V^\varepsilon_{xt}(\Phi^A_x + M) - V^\varepsilon_x\Phi^A_{xt}\\ \qquad + \varepsilon^{-\frac12}\partial_t\mathcal{R}^\varepsilon_1,\\ V^\varepsilon_{tt} = \varepsilon V^\varepsilon_{xxt} - \varepsilon^{\frac12}\Phi^\varepsilon_{xt}V^\varepsilon - \varepsilon^{\frac12}\Phi^\varepsilon_x V^\varepsilon_t - \Phi^\varepsilon_{xt}V^A - \Phi^\varepsilon_x V^A_t - \Phi^A_{xt}V^\varepsilon - (\Phi^A_x + M)V^\varepsilon_t\\ \qquad + \varepsilon^{-\frac12}\partial_t\mathcal{R}^\varepsilon_2.\end{cases} \tag{4.64}$$

Multiplying the first equation in (4.64) by Φ_t^ε, and then integrating the resulting equation over $\mathcal{I}$, after using the integration by parts we get that

(4.65)
$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{I}}|\Phi_t^\varepsilon|^2\mathrm{d}x+\int_{\mathcal{I}}|\Phi_{xt}^\varepsilon|^2\mathrm{d}x\\
&\quad=-\varepsilon^{1/2}\int_{\mathcal{I}}\Phi_{xt}^\varepsilon V_x^\varepsilon\Phi_t^\varepsilon\mathrm{d}x-\int_{\mathcal{I}}\varepsilon^{1/2}\Phi_x^\varepsilon V_{xt}^\varepsilon\Phi_t^\varepsilon\mathrm{d}x-\int_{\mathcal{I}}\Phi_x^\varepsilon V_{xt}^A\Phi_t^\varepsilon\mathrm{d}x-\int_{\mathcal{I}}V_x^\varepsilon\Phi_{xt}^A\Phi_t^\varepsilon\mathrm{d}x\\
&\qquad+\int_{\mathcal{I}}\varepsilon^{-1/2}\partial_t\mathcal{R}_1^\varepsilon\Phi_t^\varepsilon\mathrm{d}x-\int_{\mathcal{I}}\Phi_{xt}^\varepsilon V_x^A\Phi_t^\varepsilon\mathrm{d}x-\int_{\mathcal{I}}V_{xt}^\varepsilon(\Phi_x^A+M)\Phi_t^\varepsilon\mathrm{d}x=:\sum_{i=1}^{7}\mathcal{H}_i,
\end{aligned}$$

where, in view of the Cauchy–Schwarz inequality, $\mathcal{H}_i$ $(1\le i\le 5)$ can be estimated as follows:

(4.66)
$$\begin{cases}
\mathcal{H}_1\le\frac{1}{4}\|\Phi_{xt}^\varepsilon\|_{L^2}^2+c_0\varepsilon\|V_x^\varepsilon\|_{L^\infty}^2\|\Phi_t^\varepsilon\|_{L^2}^2,\ \mathcal{H}_2\le\frac{\varepsilon}{16}\|V_{xt}^\varepsilon\|_{L^2}^2+c_0\|\Phi_x^\varepsilon\|_{L^\infty}^2\|\Phi_t^\varepsilon\|_{L^2}^2,\\
\mathcal{H}_3\le c_0\|V_{xt}^A\|_{L^\infty}\left(\varepsilon^{-1/2}\|\Phi_x^\varepsilon\|_{L^2}^2+\varepsilon^{1/2}\|\Phi_t^\varepsilon\|_{L^2}^2\right),\ \mathcal{H}_4\le c_0\|\Phi_{xt}^A\|_{L^\infty}\left(\|V_x^\varepsilon\|_{L^2}^2+\|\Phi_t^\varepsilon\|_{L^2}^2\right),\\
\mathcal{H}_5\le\|\Phi_t^\varepsilon\|_{L^2}^2+c_0\varepsilon^{-1}\|\partial_t\mathcal{R}_1^\varepsilon\|_{L^2}^2.
\end{cases}$$

For $\mathcal{H}_6$ and $\mathcal{H}_7$, it follows from (3.27), (3.61), (4.3b), (4.12), (4.13), (C.4), integration by parts, and Hardy inequality (C.5) that

$$\begin{aligned}
\mathcal{H}_6&=-\int_{\mathcal{I}}\Phi_{xt}^\varepsilon\varepsilon^{-1/2}(v_z^{B,0}+v_\xi^{b,0})\Phi_t^\varepsilon\mathrm{d}x\\
&\quad-\int_{\mathcal{I}}\Phi_{xt}^\varepsilon(v_x^{I,0}+v_z^{B,1}+v_\xi^{b,1}+\varepsilon^{1/2}v_x^{I,1}+\partial_x b_v^\varepsilon)\Phi_t^\varepsilon\mathrm{d}x\\
&\le\left\|\frac{\Phi_t^\varepsilon}{x(1-x)}\right\|_{L^2}\|\Phi_{xt}^\varepsilon\|_{L^2}\left(\left\|\frac{x(1-x)}{\varepsilon^{1/2}}v_z^{B,0}\right\|_{L^\infty}+\left\|\frac{x(1-x)}{\varepsilon^{1/2}}v_\xi^{b,0}\right\|_{L^\infty}\right)\\
&\quad+\|\Phi_t^\varepsilon\|_{L^2}\|\Phi_{xt}^\varepsilon\|_{L^2}\\
&\quad\times\left(\|v_x^{I,0}\|_{L^\infty}+\|v_z^{B,1}\|_{L_z^\infty}+\|v_\xi^{b,1}\|_{L_\xi^\infty}+\varepsilon^{1/2}\|v_x^{I,1}\|_{L^\infty}+\|\partial_x b_v^\varepsilon\|_{L^\infty}\right)\\
&\le c_0\|\Phi_{xt}^\varepsilon\|_{L^2}^2\left(\|\langle z\rangle v_z^{B,0}\|_{L_z^\infty}+\|\langle\xi\rangle v_\xi^{b,0}\|_{L_\xi^\infty}\right)+c(v_*,T)\|\Phi_{xt}^\varepsilon\|_{L^2}\|\Phi_t^\varepsilon\|_{L^2}\\
&\le\left(\frac{1}{8}+K_1(T,v_*)v_*\right)\|\Phi_{xt}^\varepsilon\|_{L^2}^2+c(v_*,T)\|\Phi_t^\varepsilon\|_{L^2}^2
\end{aligned}\tag{4.67}$$

with $K_1(T,v_*)>0$ as in (4.40), and that

$$\begin{aligned}
\mathcal{H}_7&=-\int_{\mathcal{I}}V_t^\varepsilon\Phi_t^\varepsilon\Phi_{xx}^A\mathrm{d}x-\int_{\mathcal{I}}(\Phi_x^A+M)V_t^\varepsilon\Phi_{xt}^\varepsilon\mathrm{d}x\\
&\le-\varepsilon^{1/2}\int_{\mathcal{I}}V_t^\varepsilon\Phi_t^\varepsilon(\varphi_{xx}^{B,1}+\varphi_{xx}^{b,1})\mathrm{d}x\\
&\quad-\int_{\mathcal{I}}V_t^\varepsilon\Phi_t^\varepsilon\left(\varphi_{xx}^{I,0}+\varepsilon^{1/2}\varphi_{xx}^{I,1}+\varepsilon\varphi_{xx}^{B,2}+\varepsilon\varphi_{xx}^{b,2}+\partial_x^2 b_\varphi^\varepsilon\right)\mathrm{d}x\\
&\quad+\|\Phi_x^A+M\|_{L^\infty}\|V_t^\varepsilon\|_{L^2}\|\Phi_{xt}^\varepsilon\|_{L^2}
\end{aligned}\tag{4.68}$$

$$
\begin{aligned}
&\le \|V_t^\varepsilon\|_{L^2}\left\|\frac{\Phi_t^\varepsilon}{x(1-x)}\right\|_{L^2}\left(\left\|\frac{x(1-x)}{\varepsilon^{1/2}}\varphi_{zz}^{B,1}\right\|_{L^\infty}+\left\|\frac{x(1-x)}{\varepsilon^{1/2}}\varphi_{\xi\xi}^{b,1}\right\|_{L^\infty}\right)\\
&\quad + c(v_*,T)\|V_t^\varepsilon\|_{L^2}\|\Phi_{xt}^\varepsilon\|_{L^2}\\
&\quad + \|V_t^\varepsilon\|_{L^2}\|\Phi_t^\varepsilon\|_{L^2}\\
&\quad \times\left(\|\varphi_{xx}^{I,0}\|_{L^\infty}+\|\varphi_{zz}^{B,2}\|_{L_z^\infty}+\|\varphi_{\xi\xi}^{b,2}\|_{L_\xi^\infty}+\varepsilon^{1/2}\|\varphi_{xx}^{I,1}\|_{L^\infty}+\|\partial_x^2 b_\varphi^\varepsilon\|_{L^\infty}\right)\\
&\le c_0\|V_t^\varepsilon\|_{L^2}\|\Phi_{xt}^\varepsilon\|_{L^2}\left(\|\langle z\rangle\varphi_{zz}^{B,1}\|_{L_z^\infty}+\|\langle\xi\rangle\varphi_{\xi\xi}^{b,1}\|_{L_\xi^\infty}\right)\\
&\quad + c(v_*,T)\|V_t^\varepsilon\|_{L^2}\left(\|\Phi_t^\varepsilon\|_{L^2}+\|\Phi_{xt}^\varepsilon\|_{L^2}\right)\le\frac{1}{8}\|\Phi_{xt}\|_{L^2}^2+c(v_*,T)(\|\Phi_t^\varepsilon\|_{L^2}^2+\|V_t^\varepsilon\|_{L^2}^2),
\end{aligned}
$$

where the constant $C>0$ is independent of ε and δ. Plugging (4.66)–(4.68) into (4.65) followed by an integration over $(0,t)$ for any $t\in(0,T]$, it follows that

(4.69)

$$
\begin{aligned}
&\|\Phi_t^\varepsilon(\cdot,t)\|_{L^2}^2+\int_0^t\|\Phi_{x\tau}^\varepsilon\|_{L^2}^2\mathrm{d}\tau\\
&\quad\le\frac{\varepsilon}{4}\int_0^t\|V_{x\tau}\|_{L^2}^2\mathrm{d}\tau+c(v_*,T)\int_0^t\left(\varepsilon\|V_x^\varepsilon\|_{L^\infty}^2+\|\Phi_x^\varepsilon\|_{L^\infty}^2\right)\|\Phi_\tau^\varepsilon(\cdot,\tau)\|_{L^2}^2\mathrm{d}\tau+c(v_*,T)\varepsilon^{-1/2},
\end{aligned}
$$

provided that $K_1(v_*,T)v_*\le 1/16$ in (4.67), where we have used (4.14), $0<\varepsilon<1$, and Lemmas 4.2, 4.4, and 4.5.

To proceed, multiplying the second equation in (4.64) by V_t^ε, after integrating the resulting equation over $\mathcal{I}$ we get that

(4.70)

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{I}}|V_t^\varepsilon|^2\,\mathrm{d}x+\varepsilon\int_{\mathcal{I}}|V_{xt}^\varepsilon|^2\,\mathrm{d}x+\int_{\mathcal{I}}\left(\Phi_x^A+M\right)|V_t^\varepsilon|^2\,\mathrm{d}x\\
&=-\varepsilon^{1/2}\int_{\mathcal{I}}\Phi_{xt}^\varepsilon V^\varepsilon V_t^\varepsilon\mathrm{d}x-\int_{\mathcal{I}}\varepsilon^{1/2}\Phi_x^\varepsilon V_t^\varepsilon V_t^\varepsilon\mathrm{d}x-\int_{\mathcal{I}}\Phi_{xt}^\varepsilon V^A V_t^\varepsilon\mathrm{d}x-\int_{\mathcal{I}}\Phi_x^\varepsilon V_t^A V_t^\varepsilon\mathrm{d}x\\
&\quad-\int_{\mathcal{I}}\Phi_{xt}^A V^\varepsilon V_t^\varepsilon\mathrm{d}x+\varepsilon^{-1/2}\int_{\mathcal{I}}\partial_t\mathcal{R}_2^\varepsilon V_t^\varepsilon\mathrm{d}x=:\sum_{i=1}^6\mathcal{L}_i,
\end{aligned}
$$

where, similar to (4.66), $\mathcal{L}_i\,(1\le i\le 6)$ enjoy the following estimates:

(4.71)

$$
\begin{cases}
\mathcal{L}_1\le\frac{1}{4}\|\Phi_{xt}^\varepsilon\|_{L^2}^2+c(v_*,T)\varepsilon\|V^\varepsilon\|_{L^\infty}^2\|V_t^\varepsilon\|_{L^2}^2, \quad \mathcal{L}_2\le\varepsilon^{1/2}\|\Phi_x^\varepsilon\|_{L^\infty}\|V_t\|_{L^2}^2,\\
\mathcal{L}_3\le\frac{1}{4}\|\Phi_{xt}^\varepsilon\|_{L^2}^2+c(v_*,T)\|V^A\|_{L^\infty}^2\|V_t^\varepsilon\|_{L^2}^2, \quad \mathcal{L}_4\le\|V_t^A\|_{L^\infty}\left(\|\Phi_x^\varepsilon\|_{L^2}^2+\|V_t^\varepsilon\|_{L^2}^2\right),\\
\mathcal{L}_5\le\|\Phi_{xt}^A\|_{L^\infty}\left(\|V^\varepsilon\|_{L^2}^2+\|V_t^\varepsilon\|_{L^2}^2\right), \quad \mathcal{L}_6\le\|V_t^\varepsilon\|_{L^2}^2+c(v_*,T)\varepsilon^{-1}\|\partial_t\mathcal{R}_2^\varepsilon\|_{L^2}^2.
\end{cases}
$$

Therefore, we integrate (4.70) over $(0,t)\subset(0,T]$ to get

$$
\begin{aligned}
&\|V_t^\varepsilon(\cdot,t)\|_{L^2}^2+\int_0^t\int_{\mathcal{I}}\varepsilon\,|V_{x\tau}^\varepsilon|^2\,\mathrm{d}x\mathrm{d}\tau\\
&\quad\le c(v_*,T)\int_0^t\left(1+\varepsilon\|V^\varepsilon\|_{L^\infty}^2+\varepsilon^{1/2}\|\Phi_x^\varepsilon\|_{L^\infty}\right)\|V_\tau^\varepsilon(\cdot,\tau)\|_{L^2}^2\mathrm{d}\tau\\
&\qquad+\frac{1}{2}\int_0^t\|\Phi_{x\tau}^\varepsilon\|_{L^2}^2\mathrm{d}\tau+c(v_*,T)\varepsilon^{1/2}, \qquad (4.72)
\end{aligned}
$$

where we have used (4.14), (4.37), (4.45), $0<\varepsilon<1$, and Lemma 4.3. Combining (4.72) with (4.69), we arrive at

$$\begin{aligned}
&\|\Phi_t^\varepsilon(\cdot,t)\|_{L^2}^2+\|V_t^\varepsilon(\cdot,t)\|_{L^2}^2+\int_0^t\left(\|\Phi_{x\tau}^\varepsilon\|_{L^2}^2+\varepsilon\|V_{x\tau}^\varepsilon\|_{L^2}^2\right)\mathrm{d}\tau\\
&\le c(v_*,T)\varepsilon^{-1/2}+c(v_*,T)\int_0^t\left(\varepsilon\|V_x^\varepsilon\|_{L^\infty}^2+\|\Phi_x^\varepsilon\|_{L^\infty}^2\right)\|\Phi_\tau^\varepsilon(\cdot,\tau)\|_{L^2}^2\mathrm{d}\tau\\
&\quad+c(v_*,T)\int_0^t\Big(1+\varepsilon\|V^\varepsilon\|_{L^\infty}^2+\varepsilon^{1/2}\|\Phi_x^\varepsilon\|_{L^\infty}\Big)\|V_\tau^\varepsilon(\cdot,\tau)\|_{L^2}^2\mathrm{d}\tau.
\end{aligned}\tag{4.73}$$

Applying the Gronwall inequality to (4.73), alongside (4.56), (4.59), and (4.60), we get that

$$\|\Phi_t^\varepsilon(\cdot,t)\|_{L^2}^2+\|V_t^\varepsilon(\cdot,t)\|_{L^2}^2+\int_0^t\left(\|\Phi_{x\tau}^\varepsilon\|_{L^2}^2+\varepsilon\|V_{x\tau}^\varepsilon\|_{L^2}^2\right)\mathrm{d}\tau\le c(v_*,T)\varepsilon^{-1/2}.$$

This, along with (4.14), (4.37), (4.45), (4.57), and Lemmas 4.2 and 4.3, further entails that

$$\varepsilon^{3/2}\|V_{xx}^\varepsilon\|_{L_T^\infty L^2}^2\le c(v_*,T).$$

It now remains to derive the estimate for Φ_{xx}^ε. Multiplying the first equation in (4.64) by Φ_{xx}^ε, followed by an integration over $\mathcal{I}$, we get

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{I}}|\Phi_{xx}^\varepsilon|^2\mathrm{d}x\\
&\quad=\int_{\mathcal{I}}\Phi_{tt}^\varepsilon\Phi_{xx}^\varepsilon\mathrm{d}x+\varepsilon^{1/2}\int_{\mathcal{I}}\Phi_{xt}^\varepsilon V_x^\varepsilon\Phi_{xx}^\varepsilon\mathrm{d}x+\int_{\mathcal{I}}\varepsilon^{1/2}\Phi_x^\varepsilon V_{xt}^\varepsilon\Phi_{xx}^\varepsilon\mathrm{d}x+\int_{\mathcal{I}}\Phi_x^\varepsilon V_{xt}^A\Phi_{xx}^\varepsilon\mathrm{d}x\\
&\qquad+\int_{\mathcal{I}}V_x^\varepsilon\Phi_{xt}^A\Phi_{xx}^\varepsilon\mathrm{d}x-\int_{\mathcal{I}}\varepsilon^{-1/2}\partial_t\mathcal{R}_1^\varepsilon\Phi_{xx}^\varepsilon\mathrm{d}x+\int_{\mathcal{I}}\Phi_{xt}^\varepsilon V_x^A\Phi_{xx}^\varepsilon\mathrm{d}x\\
&\qquad+\int_{\mathcal{I}}V_{xt}^\varepsilon(\Phi_x^A+M)\Phi_{xx}^\varepsilon\mathrm{d}x:=\int_{\mathcal{I}}\Phi_{tt}^\varepsilon\Phi_{xx}^\varepsilon\mathrm{d}x+\sum_{i=1}^7\hat{\mathcal{H}}_i,
\end{aligned}\tag{4.74}$$

where, thanks to integration by parts, we have

$$\int_{\mathcal{I}}\Phi_{tt}^\varepsilon\Phi_{xx}^\varepsilon\mathrm{d}x=\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{I}}\Phi_t^\varepsilon\Phi_{xx}^\varepsilon\mathrm{d}x-\int_{\mathcal{I}}\Phi_t^\varepsilon\Phi_{xxt}^\varepsilon=\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{I}}\Phi_t^\varepsilon\Phi_{xx}^\varepsilon\mathrm{d}x+\int_{\mathcal{I}}|\Phi_{xt}^\varepsilon|^2\mathrm{d}x.\tag{4.75}$$

For $\hat{\mathcal{H}}_i$ $(1\le i\le 7)$, we get by the Cauchy–Schwarz inequality that

$$\begin{cases}
\hat{\mathcal{H}}_1\le c_0(\|\Phi_{xt}^\varepsilon\|_{L^2}^2+\varepsilon\|V_x^\varepsilon\|_{L^\infty}^2\|\Phi_{xx}^\varepsilon\|_{L^2}^2),\quad \hat{\mathcal{H}}_2\le c_0(\varepsilon\|V_{xt}^\varepsilon\|_{L^2}^2+\|\Phi_x^\varepsilon\|_{L^\infty}^2\|\Phi_{xx}^\varepsilon\|_{L^2}^2),\\
\hat{\mathcal{H}}_3\le c_0(\|V_{xt}^A\|_{L^\infty}^2\|\Phi_x^\varepsilon\|_{L^2}^2+\|\Phi_{xx}^\varepsilon\|_{L^2}^2),\quad \hat{\mathcal{H}}_4\le c_0(\|\Phi_{xt}^A\|_{L^\infty}^2\|V_x^\varepsilon\|_{L^2}^2+\|\Phi_{xx}^\varepsilon\|_{L^2}^2),\\
\hat{\mathcal{H}}_5\le c_0(\|\Phi_{xx}^\varepsilon\|_{L^2}^2+\varepsilon^{-1}\|\partial_t\mathcal{R}_1^\varepsilon\|_{L^2}^2),\quad \hat{\mathcal{H}}_6\le\|V_x^A\|_{L^\infty}\left(\|\Phi_{xt}^\varepsilon\|_{L^2}^2+\|\Phi_{xx}^\varepsilon\|_{L^2}^2\right),\\
\hat{\mathcal{H}}_7\le c_0(\varepsilon^{1/2}\|\Phi_x^A+M\|_{L^\infty}^2\|V_{xt}^\varepsilon\|_{L^2}^2+\varepsilon^{-1/2}\|\Phi_{xx}^\varepsilon\|_{L^2}^2).
\end{cases}$$

Inserting (4.75) and estimates on $\hat{\mathcal{H}}_i$ $(1\le i\le 7)$ into (4.74), followed by an integration in t, we get

$$
\begin{aligned}
&\|\Phi^\varepsilon_{xx}(\cdot,t)\|^2_{L^2}\\
&\le \int_{\mathcal{I}} \Phi^\varepsilon_t \Phi^\varepsilon_{xx}\mathrm{d}x + c(v_*,T)\varepsilon^{-1/2} + c(v_*,T)\varepsilon^{-1/2}\int_0^t \left(\|\Phi^\varepsilon_{x\tau}\|^2_{L^2} + \|\Phi^\varepsilon_{xx}\|^2_{L^2}\right)\mathrm{d}\tau\\
&\quad + c(v_*,T)\int_0^t \left(\varepsilon^{1/2}\|V^2_{x\tau}\|^2_{L^2} + \varepsilon^{-1/2}\|\Phi^\varepsilon_{xx}\|^2_{L^2}\right)\mathrm{d}\tau\\
&\quad + c(v_*,T)\int_0^t \left(\varepsilon\|V^\varepsilon_x\|^2_{L^\infty} + \|\Phi^\varepsilon_x\|^2_{L^\infty}\right)\|\Phi^\varepsilon_{xx}\|^2_{L^2}\mathrm{d}\tau\\
&\le \frac{1}{2}\|\Phi^\varepsilon_{xx}(\cdot,t)\|^2_{L^2} + c(v_*,T)\|\Phi^\varepsilon_t\|^2_{L^2} + c(v_*,T)\varepsilon^{-1}\\
&\quad + c(v_*,T)\int_0^t \left(\varepsilon\|V^\varepsilon_x\|^2_{L^\infty} + \|\Phi^\varepsilon_x\|^2_{L^\infty}\right)\|\Phi^\varepsilon_{xx}\|^2_{L^2}\mathrm{d}\tau\\
&\le \frac{1}{2}\|\Phi^\varepsilon_{xx}(\cdot,t)\|^2_{L^2} + c(v_*,T)\varepsilon^{-1} + c(v_*,T)\int_0^t \left(\varepsilon\|V^\varepsilon_x\|^2_{L^\infty} + \|\Phi^\varepsilon_x\|^2_{L^\infty}\right)\|\Phi^\varepsilon_{xx}\|^2_{L^2}\mathrm{d}\tau,
\end{aligned}
$$

where we have used (4.14), (4.37), (4.45), (4.60), and Lemma 4.2. That is,

$$
\|\Phi^\varepsilon_{xx}(\cdot,t)\|^2_{L^2} \le c(v_*,T)\varepsilon^{-1} + c(v_*,T)\int_0^t \left(\varepsilon\|V^\varepsilon_x\|^2_{L^\infty} + \|\Phi^\varepsilon_x\|^2_{L^\infty}\right)\|\Phi^\varepsilon_{xx}\|^2_{L^2}\mathrm{d}\tau,
$$

which, along with (4.60) and the Gronwall inequality, gives

$$
\|\Phi^\varepsilon_{xx}(\cdot,t)\|^2_{L^2} \le c(v_*,T)\varepsilon^{-1}, \quad t\in(0,T],
$$

for some constant $c(v_*,T) > 0$ independent of ε and δ and thus ends the proof of Lemma 4.7. ∎

With Lemma 4.7, we can get an improved estimate for Φ^ε_x.

Corollary 4.8. *Assume the conditions in Lemmas* 4.4–4.7 *hold. Let* $(\Phi^\varepsilon, V^\varepsilon)$ *be the solution of the problem* (4.8) *on* $[0,T]$ *satisfying* (4.36)*. Then we have*

$$
\|\Phi^\varepsilon_x(\cdot,t)\|^2_{L^2} + \int_0^t\int_{\mathcal{I}} |\Phi^\varepsilon_\tau|^2\mathrm{d}x\mathrm{d}\tau \le c(v_*,T), \quad t\in(0,T], \tag{4.76}
$$

where $c(v_*,T)>0$ *is a constant independent of* ε *and* δ*.*

Proof. Recalling (4.52), (4.53), and (4.54), we have

(4.77)

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{I}} |\Phi^\varepsilon_x|^2\mathrm{d}x + \int_{\mathcal{I}} |\Phi^\varepsilon_t|^2\mathrm{d}x\\
&= -\int_{\mathcal{I}} \Phi^\varepsilon_t\Phi^\varepsilon_x V^A_x\mathrm{d}x - \int_{\mathcal{I}} V^\varepsilon_x\Phi^\varepsilon_t(\Phi^A_x + M)\mathrm{d}x - \int_{\mathcal{I}} \varepsilon^{1/2}\Phi^\varepsilon_t\Phi^\varepsilon_x V^\varepsilon_x\mathrm{d}x + \int_{\mathcal{I}} \varepsilon^{-1/2}\mathcal{R}^\varepsilon_1\Phi^\varepsilon_t\mathrm{d}x\\
&\le -\int_{\mathcal{I}} \Phi^\varepsilon_t\Phi^\varepsilon_x V^A_x\mathrm{d}x - \int_{\mathcal{I}} V^\varepsilon_x\Phi^\varepsilon_t(\Phi^A_x + M)\mathrm{d}x + \frac{1}{4}\int_{\mathcal{I}} |\Phi^\varepsilon_t|^2\,\mathrm{d}x\\
&\quad + c(v_*,T)\varepsilon\|V^\varepsilon_x\|^2_{L^\infty}\|\Phi^\varepsilon_x\|^2_{L^2} + c(v_*,T)\varepsilon^{1/2}\\
&=: \hat{\mathcal{Q}}_1 + \hat{\mathcal{Q}}_2 + \frac{1}{4}\int_{\mathcal{I}} |\Phi^\varepsilon_t|^2\,\mathrm{d}x + c(v_*,T)\varepsilon\|V^\varepsilon_x\|^2_{L^\infty}\|\Phi^\varepsilon_x\|^2_{L^2} + c(v_*,T)\varepsilon^{1/2},
\end{aligned}
$$

where, thanks to (3.27), (3.61), (4.3b), (4.12), (4.13), (C.4), integration by parts, and Hardy inequality (C.5), $\hat{Q}_1$ and $\hat{\mathcal{Q}}_2$ enjoy the following estimates:

$$\begin{aligned}
\hat{\mathcal{Q}}_1 = &-\int_{\mathcal{I}} \Phi_t^\varepsilon \Phi_x^\varepsilon \varepsilon^{-1/2}\left(v_z^{B,0}+v_\xi^{b,0}\right)\mathrm{d}x \\
&- \int_{\mathcal{I}} \Phi_t^\varepsilon \Phi_x^\varepsilon \left(v_x^{I,0}+v_z^{B,1}+v_\xi^{b,1}+\varepsilon^{1/2}v_x^{I,1}+\partial_x b_v^\varepsilon\right)\mathrm{d}x \\
\leq &\left\|\frac{\Phi_t^\varepsilon}{x(1-x)}\right\|_{L^2}\|\Phi_x^\varepsilon\|_{L^2}\left(\left\|\frac{x(1-x)}{\varepsilon^{1/2}}v_z^{B,0}\right\|_{L^\infty}+\left\|\frac{x(1-x)}{\varepsilon^{1/2}}v_\xi^{b,0}\right\|_{L^\infty}\right) \\
&+\|\Phi_t^\varepsilon\|_{L^2}\|\Phi_x^\varepsilon\|_{L^2}\left(\|v_x^{I,0}\|_{L^\infty}+\|v_z^{B,1}\|_{L_z^\infty}+\|v_\xi^{b,1}\|_{L_\xi^\infty}+\varepsilon^{1/2}\|v_x^{I,1}\|_{L^\infty}+\|\partial_x b_v^\varepsilon\|_{L^\infty}\right) \\
\leq &\, c_0\|\Phi_{xt}^\varepsilon\|_{L^2}\|\Phi_x^\varepsilon\|_{L^2}\left(\|\langle z\rangle v_z^{B,0}\|_{L_z^\infty}+\|\langle \xi\rangle v_\xi^{b,0}\|_{L_\xi^\infty}\right)+c(v_*,T)\|\Phi_x^\varepsilon\|_{L^2}\|\Phi_t^\varepsilon\|_{L^2} \\
\leq &\,\frac{1}{8}\|\Phi_t^\varepsilon\|_{L^2}^2+c(v_*,T)\varepsilon^{1/2}\|\Phi_{xt}^\varepsilon\|_{L^2}^2+c(v_*,T)(1+\varepsilon^{-1/2})\|\Phi_x^\varepsilon\|_{L^2}^2,
\end{aligned}$$

$$\begin{aligned}
\hat{\mathcal{Q}}_2 = &\int_{\mathcal{I}} V^\varepsilon \Phi_t^\varepsilon \Phi_{xx}^A \mathrm{d}x + \int_{\mathcal{I}} (\Phi_x^A + M) V^\varepsilon \Phi_{xt}^\varepsilon \mathrm{d}x \\
\leq &\,\varepsilon^{1/2}\int_{\mathcal{I}} V^\varepsilon \Phi_t^\varepsilon(\varphi_{xx}^{B,1}+\varphi_{xx}^{b,1})\mathrm{d}x \\
&+ \int_{\mathcal{I}} V^\varepsilon \Phi_t^\varepsilon\left(\varphi_{xx}^{I,0}+\varepsilon^{1/2}\varphi_{xx}^{I,1}+\varepsilon\varphi_{xx}^{B,2}+\varepsilon\varphi_{xx}^{b,2}+\partial_x^2 b_\varphi^\varepsilon\right)\mathrm{d}x \\
&+\|\Phi_x^A+M\|_{L^\infty}\|V^\varepsilon\|_{L^2}\|\Phi_{xt}^\varepsilon\|_{L^2} \\
\leq &\,\|V^\varepsilon\|_{L^2}\left\|\frac{\Phi_t^\varepsilon}{x(1-x)}\right\|_{L^2}\left(\left\|\frac{x(1-x)}{\varepsilon^{1/2}}\varphi_{zz}^{B,1}\right\|_{L^\infty}+\left\|\frac{x(1-x)}{\varepsilon^{1/2}}\varphi_{\xi\xi}^{B,1}\right\|_{L^\infty}\right) \\
&+c(v_*,T)\|V^\varepsilon\|_{L^2}\|\Phi_{xt}^\varepsilon\|_{L^2} \\
&+c_0\|V^\varepsilon\|_{L^2}\|\Phi_t^\varepsilon\|_{L^2} \\
&\times\left(\|\varphi_{xx}^{I,0}\|_{L^\infty}+\varepsilon^{1/2}\|\varphi_{xx}^{I,1}\|_{L^\infty}+\|\varphi_{zz}^{B,2}\|_{L_z^\infty}+\|\varphi_{\xi\xi}^{b,2}\|_{L_\xi^\infty}+\|\partial_x^2 b_\varphi^\varepsilon\|_{L^\infty}\right) \\
\leq &\, c_0\|V^\varepsilon\|_{L^2}\|\Phi_{xt}^\varepsilon\|_{L^2} \\
&\times\left(\|\langle z\rangle\varphi_{zz}^{B,1}\|_{L_z^\infty}+\|\langle\xi\rangle\varphi_{\xi\xi}^{b,1}\|_{L_\xi^\infty}\right)+c(v_*,T)\|V^\varepsilon\|_{L^2}\left(\|\Phi_t^\varepsilon\|_{L^2}+\|\Phi_{xt}^\varepsilon\|_{L^2}\right) \\
\leq &\,\frac{1}{8}\|\Phi_t^\varepsilon\|_{L^2}^2+c(v_*,T)\varepsilon^{1/2}\|\Phi_{xt}^\varepsilon\|_{L^2}^2+c(v_*,T)(1+\varepsilon^{-1/2})\|V^\varepsilon\|_{L^2}^2.
\end{aligned}$$

Therefore, we update (4.77) as

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{I}}|\Phi_x^\varepsilon|^2\mathrm{d}x+\frac{1}{2}\int_{\mathcal{I}}|\Phi_t^\varepsilon|^2\mathrm{d}x \\
&\leq c(v_*,T)\varepsilon\|V_x^\varepsilon\|_{L^\infty}^2\|\Phi_x^\varepsilon\|_{L^2}^2+c(v_*,T)\varepsilon^{1/2}+c(v_*,T)\varepsilon^{1/2}\|\Phi_{xt}^\varepsilon\|_{L^2}^2 \\
&\quad+c(v_*,T)\varepsilon^{-1/2}(\|\Phi_x^\varepsilon\|_{L^2}^2+\|V^\varepsilon\|_{L^2}^2).
\end{aligned} \tag{4.78}$$

Integrating (4.78) with respect to t gives

$$\int_{\mathcal{I}}|\Phi_x^\varepsilon|^2(\cdot,t)\mathrm{d}x+\int_0^t|\Phi_\tau^\varepsilon|^2\mathrm{d}x\mathrm{d}\tau\leq c(v_*,T)+c(v_*,T)\int_0^t\varepsilon\|V_x^\varepsilon\|_{L^\infty}^2\|\Phi_x^\varepsilon\|_{L^2}^2\mathrm{d}\tau,\quad t\in[0,T],$$

for some constant $c(v_*,T)>0$ independent of ε and δ, where we have used (4.37), (4.63), and $0<\varepsilon<1$. This, alongside the Gronwall inequality and (4.56), immediately implies (4.76), and the proof is complete. ∎

Remark 4.1. In view of (4.37) and (4.76), the a priori assumption (4.36) is verified. Indeed, from (4.37), (4.76), and the Sobolev inequality (C.2), we get for $0 < \varepsilon < 1$ that

$$\begin{aligned}\sup_{t\in[0,T]} \|\Phi^\varepsilon(\cdot,t)\|_{L^\infty}^2 &\le c(v_*,T) \sup_{t\in[0,T]} \left(\|\Phi^\varepsilon(\cdot,t)\|_{L^2}^2 + \|\Phi_x^\varepsilon(\cdot,t)\|_{L^2}\|\Phi^\varepsilon(\cdot,t)\|_{L^2}\right)\\ &\le c(v_*,T)\left(\varepsilon^{1/2}+\varepsilon^{1/4}\right) \le c(v_*,T)\varepsilon^{1/4},\end{aligned} \tag{4.79}$$

where the constant $c(v_*,T) > 0$ is independent of ε and δ. Furthermore, if we take $\delta = \frac{\delta_1}{2}$ with δ_1 as in Lemma 4.4, then we have $\sup_{t\in[0,T]}\|\Phi^\varepsilon(\cdot,t)\|_{L^\infty} \le c(v_*,T)\varepsilon^{1/8} < \frac{\delta}{2}$ provided $c(v_*,T)\varepsilon^{1/8} \le \delta_1/4$. Hence, all the estimates in Lemmas 4.4–4.7 and Corollaries 4.6 and 4.8 exactly hold true with the constant $c(v_*,T)$ independent of ε.

4.3. Proof of Proposition 4.1. Thanks to the analysis and results in the preceding subsection, we know that for any $T > 0$ such that $C_1(T,v_*)v_* < 1/16$ with $K_1(T,v_*)$ presented in (4.40) and (4.67), the solution $(\Phi^\varepsilon, V^\varepsilon)$ satisfies, for any $t \in [0,T]$,

$$\|\Phi^\varepsilon(\cdot,t)\|_{L^2}^2 + \varepsilon^{\frac12}\|\Phi_x^\varepsilon(\cdot,t)\|_{L^2}^2 + \varepsilon^{3/2}\|\Phi_{xx}^\varepsilon\|_{L^2}^2 + \varepsilon^\ell\|\partial_x^\ell V^\varepsilon(\cdot,t)\|_{L^2}^2 \le c(v_*,T)\varepsilon^{1/2} \tag{4.80}$$

and

$$\begin{aligned}&\int_0^t \left(\|\Phi_x^\varepsilon\|_{L^2}^2 + \varepsilon^{1/2}\|\Phi_\tau^\varepsilon\|_{L^2}^2 + \varepsilon\|\Phi_{x\tau}^\varepsilon\|_{L^2}^2 + \varepsilon\|V_x^2\|_{L^2}^2 + \|V_\tau^\varepsilon\|_{L^2}^2 + \varepsilon^{5/2}\|V_{x\tau}\|_{L^2}^2\right)\mathrm{d}\tau\\ &\quad\le c(v_*,T)\varepsilon^{1/2},\end{aligned} \tag{4.81}$$

where $\ell = 0,1,2$, and $c(v_*,T) > 0$ is a constant depending on T but independent of ε. In particular, since $K_1(T,v_*)$ is increasing in T, if v_* is fixed, then there exists an increasing function $\phi(\cdot,v_*) = K_1^{-1}(\cdot,v_*)$ such that $K_1(T,v_*)v_* \le 1/16$ provided $T \le \phi(\frac{1}{16v_*},v_*) =: T_0$. Then the estimates (4.80) and (4.81) hold for any $t \in [0,T_0]$. This, along with the local existence result and the continuation argument, implies that the problem (4.8) admits a unique solution $(\Phi^\varepsilon,V^\varepsilon) \in L^\infty(0,T_0;H^2\times H^2)$ satisfying (4.80) and (4.81). In what follows, we shall show that $T_0 \to \infty$ as $v_* \to 0$. To achieve this, without loss of generality, we first assume that $v_* \le 1$. Then we may strengthen the condition $K_1(T,v_*)v_* \le 1/16$ for (4.40) and (4.67) as $K_1(T,1)v_* \le 1/16$. Here we write $K_1(T) := K_1(T,1)$ for simplicity. Clearly, since $K_1(T,v_*)$ is increasing in v_*, we have $K_1(T,v_*)v_* \le 1/16$ as long as $K_1(T)v_* \le 1/16$. Therefore we know that the estimates (4.80) and (4.81) hold for any $t \le K_1^{-1}(\frac{1}{16v_*}) =: T_0$ with $T_0 \to \infty$ as $v_* \to 0$ due to the increasing monotonicity of $K_1^{-1}(\cdot)$. This completes the proof of Proposition 4.1. ∎

4.4. Proof of Theorem 2.1. From Proposition 4.1, we know that for any $v_* > 0$, there exist constants $T_0 > 0$ and $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0,\varepsilon_0)$, the problem (2.2)–(2.3) admits a unique solution $(\varphi^\varepsilon, v^\varepsilon) \in L^\infty(0,T_0;H^2\times H^2)$. To finish the proof of Theorem 2.1, now it remains only to show the estimates in (2.21). Recalling (4.1) and Lemma 3.5, it suffices to show the estimates of $\mathcal{E}_1^\varepsilon$ and $\mathcal{E}_2^\varepsilon$ stated in (4.2). Thanks to (4.11), (4.12), (4.24), (4.25), Lemma 3.5, and the fact that $\partial_x b_\varphi$ is independent of x, there holds that

$$\|\partial_z^l \varphi^{B,2}\|_{L_T^\infty L_z^\infty} + \|\partial_\xi^l \varphi^{b,2}\|_{L_T^\infty L_\xi^\infty} + \|\varphi_x^{I,1}\|_{L_T^\infty L^\infty} + \varepsilon^{-1}\|\partial_x^l b_\varphi^\varepsilon\|_{L_T^\infty L^\infty} \le c(v_*,T), \tag{4.82}$$

$$\|v^{B,1}\|_{L_T^\infty L_z^\infty} + \|v^{b,1}\|_{L_T^\infty L_\xi^\infty} + \|v^{I,1}\|_{L_T^\infty L^\infty} + \varepsilon^{-1}\|b_v^\varepsilon\|_{L_T^\infty L^\infty} \le c(v_*,T), \tag{4.83}$$

where $l = 0, 1$, and the constant $c(v_*,T) > 0$ is independent of ε. Furthermore, from (4.79), (C.2), (C.3), Lemmas 4.4–4.7, and Corollary 4.8, we get

$$\|\Phi^\varepsilon\|_{L_T^\infty L^\infty} \le c(v_*,T)\left(\|\Phi^\varepsilon\|_{L_T^\infty L^2} + \|\Phi^\varepsilon\|_{L_T^\infty L^2}^{1/2}\|\Phi_x^\varepsilon\|_{L_T^\infty L^2}^{1/2}\right) \le c(v_*,T)\varepsilon^{1/8}, \tag{4.84}$$

$$\|\Phi_x^\varepsilon\|_{L_T^\infty L^\infty} \le c(v_*,T)\left(\|\Phi_x^\varepsilon\|_{L_T^\infty L^2} + \|\Phi_x^\varepsilon\|_{L_T^\infty L^2}^{1/2}\|\Phi_{xx}^\varepsilon\|_{L_T^\infty L^2}^{1/2}\right) \le c(v_*,T)\varepsilon^{-1/4}, \tag{4.85}$$

and

$$\|V^\varepsilon\|_{L_T^\infty L^\infty} \le \sqrt{2}\|V^\varepsilon\|_{L_T^\infty L^2}^{1/2}\|V_x^\varepsilon\|_{L_T^\infty L^2}^{1/2} \le c(v_*,T) \tag{4.86}$$

for some constant $c(v_*,T) > 0$ independent of ε. Therefore we get from (4.6), (4.7), and (4.82)–(4.86) that

$$\begin{aligned}\|\mathcal{E}_1^\varepsilon\|_{L_T^\infty L^\infty} &\le c_0\varepsilon\left(\|\varphi^{B,2}(z,t)\|_{L_T^\infty L^\infty} + \|\varphi^{b,2}(\xi,t)\|_{L_T^\infty L^\infty}\right)\\ &\quad + c_0\varepsilon^{1/2}\|\Phi^\varepsilon(x,t)\|_{L_T^\infty L^\infty} + c_0\|b_\varphi^\varepsilon\|_{L_T^\infty L^\infty}\\ &\le c(v_*,T)\varepsilon^{5/8},\end{aligned}$$

$$\begin{aligned}\|\mathcal{E}_2^\varepsilon\|_{L_T^\infty L^\infty} &\le c_0\varepsilon^{1/2}\Big(\|v^{I,1}(x,t)\|_{L_T^\infty L^\infty} + \|v^{B,1}(z,t)\|_{L_T^\infty L^\infty} + \|v^{b,1}(\xi,t)\|_{L_T^\infty L^\infty}\Big)\\ &\quad + c_0\varepsilon^{1/2}\|V(x,t)\|_{L_T^\infty L^\infty} + c_0\|b_v^\varepsilon(x,t)\|_{L_T^\infty L^\infty}\\ &\le c(v_*,T)\varepsilon^{1/2},\end{aligned}$$

and

$$\begin{aligned}\|\partial_x\mathcal{E}_1^\varepsilon\|_{L_T^\infty L^\infty} &\le c_0\varepsilon^{1/2}\left(\|\varphi_z^{B,2}(z,t)\|_{L_T^\infty L_z^\infty} + \|\varphi_\xi^{b,2}(\xi,t)\|_{L_T^\infty L_\xi^\infty}\right)\\ &\quad + c_0\varepsilon^{1/2}\|\Phi_x^\varepsilon(x,t)\|_{L_T^\infty L^\infty} + c_0\|\partial_x b_\varphi^\varepsilon\|_{L_T^\infty L^\infty}\\ &\le c(v_*,T)\varepsilon^{1/4},\end{aligned} \tag{4.87}$$

where $0 < \varepsilon < 1$ has been used. Combining the above estimates on $\mathcal{E}_i^\varepsilon$ $(i = 1, 2)$, we get (2.21) and thus finish the proof of Theorem 2.1. ∎

4.5. Proof of Theorem 2.2. Theorem 2.2 follows directly from Theorem 2.1, except for the estimate (2.24). To prove (2.24), we first notice from (4.1) that

$$\varphi_x^\varepsilon = \varphi_x^{I,0} + \varphi_z^{B,1}(z,t) + \varphi_\xi^{b,1}(\xi,t) + \varepsilon^{1/2}\varphi_x^{I,1} + \partial_x\mathcal{E}_1^\varepsilon,$$

which implies that

$$u^\varepsilon = u^{I,0} + u^{B,0} + u^{b,0} + \varepsilon^{1/2}\varphi_x^{I,1} + \partial_x\mathcal{E}_1^\varepsilon. \tag{4.88}$$

On the other hand, from (4.82) and (4.87), we have $\|\varepsilon^{1/2}\varphi_x^{I,1}\|_{L_T^\infty L^\infty} + \|\partial_x\mathcal{E}_1^\varepsilon\|_{L_T^\infty L^\infty} \le c(v_*,T)\varepsilon^{1/4}$ with the constant $c(v_*,T) > 0$ independent of ε. This, along with (4.88), gives rise to (2.24). ∎

Appendix A. Local existence result on $v^{B,0}$. In this appendix, we detail the proof of local existence and uniqueness of solutions to the problem (2.12) for the leading-order boundary layer profile $v^{B,0}$. Equivalently, we study the reformulated problem (3.34), i.e.,

$$\begin{cases} \vartheta_t = \vartheta_{zz} - \overline{u^{I,0}} \mathrm{e}^{\vartheta+\phi}(\vartheta+\phi) - \overline{u^{I,0}}\, v^{I,0}(0,t)\left(\mathrm{e}^{\vartheta+\phi}-1)\right) + \varrho, \\ \vartheta(0,t)=0, \quad \vartheta(+\infty,t)=0, \\ \vartheta(z,0)=0. \end{cases} \tag{A.1}$$

The solution space for the problem reads

$$\mathcal{X}_T = \left\{ u \in L^2_T L^2_z | \, \partial_t^l u|_{t=0} = \theta_l, \ \partial_t^k u \in L^2_T H_z^{6-2k}, \ l=0,1,2, \ k=0,1,2,3 \right\}$$

for some $T>0$, where $\theta_0 \equiv 0$, and $\theta_l := \partial_t^l \vartheta|_{t=0}$ $(l=1,2)$ are determined by u_0, v_0 and $\vartheta(z,0)$ through the equation $(\mathrm{A.1})_1$. By (2.20), we know that the initial datum is compatible up to order two. We shall divide the proof into three steps in the following.

Step 1: *Linearization.* Given $\omega \in \mathcal{X}_T$, we first consider the following linearized problem for (A.1):

$$\begin{cases} v_t = v_{zz} - \overline{u^{I,0}} v - \overline{u^{I,0}} \mathrm{e}^{w+\phi}\phi - \overline{u^{I,0}}(\mathrm{e}^{w+\phi}-1)(v^{I,0}(0,t)+w) + \varrho \\ \qquad =: v_{zz} - \overline{u^{I,0}} v + F + \varrho, \\ v(0,t)=0, \quad v(+\infty,t)=0, \\ v(z,0)=0, \end{cases} \tag{A.2}$$

where $\overline{u^{I,0}}$ is as in (3.34). Let $V = \mathrm{e}^{\int_0^t \overline{u^{I,0}} \mathrm{d}s} v$. Then V satisfies

$$\begin{cases} V_t = V_{zz} + F \mathrm{e}^{\int_0^t \overline{u^{I,0}} \mathrm{d}s} + \varrho \mathrm{e}^{\int_0^t \overline{u^{I,0}} \mathrm{d}s}, \\ V(0,t)=0, \quad V(+\infty,t)=0, \\ V(z,0)=0, \end{cases}$$

which can be solved explicitly by the reflection method:

$$\begin{aligned} V = &\int_0^t \int_0^\infty \Gamma(z-y,t-\tau)\left[F(y,\tau)\mathrm{e}^{\int_0^\tau \overline{u^{I,0}} \mathrm{d}s} + \varrho(y,\tau)\mathrm{e}^{\int_0^\tau \overline{u^{I,0}} \mathrm{d}s}\right] \mathrm{d}y \mathrm{d}\tau \\ &- \int_0^t \int_{-\infty}^0 \Gamma(z-y,t-\tau)\left[F(-y,\tau)\mathrm{e}^{\int_0^\tau \overline{u^{I,0}} \mathrm{d}s} + \varrho(-y,\tau)\mathrm{e}^{\int_0^\tau \overline{u^{I,0}} \mathrm{d}s}\right] \mathrm{d}y \mathrm{d}\tau, \end{aligned}$$

where $\Gamma(z,t) = \frac{1}{\sqrt{4\pi t}} \mathrm{e}^{-\frac{z^2}{4t}}$ is the heat kernel. Hence one can recover v from the above identity along with the definition of V. The uniqueness of solutions to the problem (A.2) is standard, so the details are omitted here.

Step 2: *A priori estimates.* We shall show for (A.2) that there exists a suitably large constant $K>0$ and a small $T_0>0$ such that if

$$\sum_{k=0}^{3} \|\partial_t^k w\|^2_{L^2_T H_z^{6-2k}} \le 2K \quad \text{for } T \le T_0, \tag{A.3}$$

then it holds that

$$\sum_{k=0}^{3} \|\partial_t^k v\|^2_{L^2_T H_z^{6-2k}} \le K \quad \forall T \le T_0. \tag{A.4}$$

First, by the Sobolev embedding theorem and (A.3), we have

$$\partial_t^k w \in C([0,T];H_z^{5-2k}), \quad k=0,1,2. \tag{A.5}$$

Furthermore, for any $t\in[0,T]$ and for $k=0,1,2$, it follows that

$$\begin{aligned}\|\partial_t^k w(\cdot,t)\|_{H^{4-2k}} &= \left\|\theta_k + \int_0^t \partial_\tau^{k+1} w(\cdot,\tau)\mathrm{d}\tau\right\|_{H^{4-2k}} \le C\|\theta_k\|_{H^{4-2k}} \\ &\quad + \int_0^T \|\partial_t^{k+1} w(\cdot,t)\|_{H^{4-2k}}\mathrm{d}t \\ &\le \hat{C}_0 + T^{1/2}\left(\int_0^T \|\partial_t^{k+1} w(\cdot,t)\|_{H^{4-2k}}^2\mathrm{d}t\right)^{1/2} \quad \text{(A.6)} \\ &\le \hat{C}_0 + \tilde{C}K^{1/2}T^{1/2}, \quad \text{(A.7)}\end{aligned}$$

where $\hat{C}_0$ is a positive constant depending only on the initial data θ_j. Hereafter $\tilde{C}>0$ is a generic constant independent of T. With (3.27b), (3.35), (A.3), (A.5), and (A.6), similar to the proofs of (3.54) and (3.57), we proceed to derive for $k=0,1,2$ that

$$\begin{aligned}&\left\|\partial_t^k F\right\|^2_{L_T^2 H_z^{4-2k}} \\ &\le \sum_{j=0}^k \Big\{ \|\partial_t^j \overline{u^{I,0}} \partial_t^{k-j}(\mathrm{e}^{w\phi}\phi)\|^2_{L_T^2 H_z^{4-2k}} \|\partial_t^j \overline{u^{I,0}} \partial_t^{k-j}[w(\mathrm{e}^{w\phi}-1)]\|^2_{L_T^2 H_z^{4-2k}} \\ &\quad + \|\partial_t^j(\overline{u^{I,0}} v^{I,0}(0,t))\partial_t^{k-j}(\mathrm{e}^{w\phi}-1)\|^2_{L_T^2 H_z^{4-2k}} \Big\} \\ &\le \tilde{C}T \sum_{j=0}^k \|\partial_t^j \overline{u^{I,0}}\|^2_{L^\infty(0,T)} \left(\|\partial_t^{k-j}(\mathrm{e}^{w\phi}\phi)\|^2_{L_T^\infty H_z^{4-2k}} \|\partial_t^{k-j}(\mathrm{e}^{w\phi}-1)\|^2_{L_T^\infty H_z^{4-2k}} \right) \\ &\quad + \tilde{C}T \sum_{j=0}^k \|\partial_t^j \overline{u^{I,0}}(v^{I,0}(0,t))\|^2_{L^\infty(0,T)} \|\partial_t^{k-j}[w(\mathrm{e}^{w\phi}-1)]\|^2_{L_T^\infty H_z^{4-2k}} \\ &\le \tilde{C}K\mathrm{e}^{\tilde{C}K} T\left(1\sum_{j=0}^k \|\partial_t^j w\|^2_{L_T^\infty H_z^{4-2k}}\right) \le \tilde{C}TK\mathrm{e}^{\tilde{C}K}(1KT), \quad \text{(A.8)}\end{aligned}$$

where we have used the fact $\mathrm{e}^{w+\phi} \le \mathrm{e}^{\tilde{C}K}$ due to (A.3), the constant $\tilde{C}>0$ may depend on $\overline{u^{I,0}}$, $v^{I,0}$ and ϕ, but independent of T and K. Thanks to (3.27b), we get for $k=0,1,2$ that

$$\left\|\partial_t^k \varrho\right\|^2_{L_T^2 H_z^{4-2k}} \le \tilde{C}T + \tilde{C}\sum_{j=0}^k \|\partial_t^j v^{I,0}(0,t)\|^2_{L^2(0,T)} \le \tilde{C}T, \tag{A.9}$$

where $\tilde{C}>0$ is a constant independent of K and T. By a procedure similar to the one in the proof of Lemma 3.3, one can deduce for $T\le 1$ that

$$
\begin{aligned}
&\sum_{k=0}^{3}\|\partial_t^k V\|^2_{L^2_T H^{6-2k}_z}\\
&\quad\le \tilde{C}+\sum_{k=0}^{2}\left(\left\|\partial_t^k\left(F\mathrm{e}^{\int_0^t \overline{u^{I,0}}\mathrm{d}s}\right)\right\|^2_{L^2_T H^{4-2k}_z}+\left\|\partial_t^k\left(\varrho\mathrm{e}^{\int_0^t \overline{u^{I,0}}\mathrm{d}s}\right)\right\|^2_{L^2_T H^{4-2k}_z}\right)\\
&\quad\le \tilde{C}\sum_{k=0}^{2}\mathrm{e}^{\tilde{C}T}\left(\left\|\partial_t^k F\right\|^2_{L^2_T H^{4-2k}_z}+\|\partial_t^k\varrho\|^2_{L^2_T H^{4-2k}_z}\right)\\
&\quad\le \tilde{C}+\tilde{C}T\Big(1+K^2\mathrm{e}^{\tilde{C}T+\tilde{C}K}\Big),
\end{aligned}
$$

where we have used (A.8) and (A.9), and the constant $\tilde{C}>0$ is independent of K and T. In view of the definition of V, there holds that

$$
\sum_{k=0}^{2}\|\partial_t^k v\|^2_{L^2_T H^{6-2k}_z}\le \hat{C}_1+\tilde{C}T\Big(1+K^2\mathrm{e}^{CT+CK}\Big)
$$

for some constants $\hat{C}_1$ and $\tilde{C}$ independent of K and T. Hence, we get

$$
\sum_{k=0}^{2}\|\partial_t^k v\|^2_{L^2_T H^{6-2k}_z}\le 2\hat{C}_1=:K, \tag{A.10}
$$

provided

$$
T\le \min\left\{1,\left[\tilde{C}+(2\hat{C}_1)^2\mathrm{e}^{\tilde{C}+2\tilde{C}\hat{C}_1}\right]^{-1}\right\}=:T_0.
$$

This gives (A.4).

Step 3: *Contraction.* Denote

$$
\mathcal{Y}_T:=\left\{u\in\mathcal{X}_T\,\Big|\,\sum_{k=0}^{3}\|\partial_t^k u\|^2_{L^2_T H^{6-2k}_z}\le K\right\}
$$

with K as in (A.10). In the previous steps, we have proved for $T\le T_0$ that the solution map $\Theta:\ \mathcal{Y}_T\to\mathcal{Y}_T$ for the linearized problem (A.2) is well-defined. To prove the existence of solutions to (A.1), it now suffices to show the contraction of Θ in the norm $\|\cdot\|_{C(0,T;L^2_z)}$ for suitably small $T>0$. For any $w_1, w_2\in\mathcal{Y}_T$, denote $v_i=\Theta(w_i)$ $(i=1,2)$ and

$$
W=w_1-w_2,\quad V=v_1-v_2.
$$

Then we have from (A.2) that

$$
\begin{cases}
V_t=V_{zz}-\overline{u^{I,0}}V-\overline{u^{I,0}}\mathrm{e}^{\phi}\left(\mathrm{e}^{w_1}-\mathrm{e}^{w_2}\right)\left(\phi+v^{I,0}(0,t)+w_1\right)-\overline{u^{I,0}}(\mathrm{e}^{w_2+\phi}-1)W,\\
V(0,t)=0,\quad V(+\infty,t)=0,\\
V(z,0)=0.
\end{cases}
$$

The standard L^2 estimate implies that

$$
\frac{\mathrm{d}}{\mathrm{d}t}\|V\|^2_{L^2_z}+\int_{\mathbb{R}_+}\left(V_z^2+V^2\right)\mathrm{d}z\le \tilde{C}\mathrm{e}^{\tilde{C}K}\|W\|^2_{L^2}.
$$

It thus holds that

$$\sup_{t\in[0,T]}\|V\|_{L_z^2}^2+\|V\|_{L_T^2H_z^1}^2\le \tilde{C}T\mathrm{e}^{K}\sup_{t\in[0,T]}\|W\|_{L_z^2}^2\le \frac{1}{2}\sup_{t\in[0,T]}\left(\|W\|_{L_z^2}^2+\|W\|_{L_T^2H_z^1}^2\right),$$

provided

$$T\le \min\left\{T_0,\frac{1}{2}\left[\tilde{C}\mathrm{e}^{\tilde{C}K}\right]^{-1}\right\}=:T_1.$$

Hence the desired contraction of Θ is proved.

Finally, based on the analysis in Steps 1–3, we conclude that the problem (A.1) admits a solution $\vartheta\in\mathcal{Y}_{T_1}$. The uniqueness of the solution is standard, so we omit the details here. The proof is complete. ∎

Appendix B. Proof of (3.98). We shall prove that

$$\|\langle z\rangle^l\partial_t^k g\|_{L_T^2H_z^{4-2k}}\le c(v_*,T)\quad \text{for } k=0,1,2, \tag{B.1}$$

where the constant is independent of ε and δ. For this, we first split the function g into three parts

$$g=\mathcal{J}_1+\mathcal{J}_2+\mathcal{J}_3,$$

where

$$\begin{aligned}
\mathcal{J}_1=&(\mathrm{e}^{v^{B,0}}-1)\int_z^\infty \eta v^{I,1}(0,t)\partial_y\left[(\varphi_x^{I,0}(0,t)+M+\varphi_y^{B,1})\mathrm{e}^{-v^{B,0}}\right]\mathrm{d}s(v^{I,0}(0,t)+v^{B,0})\\
&-\int_z^\infty \eta v^{I,1}(0,t)\partial_y\left[(\varphi_x^{I,0}(0,t)+M+\varphi_y^{B,1})\mathrm{e}^{-v^{B,0}}\right]\mathrm{d}y(v^{I,0}(0,t)+v^{B,0}),\\
\mathcal{J}_2=&(\mathrm{e}^{v^{B,0}}-1)\int_z^\infty \left[v_y^{B,0}(\varphi_{xx}^{I,0}(0,t)y+\varphi_x^{I,1}(0,t))+\varphi_y^{B,1}v_x^{I,0}(0,t)\right]\mathrm{e}^{-v^{B,0}}\\
&\times\,\mathrm{d}y(v^{I,0}(0,t)+v^{B,0})\\
&+\int_z^\infty \left[v_y^{B,0}(\varphi_{xx}^{I,0}(0,t)y+\varphi_x^{I,1}(0,t))+\varphi_y^{B,1}v_x^{I,0}(0,t)\right]\mathrm{e}^{-v^{B,0}}\mathrm{d}y(v^{I,0}(0,t)+v^{B,0}),\\
\mathcal{J}_3=&(\varphi_x^{I,0}(0,t)+M)\eta(z)v^{I,1}(0,t)+\eta(z)v_t^{I,1}(0,t)-\varphi_z^{B,1}(v_x^{I,0}(0,t)z+v^{I,1}(0,t))\\
&-(\varphi_{xx}^{I,0}(0,t)z+\varphi_x^{I,1}(0,t))v^{B,0}+\eta(z)v^{I,1}(0,t)(\varphi_x^{I,0}(0,t)+M+\varphi_z^{B,1})\\
&\times\,(v^{I,0}(0,t)+v^{B,0})\\
&-\eta''(z)v^{I,1}(0,t)+\eta(z)v^{I,1}(0,t)\varphi_z^{B,1}.
\end{aligned}$$

Thanks to (3.27), (3.35), (3.55), (3.57), (3.61b), (4.10), (4.12), and the Hölder inequality, we get for $k=0,1,2$ and $l\in\mathbb{N}$ that

$$\begin{aligned}
&\left\|\langle z\rangle^l\partial_t^k\int_z^\infty \eta v^{I,1}(0,t)\partial_y\left[(\varphi_x^{I,0}(0,t)+M+\varphi_y^{B,1})\mathrm{e}^{-v^{B,0}}\right]\mathrm{d}y\right\|_{L_T^2H_z^{4-2k}}^2\\
&\quad\le\left\|\langle z\rangle^l\partial_t^k\int_z^\infty \eta v^{I,1}(0,t)\partial_y\left[(\varphi_x^{I,0}(0,t)+M+\varphi_y^{B,1})\mathrm{e}^{-v^{B,0}}\right]\mathrm{d}y\right\|_{L_T^2L_z^2}^2
\end{aligned} \tag{B.2}$$

$$
\begin{aligned}
&+c(v_*,T)\left\|\langle z\rangle^l\partial_t^k\left[\eta v^{I,1}(0,t)\partial_z\left((\varphi_x^{I,0}(0,t)+M+\varphi_z^{B,1})\mathrm{e}^{-v^{B,0}}\right)\right]\right\|^2_{L^2_TH_z^{3-2k}}\\
&\le c(v_*,T)\sum_{j=0}^k\left(\|\langle z\rangle^{l+2}\partial_t^jv^{B,0}\|^2_{L^2_TH_z^1}+\|\langle z\rangle^{l+2}\partial_t^j\varphi^{B,1}\|^2_{L^2_TH_z^2}\right)\int_{\mathbb{R}_+}\langle y\rangle^{-2}\mathrm{d}y\\
&\quad+c(v_*,T)\sum_{j=0}^k\left(\|\langle z\rangle^{l+2}\partial_t^jv^{B,0}\|^2_{L^2_TH_z^{4-2k}}+\|\langle z\rangle^{l+2}\partial_t^j\varphi^{B,1}\|^2_{L^2_TH_z^{5-2k}}\right)\int_{\mathbb{R}_+}\langle y\rangle^{-2}\mathrm{d}y\\
&\le c(v_*,T),
\end{aligned}
$$

where we have used $\partial_t^kv^{I,1}(0,t)\in L^\infty(0,T)\,(k=0,1,2)$ due to (3.61b) and taken the space-time L^∞-norms for the terms involving lower-order spatial derivatives which are bounded according to (4.10) and (4.12). Therefore, $\mathcal{J}_1$ can be estimated as follows:

(B.3)

$$
\begin{aligned}
&\left\|\langle z\rangle^l\partial_t^k\mathcal{J}_1\right\|_{L^2_TH_z^{4-2k}}\\
&\quad\le c(v_*,T)\left\|\langle z\rangle^l\partial_t^k\int_z^\infty\eta v^{I,1}(0,t)\partial_y\left[(\varphi_x^{I,0}(0,t)+M+\varphi_y^{B,1})\mathrm{e}^{-v^{B,0}}\right]\mathrm{d}y\right\|^2_{L^2_TH_z^{4-2k}}\\
&\qquad\times\left(\|\mathrm{e}^{v^{B,0}}-1\|^2_{L^\infty_TH_z^{4-2k}}+1\right)\left(1+\|v^{B,0}\|^2_{L^\infty_TH_z^{4-2k}}\right)\\
&\quad\le c(v_*,T)
\end{aligned}
$$

for $k=0,1,2$, where we have used (3.56), (3.57), (4.10), and (4.12). By arguments similar to those proving (B.2), we get

$$
\begin{aligned}
&\left\|\langle z\rangle^l\partial_t^k\int_z^\infty\left[v_y^{B,0}(\varphi_{xx}^{I,0}(0,t)y+\varphi_x^{I,1}(0,t))+\varphi_y^{B,1}v_x^{I,0}(0,t)\right]\mathrm{e}^{-v^{B,0}}\mathrm{d}y\right\|^2_{L^2_TH_z^{4-2k}}\\
&\le c(v_*,T)\sum_{j=0}^k\|\langle z\rangle^l\int_z^\infty\left[\partial_t^{k-j}(v_y^{B,0}\mathrm{e}^{-v^{B,0}})\partial_t^j(\varphi_{xx}^{I,0}(0,t)y+\varphi_x^{I,1}(0,t))\mathrm{d}y\right\|^2_{L^2_TH_z^{4-2k}}\\
&\quad+c(v_*,T)\sum_{j=0}^k\left\|\langle z\rangle^l\int_z^\infty\partial_t^jv_x^{I,0}(0,t)\partial_t^{k-j}(\varphi_y^{B,1}\mathrm{e}^{-v^{B,0}})\mathrm{d}y\right\|^2_{L^2_TH_z^{4-2k}}\\
&\le c(v_*,T)\int_{\mathbb{R}_+}\langle y\rangle^{-2}\mathrm{d}z\sum_{i,j=0}^k\|\langle z\rangle^{l+3}\partial_t^iv^{B,0}\|^2_{L^\infty_TH_z^{5-2k}}\int_0^T\left(\left|\partial_t^j\varphi_{xx}^{I,0}(0,t)\right|^2+\left|\partial_t^j\varphi_x^{I,1}\right|^2\right)\mathrm{d}t\\
&\quad+c(v_*,T)\sum_{i,j=0}^k\left(\|\langle z\rangle^{l+2}\partial_t^iv^{B,0}\|^2_{L^\infty_TH_z^{4-2k}}+\|\langle z\rangle^{l+2}\partial_t^i\varphi^{B,1}\|^2_{L^\infty_TH_z^{4-2k}}\right)\int_{\mathbb{R}_+}\langle y\rangle^{-2}\mathrm{d}z\\
&\quad\times\int_0^T\left|\partial_t^j\partial_xv^{I,0}(0,t)\right|^2\mathrm{d}t\\
&\le c(v_*,T),
\end{aligned}
$$

due to (3.27), (3.61a), (4.10), (4.12), and the Hölder inequality. Therefore, for $\mathcal{J}_2$, we get

(B.4)

$$\begin{aligned}
&\left\|\langle z\rangle^l \mathcal{J}_2\right\|_{L^2_T H^{4-2k}_z}\\
&\le c(v_*,T)\left\|\langle z\rangle^l \partial_t^k \int_z^\infty \left[v_y^{B,0}(\varphi_{xx}^{I,0}(0,t)y+\varphi_x^{I,1}(0,t))+\varphi_y^{B,1}v_x^{I,0}(0,t)\right]\mathrm{e}^{-v^{B,0}}\mathrm{d}y\right\|^2_{L^2_T H^{4-2k}_z}\\
&\quad\times\left(\|\mathrm{e}^{v^{B,0}}-1\|^2_{L^\infty_T H^{4-2k}_z}+1\right)\left(1+\|v^{B,0}\|^2_{L^\infty_T H^{4-2k}_z}\right)\le c(v_*,T).
\end{aligned}$$

Now let us turn to $\mathcal{J}_3$. With (3.35), (3.61a), (3.61b), (4.10), (4.12), and the fact that η is a smooth function with compact support, we deduce for $k=0,1,2$ that

$$\begin{aligned}
\left\|\langle z\rangle^l \partial_t^k \mathcal{J}_3\right\|^2_{L^2_T H^{4-2k}_z} &\le c(v_*,T)\sum_{i,j=0}^k\left(1+\|\langle z\rangle\partial_t^j\varphi^{B,1}\|^2_{L^\infty_T H^{5-2k}_z}+\|\langle z\rangle\partial_t^j v^{B,0}\|^2_{L^\infty_T H^{4-2k}_z}\right)\\
&\qquad\times\left(1+\int_0^T(|\partial_t^i v^{I,1}(0,t)|^2+|\partial_t^i\varphi_x^{I,1}(0,t)|^2+|\partial_t^2\varphi_{xx}^{I,0}(0,t)|^2)\mathrm{d}t\right)\\
&\le c(v_*,T).
\end{aligned}$$

This combined with (B.3) and (B.4) gives (B.1). The proof of (3.98) is complete. ◻

Appendix C. Some analytic tools. In this appendix, we collect some basic results used in this paper, which include some Sobolev-type inequalities and an embedding theorem on space-time Sobolev spaces. Let us begin with the Sobolev inequalities.

LEMMA C.1 (see [3, p. 236]). *Let $p>1$. Then for any $\epsilon>0$, there exists a positive constant $C=C(\epsilon,p)$ such that*

$$\|h\|_{L^\infty(\mathcal{I})}\le\epsilon\|h_x\|_{L^p(\mathcal{I})}+C\|h\|_{L^1(\mathcal{I})} \tag{C.1}$$

for any $h\in W^{1,p}(\mathcal{I})$.

LEMMA C.2. *For any $h\in H^1(\mathcal{I})$, it holds that*

$$\|h\|_{L^\infty(\mathcal{I})}\le C\left(\|h\|_{L^2}+\|h\|^{1/2}_{L^2}\|h_x\|^{1/2}_{L^2}\right), \tag{C.2}$$

where $C>0$ is a constant independent of h.

We also remark that if $h\in H^1_0(\mathcal{I})$, then

$$\|h\|_{L^\infty}\le\sqrt{2}\|h\|^{1/2}_{L^2}\|h_x\|^{1/2}_{L^2}\quad\text{and}\quad\|h\|_{L^\infty}\le C\|h_x(\cdot,t)\|_{L^2}, \tag{C.3}$$

and that if $h\in H^1_z$ (resp., H^1_ξ), then

(C.4)

$$\|h\|_{L^\infty_z}\le C\|h\|^{1/2}_{L^2_z}\|h_z\|^{1/2}_{L^2_z}\le C\|h\|_{H^1_z}\left(\text{resp., }\|h\|_{L^\infty_\xi}\le C\|h\|^{1/2}_{L^2_z}\|h_\xi\|^{1/2}_{L^2_\xi}\le C\|h\|_{H^1_\xi}\right),$$

where the constant $C>0$ is independent of h.

Next, we introduce the Hardy's inequality.

LEMMA C.3 (cf. [3, p. 233]). *Let $u\in W^{1,p}_0(\mathcal{I})$ with $1<p<\infty$. Then*

$$\left\|\frac{u}{x(1-x)}\right\|_{L^p(\mathcal{I})}\le C_p\|u_x\|_{L^p(\mathcal{I})}, \tag{C.5}$$

where $C_p>0$ is a constant depending only on p.

The following embedding theorem is also frequently used in our analysis.

PROPOSITION C.4 (cf. [48, Lemma 1.2]). *Let V, H, and V' be three Hilbert spaces satisfying $V \subset H \subset V'$, with V' being the dual of V. If a function u belongs to $L^2(0,T;V)$ and its time derivatives u_t belongs to $L^2(0,T;V')$, then*

$$u \in C([0,T];H) \quad \text{and} \quad \|u\|_{L^\infty(0,T;H)} \le C\left(\|u\|_{L^2(0,T;V)} + \|u_t\|_{L^2(0,T;V')}\right),$$

where the constant $C>0$ depends on T but is independent of u.

Remark C.1. Proposition C.4 implies the following fact for any $m \in \mathbb{N}$:

$$\left\{u \,|\, u \in L^2(0,T;X^{m+2}), u_t \in L^2(0,T;X^m)\right\} \hookrightarrow C([0,T];X^{m+1}) \text{ continuously},$$

where $X^m := H^m$, H_z^m, or H_ξ^m.

Finally, by the change of variables in (2.4), for any $G_1(z,t) \in H_z^m$ and $G_2(\xi,t) \in H_\xi^m$ with $m \in \mathbb{N}$, we have the following inequalities:

$$\left\|\partial_x^m G_1\left(\frac{x}{\varepsilon^{1/2}},t\right)\right\|_{L^2} = \varepsilon^{\frac{1-2m}{4}} \|\partial_z^m G_1(z,t)\|_{L_z^2}, \quad \|\partial_x^m G_1(z,t)\|_{L^\infty} = \varepsilon^{-\frac{m}{2}} \|\partial_z^m G_1(z,t)\|_{L_z^\infty}, \tag{C.6a}$$

$$\left\|\partial_x^m G_2\left(\frac{x-1}{\varepsilon^{1/2}},t\right)\right\|_{L^2} = \varepsilon^{\frac{1-2m}{4}} \|\partial_\xi^m G_2(\xi,t)\|_{L_\xi^2}, \quad \|\partial_x^m G_2(\xi,t)\|_{L^\infty} = \varepsilon^{-\frac{m}{2}} \|\partial_\xi^m G_2(\xi,t)\|_{L_\xi^\infty}. \tag{C.6b}$$

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