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THE UNIFORM SPREADING SPEED IN COOPERATIVE SYSTEMS WITH NON-UNIFORM INITIAL DATA

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Abstract. This paper considers the spreading speed of cooperative nonlocal dispersal systems with irreducible reaction functions and non-uniform initial data. Here the non-uniformity means that all components of initial data decay exponentially but their decay rates are different. It is well-known that in a monostable reaction-diffusion or nonlocal dispersal equation, different decay rates of initial data yield different spreading speeds. In this paper, we show that due to the cooperation and irreducibility of reaction functions, all components of the solution with non-uniform initial data will possess a uniform spreading speed which decreasingly depends only on the smallest decay rate of initial data. The decreasing property of the uniform spreading speed in the smallest decay rate further implies that the component with the smallest decay rate can accelerate the spatial propagation of other components.

1. **Introduction.** The long-range dispersal, such as the spread of infectious disease across countries and continents by the travel of infected humans [22], has increasingly become an important phenomenon nowadays, and it has attracted extensive attention of researchers (see [7, 36, 40]). Mathematically the long-range dispersal can be modelled by a nonlocal dispersal operator that describes the movements between not only adjacent but also nonadjacent spatial locations. A typical nonlocal dispersal equation with reaction is given by

$$u_t = k * u - u + f(u), \quad t > 0, \; x \in \mathbb{R}, \tag{1}$$

where $u(t,x)$ stands for the population density at time t and location x, $f(u)$ is a reaction function, and the nonlocal dispersal operator is represented by

$$k * u(t,x) - u(t,x) = \int_{\mathbb{R}} k(x-y)u(t,y)dy - u(t,x).$$

Here $k : \mathbb{R} \to \mathbb{R}$ is a nonnegative and continuous function with $\int_{\mathbb{R}} k(x)dx = 1$. As stated in [18], $k(x-y)$ can be viewed as the probability for individuals to move

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from location y to location x, $k * u(t,x) = \int_{\mathbb{R}} k(x-y)u(t,y)dy$ stands for the rate at which individuals arrive at location x from other locations, and $-u(t,x) = -\int_{\mathbb{R}} k(y-x)u(t,x)dy$ is the rate at which individuals leave location x and move to other locations. One of the significant research topics in the literature for (1) is the wave propagation phenomena which are associated with the studies of traveling wave solutions, entire solutions and spreading speeds. These results can be used to describe the spreading process of populations, such as the spatial spread of infectious diseases and the invasion of species. For the traveling wave solutions of (1), we refer to the classical works by Bates et al. [5], Carr and Chmaj [9], Chen [10], Chen and Guo [11], Coville, Dávila and Martínez [14], Schumacher [43], Yagisita [55], etc. For the entire solutions of (1), we refer to, for example, Li, Sun and Wang [29]. For the spreading speeds of (1), we refer to the works by Lutscher, Pachepsky and Lewis [35], Shen and Zhang [44], Zhang, Li and Wang [57], Rawal, Shen and Zhang[41], Finkelshtein, Kondratiev and Tkachov [19, 20], Liang and Zhou [33], etc. For the results on nonlocal dispersal equations with free boundary, we refer to Cao et al. [8], Du, Li and Zhou [15], and Du and Ni [16], etc.

In this paper, we are concerned with the spreading speed of the following m-component nonlocal dispersal system

$$\begin{cases} U_t = D(K * U - U) + F(U), & t > 0,\ x \in \mathbb{R}, \\ U(0,x) = U_0(x) = (u_{1,0}(x), \dots u_{m,0}(x)), & x \in \mathbb{R}, \end{cases} \tag{2}$$

where $U = (u_1, \dots, u_m)$, $K = (k_1, \dots, k_m)$, $F = (f_1, \dots, f_m)$, $D = \operatorname{diag}\{d_1, \dots, d_m\}$ with $d_j > 0$, and $2 \leqslant m \in \mathbb{Z}^+$. The nonlocal dispersal is represented by

$$K * U(t,x) - U(t,x) \triangleq (k_1 * u_1(t,x) - u_1(t,x), \dots, k_m * u_m(t,x) - u_m(t,x)).$$

We assume that $F(U)$ is cooperative (namely $\frac{\partial}{\partial u_i} f_j(U) \geqslant 0$ for any $j \neq i$) and monostable with an unstable equilibrium $U \equiv \mathbf{0} \in \mathbb{R}^m$ and a stable equilibrium $U \equiv P \in (\mathbb{R}^+)^m$. Assume that

$$U_0(\cdot) \not\equiv \mathbf{0},\ \mathbf{0} \leqslant U_0(x) \leqslant P \text{ for all } x \in \mathbb{R}.$$

The kernel $K \in C(\mathbb{R}, \mathbb{R}^m)$ is symmetric on $\mathbb{R}$ and satisfies the Mollison condition (see [14, 39, 40]), in the sense that, there exists $\Lambda > 0$ such that

$$\int_{\mathbb{R}} k_j(x) e^{\Lambda |x|} dx < +\infty,\ j \in \{1, \dots, m\}.$$

The local dispersal system, as a counterpart of (2), is called the reaction-diffusion system which reads as

$$\begin{cases} U_t = D\Delta U + F(U), & t > 0,\ x \in \mathbb{R}, \\ U(0,x) = U_0(x), & x \in \mathbb{R}. \end{cases} \tag{3}$$

When $m = 2$, traveling wave solutions and entire solutions were obtained for (2) by Li, Xu and Zhang [30], Meng, Yu and Hsu [38], and for (3) by Hsu and Yang [24], Zhao and Wang [58], Xu and Zhao [50], Wu and Hsu [49]. When the initial data U_0 are compactly supported (or equivalently $U_0(x) \equiv \mathbf{0}$ for large $x > 0$), there are numerous results on the spreading spread of (2) and (3). For the nonlocal dispersal system (2), we refer to Bao et al [3], Bao, Shen and Shen [4], Hu et al. [25]. For the local dispersal system (3) and its discrete-time counterpart, we refer to Kolmogorov, Petrovsky and Piskunov [26] and Aronson and Weinberger [1, 2] for the case $m = 1$ (i.e. classical reaction-diffusion equation), and Weinberger [47], Lui [34], Weinberger, Lewis and Li [48], Li, Weinberger and Lewis [28], Liang and Zhao [31, 32], Fang and Zhao [17], and Wang [46] for the case $m \geqslant 2$.

Note that the aforementioned existing results on the spreading speeds of (2) and (3) essentially assume that the initial data $U_0(x)$ are compactly supported. However, when the initial data $U_0(x)$ are not compactly supported, the results of spreading speed are much fewer. Especially, when the initial value function decays exponentially, namely

$$u(0,x) \sim C\mathrm{e}^{-\sigma|x|} \text{ as } |x| \to +\infty \text{ with } \sigma > 0,\ C > 0, \tag{4}$$

the system (2) with $m = 1$, namely (1), has a spreading speed

$$s(\sigma) = \frac{1}{\sigma}\left\{\int_{\mathbb{R}} k(x)e^{\sigma x}dx - 1 + f'(0)\right\} \text{ for } \sigma \in (0,\sigma^*), \tag{5}$$

where $\sigma^* = \min\{\sigma > 0 \mid s(\sigma) = \min\{s(\sigma); \sigma > 0\}\}$, see e.g. [13, 45, 54]. In (4), the symbol "$\sim$" means $\lim_{|x|\to\infty} u(0,x)/(C\mathrm{e}^{-\sigma|x|}) = 1$. Similar results for (3) with $m = 1$ (i.e. reaction-diffusion equation) and exponentially decaying initial data were previously obtained by Booty, Haberman and Minzon [6], Hamel and Nadin [23], McKean [37], and Sattinger [42], etc. When $m = 2$, a recent work by Xu, Li and Ruan [53] studied the spreading speed of (2) for initial data $u_{1,0}(x)$ and $u_{2,0}(x)$ decaying exponentially with the same decay rate.

The purpose of this paper is to study the spreading speed of (2) where $m \geqslant 2$ and all components of initial data U_0 decay exponentially but their decay rates may be different. We assume that each component of $U_0(x)$ has its own decay rate, namely

$$u_{j,0}(x) \sim C_j e^{-\lambda_j|x|} \text{ as } |x| \to +\infty \text{ with } C_j > 0 \text{ for any } j \in J \triangleq \{1,\ldots,m\}. \tag{6}$$

We call the initial data $U_0(x)$ are non-uniform if there exist some $i, j \in J$ with $i \neq j$ such that $\lambda_i \neq \lambda_j$. The case of non-uniform initial data considered in this paper is essentially different from the case in [53] where $m = 2$ and $\lambda_1 = \lambda_2$. From (5) and other results mentioned above, we conclude that the spreading speed of scalar dispersal equations essentially depends on the decay rate of exponentially decaying initial data. For the dispersal system, if all components of initial data U_0 have the same decay rate (i.e. uniform initial data), the spreading speed can still be determined by this single decay rate as shown in [53] for $m = 2$. But now if the initial data are non-uniform, an immediate question is whether all components of (2) have the same spreading speed, and if so, which component will paly a prevailing role in determining this spreading speed. To proceed, we give the definition of uniform spreading speed of (2).

Definition 1.1 (Uniform spreading speed)**.** Given initial data U_0 satisfying (6), a positive constant c_0 is called the uniform spreading speed of the solution of (2), if for any $j \in J$ and $\varepsilon \in (0, c_0)$, there is a constant $\nu > 0$ such that

$$\begin{cases} \lim\limits_{t\to+\infty} \sup\limits_{|x|\geqslant(c_0+\varepsilon)t} u_j(t,x) = 0, \\ \liminf\limits_{t\to+\infty} \inf\limits_{|x|\leqslant(c_0-\varepsilon)t} u_j(t,x) \geqslant \nu. \end{cases}$$

We will show that when the reaction function F is cooperative and $F'(\mathbf{0})$ is irreducible, all components of the solution U of (2) with non-uniform initial data (different decay rates) satisfying (6) have a uniform spreading speed (the same spreading speed), see Theorem 2.2. Furthermore, this uniform spreading speed depends only on the smallest decay rate $\lambda_0 \triangleq \min\{\lambda_j, j \in J\}$ and is decreasing with respect to λ_0, which implies that the component with the smallest decay rate can accelerate the spatial propagation of other components of U (see details in Section 2). We also refer to a recent work by Xu, Li, and Ruan [52] where the acceleration propagation of (3) was obtained for non-uniform non-exponentially decaying initial data, and other works by Coulon and Yangari [12], Yangari [56], and Xu, Li and Lin [51] for the acceleration propagation with non-uniform nonlocal dispersal kernels and compactly supported initial data.

Remark 1.2. The problem how the dispersal kernel affects the spreading speed has attracted wide attention of researchers, and there are mainly two directions. First, some works study the effect of the decaying behavior of the dispersal kernel as $x \to \pm\infty$. In (2), it is shown that the spreading speed is finite when K satisfies $\int_{\mathbb{R}} k_j(x)e^{\Lambda|x|}dx < +\infty$ for $j \in J$ and some $\Lambda > 0$ (see [25, 35]), but infinite when the condition fails (see [21, 51, 55]). Second, the authors of [53, 54] study the effect of asymmetric dispersal kernel on the sign of spreading speed, and obtain a direct relationship between them.

The rest of this paper is organized as follows. In Section 2, we present the main assumptions and results. In Section 3, we study a special case where all components of initial data have the same decay rate λ, and prove that (2) has a uniform spreading speed dependent on λ. In Section 4, we focus on the general case that the initial data satisfy (6) and complete the proof of our main result.

2. **Main assumptions and results.**

In this section, we give the main assumptions and results. Let us introduce some notations first. For $U = (u_1, \dots, u_m) \in \mathbb{R}^m$, $V = (v_1, \dots, v_m) \in \mathbb{R}^m$, we write $U \geqslant V$ if $u_j \geqslant v_j$ for any $j \in J$; $U \gg V$ if $u_j > v_j$ for any $j \in J$. Denote

$$[U, V] = \{\phi \in \mathbb{R}^m; U \leqslant \phi \leqslant V\}.$$

Let $\|U\| = \sqrt{u_1^2 + \ldots + u_m^2}$ denote the norm of $\mathbb{R}^m$. We write $\mathbf{0} = (0, \dots, 0) \in \mathbb{R}^m$ and $\mathbf{1} = (1, \dots, 1) \in \mathbb{R}^m$. Assume that

(A1) **(a):** there is a strictly positive equilibrium $P = (p_1, p_2, ..., p_m)$ such that $F(\mathbf{0}) = F(P) = \mathbf{0}$ and $F \in C^1([\mathbf{0}, P], \mathbb{R}^m)$; there is no other equilibrium ϕ in $[\mathbf{0}, P]$ such that $F(\phi) = \mathbf{0}$.

(b): F is cooperative in $[\mathbf{0}, P]$, namely $\frac{\partial}{\partial u_i} f_j(U) \geqslant 0$ for any $U \in [\mathbf{0}, P]$ and $j \neq i$.

(c): $F'(\mathbf{0})$ is an irreducible matrix satisfying

$$\max\{\text{Re } \lambda | \det(\lambda I - F'(\mathbf{0})) = 0\} > 0.$$

(d): for any $j \in J$, the function k_j is nonnegative, continuous, symmetric on $\mathbb{R}$, and decreasing on $\mathbb{R}^+$. Moreover, $\int_{\mathbb{R}} k_j(x)dx = 1$ and there exists $\Lambda > 0$ such that

$$\int_{\mathbb{R}} k_j(x)e^{\Lambda|x|}dx < +\infty. \tag{7}$$

Note that (2) is monostable on $[\mathbf{0}, P]$ under (A1)(a) and (c); namely, the equilibrium $U \equiv \mathbf{0}$ is unstable and $U \equiv P$ is stable. From (A1)(b), the matrix $F'(\mathbf{0})$ is essentially nonnegative. Note that a matrix $A = (a_{ij})_{m\times m}$ is essentially nonnegative if all coefficients of the matrix $(A - \min_{i\in\{1,\dots,m\}}\{a_{ii}\}\mathbf{I}_m)$ are nonnegative.

We define

$$\overline{\Lambda} = \sup\left\{\lambda > 0 \;\middle|\; \int_{\mathbb{R}} k_j(x)e^{\lambda x}dx < +\infty,\ j \in \{1,\dots,m\}\right\} \in (0,+\infty)\cup\{+\infty\}.$$

For $\lambda \in (0,\overline{\Lambda})$, let $\mathcal{K}(\lambda)$ denote the $m\times m$ matrix as follows

$$\mathcal{K}(\lambda) = D\cdot \mathrm{diag}\left\{\int_{\mathbb{R}} k_1(y)e^{\lambda y}dy,\dots,\int_{\mathbb{R}} k_m(y)e^{\lambda y}dy\right\} - D + F'(\mathbf{0}). \tag{8}$$

Since $F'(\mathbf{0})$ is irreducible, so is $\mathcal{K}(\lambda)$. By the Perron-Frobenius theorem (see [27]), $\mathcal{K}(\lambda)$ has an eigenvalue $\gamma(\lambda)$ with algebraic multiplicity one, and we denote by $V(\lambda)$ the positive unit eigenvector corresponding to $\gamma(\lambda)$, namely $\mathcal{K}(\lambda)V(\lambda) = \gamma(\lambda)V(\lambda)$ and

$$V(\lambda) \gg \mathbf{0} \quad \text{for } \lambda \in (0,\overline{\Lambda}). \tag{9}$$

From the symmetry of k_j, it follows that $\int_{\mathbb{R}} k_j(y)e^{\lambda y}dy \geqslant 1$ for any $\lambda\in(0,\overline{\Lambda})$. Then (A1)(c) implies that $\gamma(\lambda) > 0$. For $\lambda \in (0,\overline{\Lambda})$, denote

$$c(\lambda) = \gamma(\lambda)/\lambda > 0. \tag{10}$$

Obviously, $c(\lambda)$ is continuous on $(0,\overline{\Lambda})$ and

$$c(\lambda)\lambda V(\lambda) - \mathcal{K}(\lambda)V(\lambda) = \mathbf{0} \text{ for any } \lambda \in (0,\overline{\Lambda}). \tag{11}$$

Define

$$c^* \triangleq \inf_{\lambda\in(0,\overline{\Lambda})}\{c(\lambda)\} < +\infty. \tag{12}$$

It was shown in [25, Lemma 2.4] that $\lambda^* < +\infty$, where λ^* is the smallest positive number at which the above infimum is attained, namely

$$c^* = c(\lambda^*) = \gamma(\lambda^*)/\lambda^* > 0.$$

Remark 2.1. The function $c(\cdot)$ defined by (10) is strictly decreasing on $(0,\lambda^*)$. Indeed, by Lemma 6.5 and (6.5) in Lui [34], $c(\lambda)$ is twice continuously differentiable and decreasing (i.e. $c'(\lambda)\leqslant 0$) on $(0,\lambda^*)$, and it satisfies

$$(\lambda^2 c')' = 2\lambda c' + \lambda^2 c'' \geqslant 0.$$

Then $c''(\lambda) \geqslant 0$ for $\lambda \in (0,\lambda^*)$. Suppose that $c(\lambda)$ is decreasing but not strictly decreasing on $(0,\lambda^*)$. Then there exists $\mu\in(0,\lambda^*)$ such that $c'(\mu) = 0$. From $c'(\lambda)\leqslant 0$ and $c''(\lambda)\geqslant 0$ on $(0,\lambda^*)$, we get that $c'(\lambda) = 0$ for any $\lambda\in[\mu,\lambda^*)$, which implies by the continuity of $c(\lambda)$ that $c(\lambda) = c(\lambda^*)$ for $\lambda\in[\mu,\lambda^*]$. On the other hand, recall that λ^* is the smallest positive number at which $\inf_{\lambda>0}\{c(\lambda)\}$ is attained, and we have $c(\lambda) > c(\lambda^*)$ for $\lambda\in(0,\lambda^*)$. It is a contradiction.

There are some additional assumptions on F.

(A2): for $\lambda\in(0,\lambda^*]$, $F(\min\{P, qV(\lambda)\}) \leqslant qF'(\mathbf{0})V(\lambda)$ for any $q > 0$.

(A3): there are positive numbers q_0, δ_0, and M such that

$$F(U) \geqslant F'(\mathbf{0})U - MU^{1+\delta_0} \text{ for any } U\in[\mathbf{0},P] \text{ with } \|U\|\leqslant q_0,$$

where $U^{1+\delta_0} = (u_1^{1+\delta_0},\dots,u_m^{1+\delta_0}) \in \mathbb{R}^m$.

The assumptions (A2) and (A3) correspond to the Fisher-KPP assumption in the scalar case, namely $f'(0)u - Mu^{1+\delta_0} \leqslant f(u) \leqslant f'(0)u$. The assumption (A3) can be easily satisfied, for example, when $F\in C^{1+\delta_0}[\mathbf{0}, q_0\mathbf{1}]$. As stated in [25], under (A1)-(A3), c^* is the spreading speed of (2) with compactly supported initial data. Denote

$$\lambda_0 \triangleq \min\{\lambda_j \mid j\in J\}.$$

The following theorem about the uniform spreading speed for non-uniform initial data is the main result of this paper.

Theorem 2.2. *Assume (A1), (A2), and (A3) hold. For the non-uniform initial data $U_0(x)$ satisfying (6) with $\lambda_0 \in (0,\lambda^*)$, the solution of (2) has a uniform spreading speed $c(\lambda_0)$, which is independent of the decay rate λ_j satisfying $\lambda_j > \lambda_0$. Moreover, $c(\lambda_0)$ is strictly decreasing with respect to $\lambda_0 \in (0,\lambda^*)$.*

From Theorem 2.2, the cooperation and irreducibility of reaction functions can ensure that all components of the solution of (2) with non-uniform initial data have a uniform spreading speed. In fact, if $F \in C^1[\mathbf{0}, P]$ and $\frac{\partial}{\partial u_i} f_j(\mathbf{0}) > 0$ with $i \neq j$, then as seen from the jth equation of (2), namely

$$\frac{\partial}{\partial t} u_j = d_j(k_j * u_j - u_j) + f_j(U),$$

the component u_i of U has a direct positive effect on the growth of the component u_j, when u_j is small enough. We say u_i has an indirect positive effect on the growth of u_j, if u_i does not directly affect the growth of u_j (i.e. $\frac{\partial}{\partial u_i} f_j(\mathbf{0}) = 0$), but through other components of U, in the sense that there exists a set $\{j_1, j_2, \dots, j_k\}$ with $j_1 = i$ and $j_k = j$ such that $\frac{\partial}{\partial u_{j_{p-1}}} f_{j_p}(\mathbf{0}) > 0$ for any $p = 2, \dots, k$. The irreducibility of $F'(\mathbf{0}) = (\frac{\partial}{\partial u_i} f_j(\mathbf{0}))_{m\times m}$ means that a direct or indirect positive effect exists between any two components of U, and hence, all components of the solution with non-uniform initial data can have a uniform spreading speed.

Theorem 2.2 shows that the uniform spreading speed depends only on the smallest decay rate λ_0. This conclusion, along with the fact that the spreading speed $c(\lambda_0)$ is strictly decreasing on $(0,\lambda^*)$ in Remark 2.1, means that the component with the smallest decay rate can accelerate the spatial propagation of other components. To understand this, we assume the j_0th component of initial data U_0 has the smallest decay rate, namely $\lambda_{j_0} = \lambda_0 \in (0,\lambda^*)$. Let the decay rate of the j_0th component $u_{j_0,0}$ become smaller and fix the decay rates of other components of initial data U_0. We denote the new decay rate of $u_{j_0,0}$ by $\lambda' \in (0,\lambda_0)$. Then the uniform spreading speed becomes $c(\lambda')$ from $c(\lambda_0)$. Since $c(\cdot)$ is strictly decreasing on $(0,\lambda^*)$, we have that $c(\lambda') > c(\lambda_0)$, which means that the decrease of the smallest decay rate in the initial data can increase the spreading speed of other components of the solution.

Our idea to prove Theorem 2.2 consists of two steps. First, we focus on the special case that all components of initial data U_0 have the same decay rate $\lambda \in (0,\lambda^*)$ (namely $\lambda_j = \lambda$ for any $j \in J$) and prove that the solution has a uniform spreading speed $c(\lambda)$ in Section 3. Second, the general case that U_0 satisfies (6) with $\lambda_0 \in (0,\lambda^*)$ is considered in Section 4. By constructing a lower solution, we show that after a period of time $T > 0$, all components of $U(T,\cdot)$ are larger than an exponentially decaying function with the decay rate λ_0. This case is then transformed into the special case considered in Section 3 as long as $u_j(T,x)$ is set as the new initial data.

Moreover, from Theorem 2.2 and its proof in Section 4, the components of initial data U_0 whose decay rates are not λ_0 affect neither the result of uniform spreading speed nor its proof method. Therefore, Theorem 2.2 also holds if (6) is changed to the following assumption

(H): there exist $j_0 \in \{1,2,\dots,m\}$ and $\lambda_0 > 0$ such that

$$u_{j_0,0}(x) \sim Ce^{-\lambda_0|x|},\ u_{j,0}(x) \leqslant e^{-\lambda_0|x|} \text{ for } j \neq j_0 \text{ and } |x| \text{ large enough.}$$

In this assumption, the component $u_{j,0}$ of U_0 with $j \neq j_0$ is not restricted to exponentially decaying functions, but any function that is smaller than $e^{-\lambda_0|x|}$ when $|x|$ is large enough.

Remark 2.3. The methods in this paper are also applicable to the reaction-diffusion cooperative system (3). Therefore, no matter whether we consider a non-local or local dispersal system, the cooperation and irreducibility of F can ensure that the solution has a uniform spreading speed and the component of U with the smallest decay rate can accelerate the spatial propagation of other components.

3. **Case of the same decay rate.**

In this section, we consider the case that all components of initial data have the same decay rate $\lambda \in (0, \lambda^*)$. First, we state two important lemmas that are proved in [53, Theorem 4.1] (for Lemma 3.1) and [25, Theorem 4.5] (for Lemma 3.2).

Lemma 3.1. *(Symmetry and monotone property) If the functions $k_j(\cdot)$ and $u_{j,0}(\cdot)$ are symmetric on $\mathbb{R}$ and decreasing on $\mathbb{R}^+$ for any $j \in J$, so is $u_j(t, \cdot)$ for any $t > 0$ and $j \in J$.*

Lemma 3.2. *(Comparison principle) Assume that $\bar{U}$ is an upper solution and $\underline{U}$ is a lower solution of (2); namely $\frac{\partial}{\partial t}\bar{U}(t,x)$ and $\frac{\partial}{\partial t}\underline{U}(t,x)$ exist and*

$$\frac{\partial}{\partial t}\bar{U} - DK * \bar{U} + D\bar{U} - F(\bar{U}) \geqslant \mathbf{0} \quad \text{for } t > 0,\ x \in \mathbb{R},$$
$$\frac{\partial}{\partial t}\underline{U} - DK * \underline{U} + D\underline{U} - F(\underline{U}) \leqslant \mathbf{0} \quad \text{for } t > 0,\ x \in \mathbb{R}.$$

If $\bar{U}(0,x) \geqslant \underline{U}(0,x)$ for $x \in \mathbb{R}$, then $\bar{U}(t,x) \geqslant \underline{U}(t,x)$ for any $t \geqslant 0$ and $x \in \mathbb{R}$.

The following result is a special case of Theorem 2.2 where all components of U_0 have the same decay rate $\lambda \in (0, \lambda^*)$.

Proposition 3.3. *Assume (A1), (A2), and (A3) hold. Let $U_0(x)$ satisfy (6) with $\lambda_j = \lambda \in (0, \lambda^*)$ for any $j \in J$. Then the solution of (2) has a uniform spreading speed $c(\lambda)$.*

Proof. Let $U = (u_1, \dots, u_m)$ be the solution of (2) with initial data U_0. By (6) and (9), there is a constant $\Gamma > 0$ large enough such that

$$U_0(x) \ll \Gamma e^{-\lambda|x|} V(\lambda).$$

For $\lambda \in (0, \lambda^*)$, define

$$\bar{U}(t,x) = \min\left\{P,\ \Gamma e^{-\lambda z} V(\lambda)\right\} \quad \text{with } z = |x| - c(\lambda)t,\ t \geqslant 0,\ x \in \mathbb{R}. \tag{13}$$

Now we check that $\bar{U} = (\bar{u}_1, \dots, \bar{u}_m)$ is an upper solution. Let $v_j(\lambda)$ denote the jth component of $V(\lambda)$, namely $V(\lambda) = (v_1(\lambda), \dots, v_m(\lambda))$. For any $j \in J$, when $z < \lambda^{-1}\ln(\Gamma v_j(\lambda)/p_j)$, we have that $\bar{u}_j(t,x) = p_j$. Then by (A1)(b), from $\bar{u}_i(t,x) \leqslant p_i$ for any $i \in J$ we can get that

$$\frac{\partial}{\partial t}\bar{u}_j - d_j k_j * \bar{u}_j + d_j \bar{u}_j - f_j(\bar{U}) \geqslant -f_j(P) = 0.$$

When $z \geqslant \lambda^{-1}\ln(\Gamma v_j(\lambda)/p_j)$, it holds that $\bar{u}_j(t,x) = \Gamma e^{-\lambda z} v_j(\lambda)$. We denote $f_{j,i} = \frac{\partial}{\partial u_i} f_j(\mathbf{0})$ and (A1)(b) implies $f_{j,i} \geqslant 0$ for $i \neq j$. By (A2) and (11), we have that

$$\begin{aligned}&\frac{\partial}{\partial t}\bar{u}_j - d_j k_j * \bar{u}_j + d_j\bar{u}_j - f_j(\bar{U})\\ &\geqslant \Gamma e^{-\lambda z}\Big[\Big(c(\lambda)\lambda - d_j\int_{\mathbb{R}} k_j(y)e^{\lambda y}dy + d_j\Big)v_j(\lambda) - \sum_{i=1}^{m} f_{j,i}v_i(\lambda)\Big] = 0.\end{aligned}$$

Thus $\bar{U} = (\bar{u}_1, \dots, \bar{u}_m)$ is an upper solution of (2). Lemma 3.2 implies that

$$U(t,x) \leqslant \bar{U}(t,x) \leqslant \Gamma e^{-\lambda z}V(\lambda) \text{ for any } t \geqslant 0 \text{ and } x \in \mathbb{R}.$$

Then for any $\varepsilon > 0$ and $j \in J$, we have that

$$\begin{aligned}\lim_{t\to+\infty}\sup_{|x|\geqslant(c(\lambda)+\varepsilon)t} u_j(t,x) &\leqslant \lim_{t\to+\infty}\sup_{|x|\geqslant(c(\lambda)+\varepsilon)t} \Gamma e^{-\lambda(|x|-c(\lambda)t)}v_j(\lambda)\\ &\leqslant \lim_{t\to+\infty}\Gamma e^{-\lambda\varepsilon t}v_j(\lambda)\\ &= 0.\end{aligned}$$

Now we just need to prove that for any $\varepsilon \in (0, c(\lambda))$ and $j \in J$, there exists $\nu > 0$ such that

$$\lim_{t\to+\infty}\inf_{|x|\leqslant(c(\lambda)-\varepsilon)t} u_j(t,x) \geqslant \nu. \tag{14}$$

The proof of (14) consists of the following two steps.

First, we prove that there exist two positive constants γ and y_0 such that

$$U(1,x) \geqslant \gamma\min\left\{e^{-\lambda|x|}, e^{-\lambda y_0}\right\}V(\lambda),\ x \in \mathbb{R}. \tag{15}$$

From (6) it follows that $U_0(x) \gg \mathbf{0}$ for sufficiently large $|x|$. Then by (A1)(d), there exists $N_0 \in \mathbb{N}^+$ such that

$$\underbrace{K * K * \ldots * K}_{N_0} * U_0(x) \gg \mathbf{0} \text{ for any } x \in \mathbb{R}. \tag{16}$$

For $j \in J$, let $\pi_j : \mathbb{R}^m \to \mathbb{R}^m$ denote the function

$$\pi_j : (u_1, \dots, u_m) \mapsto (0, \dots, u_j, \dots, 0);$$

namely the jth component of $\pi_j(U)$ is u_j while others are zero. We define

$$b_j = \inf_{u_j\in(0,p_j]}\{f_j(\pi_j(U))/u_j\}. \tag{17}$$

Let $n \in \mathbb{N}^+$ and we divide equally the time period of $[0,\tau]$ into n parts, namely $[0,\tau/n]$, $[\tau/n, 2\tau/n]$,..., and $[(n-1)\tau/n, \tau]$. In $[0,\tau/n]$, we consider

$$\underline{W}(t,x) = (\underline{w}_1(t,x), \dots, \underline{w}_m(t,x)),\ t \in [0,\tau/n],\ x \in \mathbb{R},$$

where

$$\underline{w}_j(t,x) = M_j\left[u_{j,0}(x) + td_jk_j * u_{j,0}(x)\right]e^{(b_j-d_j)t},\ j \in J \tag{18}$$

and

$$M_j = (1 + d_j\tau/n)^{-1}(1 + e^{(b_j-d_j)\tau/n})^{-1},\ j \in J.$$

It is easy to check that

$$\partial_t\underline{w}_j - d_jk_j * \underline{w}_j + d_j\underline{w}_j - b_j\underline{w}_j \leqslant 0 \quad \text{for } j \in J.$$

For $t \in [0,\tau/n]$, by $u_{j,0}(x) \leqslant p_j$ we have that

$$\underline{w}_j(t,x) \leqslant M_jp_j[1 + d_j\tau/n]e^{(b_j-d_j)\tau/n} \leqslant p_j \quad \text{for } x \in \mathbb{R}.$$

From (A1)(b) and (17), it follows that

$$\begin{aligned}&\underline{W}_t - DK * \underline{W} + D\underline{W} - F(\underline{W})\\ \leqslant\ & \underline{W}_t - DK * \underline{W} + D\underline{W} - (f_1(\pi_1(\underline{W})), \ldots, f_m(\pi_m(\underline{W})))\\ \leqslant\ & \underline{W}_t - DK * \underline{W} + D\underline{W} - \mathrm{diag}\{b_1, \ldots, b_m\}\underline{W} \leqslant \mathbf{0}.\end{aligned}$$

By $\underline{W}(0,x) \leqslant U_0(x)$ for $x \in \mathbb{R}$, from Lemma 3.2 we get that

$$U(\tau/n, x) \geqslant \underline{W}(\tau/n, x). \tag{19}$$

Denote $C_1 \triangleq \min_{j\in J}\{M_j d_j e^{(b_j-d_j)\tau/n}\tau/n\}$ and then

$$U(\tau/n, x) \geqslant C_1 K * U_0(x).$$

Repeat this argument for $t \in [\tau/n, 2\tau/n]$ and substitute $K * U_0(x)$ for $U_0(x)$. We can find a constant $C_2 > 0$ such that

$$U(2\tau/n, x) \geqslant C_2 K * K * U_0(x).$$

Similarly, there exists $C_n > 0$ such that

$$U(\tau, x) \geqslant C_n \underbrace{K * K * \ldots * K}_{n} * U_0(x) \text{ for } x \in \mathbb{R}.$$

When $n = N_0$, it follows from (16) that $U(\tau, x) \gg \mathbf{0}$. When $n = 1$, we get from (18) and (19) that

$$U(\tau, x) \geqslant \underline{W}(\tau, x) \geqslant C_\tau U_0(x) \text{ with } C_\tau = \min_{j\in J}\{M_j e^{(b_j-d_j)\tau}\}.$$

Then for any $\tau > 0$ there exists $C_\tau > 0$ such that

$$U(\tau, x) \gg \mathbf{0} \text{ and } U(\tau, x) \geqslant C_\tau U_0(x) \text{ for } x \in \mathbb{R}. \tag{20}$$

When $\tau = 1$, by (6) with $\lambda_j = \lambda$ we can find $\gamma > 0$ and $y_0 > 0$ satisfying (15).

Let γ be smaller (if necessary) such that $\gamma e^{-\lambda y_0} \leqslant q_0$, where q_0 is given by assumption (A3). Define $W_0(x) = \gamma \min\left\{e^{-\lambda|x|}, e^{-\lambda y_0}\right\} V(\lambda)$, $x \in \mathbb{R}$. Then $\|W_0(x)\| \leqslant q_0$ for $x \in \mathbb{R}$ and

$$U(1, x) \geqslant W_0(x) = \begin{cases}\gamma e^{-\lambda|x|} V(\lambda) & \text{for } |x| \geqslant y_0,\\ \gamma e^{-\lambda y_0} V(\lambda) & \text{for } |x| \leqslant y_0,\end{cases} \tag{21}$$

Let $W(t,x)$ be the solution of (2) with initial data $W(0,x) = W_0(x)$. Then we get from Lemma 3.2 that

$$U(t+1, x) \geqslant W(t, x) \text{ for } t \geqslant 0,\ x \in \mathbb{R}. \tag{22}$$

Since $W_0(\cdot)$ is symmetric and decreasing on $\mathbb{R}^+$, so is $W(t, \cdot)$ by Lemma 3.1.

Second, we construct a lower solution and prove (14). Now define some notations. By Remark 2.1, for any $\lambda \in (0, \lambda^*)$, there is a constant $\delta_\lambda = \lambda^*/\lambda - 1 > 0$ such that

$$c(\lambda + \lambda s) < c(\lambda) \text{ for any } s \in (0, \delta_\lambda).$$

Denote

$$\mu = \lambda(1+\delta) > 0 \quad \text{with} \quad \delta \triangleq \min\{\delta_0, \delta_\lambda/2\} > 0,$$

where the positive constant δ_0 is given by (A3). Then it follows that

$$c(\mu) < c(\lambda). \tag{23}$$

For $j \in J$, let $G(c, \lambda; j)$ be the jth component of the vector $c\lambda V(\lambda) - \mathcal{K}(\lambda)V(\lambda)$, namely

$$G(c, \lambda; j) \triangleq \left(c\lambda - d_j \int_{\mathbb{R}} k_j(y)e^{\lambda y}dy + d_j\right) v_j(\lambda) - \sum_{i=1}^{m} f_{j,i}v_i(\lambda), \quad c > 0,\ \lambda > 0,$$

where $f_{j,i} = \frac{\partial}{\partial u_i} f_j(\mathbf{0})$ and $v_j(\lambda)$ is the jth component of $V(\lambda) \gg \mathbf{0}$. For $\lambda \in (0, \lambda^*)$, it follows from (11) that

$$G(c(\lambda), \lambda; j) = \left(c(\lambda)\lambda - d_j \int_{\mathbb{R}} k_j(y)e^{\lambda y}dy + d_j\right) v_j(\lambda) - \sum_{i=1}^{m} f_{j,i}v_i(\lambda) = 0. \tag{24}$$

By (23) we get that

$$\begin{aligned} G(c(\lambda), \mu; j) &= \left(c(\lambda)\mu - d_j \int_{\mathbb{R}} k_j(y)e^{\mu y}dy + d_j\right) v_j(\mu) - \sum_{i=1}^{m} f_{j,i}v_i(\mu) \\ &> G(c(\mu), \mu; j) = 0. \end{aligned} \tag{25}$$

For $\lambda \in (0, \lambda^*)$, we define $\underline{U} = (\underline{u}_1, \ldots, \underline{u}_m)$ as follows

$$\underline{U}(t, x) = \max\left\{\mathbf{0},\ \gamma e^{-\lambda z}V(\lambda) - Le^{-\mu z}V(\mu)\right\} \text{ with } z = |x| - c(\lambda)t,\ t \geqslant 0,\ x \in \mathbb{R},$$

where L is a positive constant large enough such that

$$L \geqslant \max\left\{\frac{\gamma e^{\lambda\delta y_0}}{1+\delta} \max_{j \in J}\left(\frac{v_j(\lambda)}{v_j(\mu)}\right),\ M\gamma^{1+\delta} \max_{j \in J}\left(\frac{v_j^{1+\delta}(\lambda)}{G(c(\lambda), \mu; j)}\right)\right\}. \tag{26}$$

Denote

$$y_j = \lambda^{-1}\delta^{-1} \ln\left(\frac{L(1+\delta)v_j(\mu)}{\gamma v_j(\lambda)}\right) \text{ and } z_j = \lambda^{-1}\delta^{-1} \ln\left(\frac{Lv_j(\mu)}{\gamma v_j(\lambda)}\right) \quad \text{for } j \in J.$$

Then $y_j > z_j$ for any $j \in J$. Note that y_j and z_j correspond respectively to the maximum point of $z \mapsto \gamma e^{-\lambda z}v_j(\lambda) - Le^{-\mu z}v_j(\mu)$ and the root of $\gamma e^{-\lambda z}v_j(\lambda) - Le^{-\mu z}v_j(\mu) = 0$, that is

$$\underline{u}_j(t, x) = \begin{cases} 0, & \text{when } z < z_j, \\ \gamma e^{-\lambda z}v_j(\lambda) - Le^{-\mu z}v_j(\mu) = 0, & \text{when } z = z_j, \\ \gamma e^{-\lambda z}v_j(\lambda) - Le^{-\mu z}v_j(\mu) > 0, & \text{when } z > z_j, \end{cases} \tag{27}$$

and

$$\max_{z \in \mathbb{R}}\{\gamma e^{-\lambda z}v_j(\lambda) - Le^{-\mu z}v_j(\mu)\} = \gamma e^{-\lambda y_j}v_j(\lambda) - Le^{-\mu y_j}v_j(\mu) > 0.$$

From (26), it follows that $y_j \geqslant y_0$ for any $j \in J$. Then we have that

$$\sup_{t \geqslant 0, x \in \mathbb{R}} \underline{u}_j(t, x) = \underline{u}_j(t, c(\lambda)t + y_j) = \gamma e^{-\lambda y_j}v_j(\lambda) - Le^{-\mu y_j}v_j(\mu) \leqslant \gamma e^{-\lambda y_0}v_j(\lambda). \tag{28}$$

Since $V(\lambda)$ is a unit vector, it holds that

$$\|\underline{U}(t, x)\| \leqslant \gamma e^{-\lambda y_0} \leqslant q_0 \quad \text{for any } t \geqslant 0,\ x \in \mathbb{R}. \tag{29}$$

Particularly, when $t = 0$, it follows from (28) that $\underline{U}(0, x) \leqslant \gamma e^{-\lambda y_0}V(\lambda)$ for any $x \in \mathbb{R}$. The definition of $\underline{U}(t, x)$ implies that $\underline{U}(0, x) \leqslant \gamma e^{-\lambda|x|}V(\lambda)$ for $x \in \mathbb{R}$. Then we get from (21) that

$$\underline{U}(0, x) \leqslant W_0(x) \text{ for } x \in \mathbb{R}. \tag{30}$$

In order to verify $\underline{U}(t,x)$ is a lower solution, namely

$$\underline{U}_t - DK * \underline{U} + D\underline{U} - F(\underline{U}) \leqslant \mathbf{0},$$

we check it holds for each component. For any $j \in J$, when $z < z_j$, since $\underline{u}_j(t,x) = 0$, it is easy to check that

$$\frac{\partial}{\partial_t}\underline{u}_j - d_j k_j * \underline{u}_j + d_j \underline{u}_j - f_j(\underline{U}) \leqslant 0.$$

When $z \geqslant z_j$, we get that

$$\underline{u}_j(t,x) = \gamma e^{-\lambda z} v_j(\lambda) - L e^{-\mu z} v_j(\mu),$$
$$\underline{u}_i(t,x) \geqslant \gamma e^{-\lambda z} v_i(\lambda) - L e^{-\mu z} v_i(\mu) \text{ for } i \neq j.$$

From (A3) and (29), it follows that

$$\begin{aligned} f_j(\underline{U}) &\geqslant \sum_{i=1}^m f_{j,i} \underline{u}_i(t,x) - M \underline{u}_j^{1+\delta}(t,x) \\ &\geqslant \sum_{i=1}^m f_{j,i} \left[\gamma e^{-\lambda z} v_i(\lambda) - L e^{-\mu z} v_i(\mu)\right] - M \gamma^{1+\delta} e^{-\mu z} v_j^{1+\delta}(\lambda). \end{aligned}$$

Then some calculations show that

$$\begin{aligned} &\frac{\partial}{\partial t}\underline{u}_j - d_j k_j * \underline{u}_j + d_j \underline{u}_j - f_j(\underline{U}) \\ &\leqslant \gamma e^{-\lambda z}\Big[\Big(c(\lambda)\lambda - d_j \int_{\mathbb{R}} k_j(y) e^{\lambda y} dy + d_j\Big) v_j(\lambda) - \sum_{i=1}^m f_{j,i} v_i(\lambda)\Big] \\ &\quad - L e^{-\mu z}\Big[\Big(c(\lambda)\mu - d_j \int_{\mathbb{R}} k_j(y) e^{\mu y} dy + d_j\Big) v_j(\mu) - \sum_{i=1}^m f_{j,i} v_i(\mu)\Big] \\ &\quad + M \gamma^{1+\delta} e^{-\mu z} v_j^{1+\delta}(\lambda) \\ &= \gamma e^{-\lambda z} G(c(\lambda), \lambda; j) - e^{-\mu z}\left[L G(c(\lambda), \mu; j) - M \gamma^{1+\delta} v_j^{1+\delta}(\lambda)\right]. \end{aligned}$$

By (24), (25), and (26), for $z \geqslant z_j$, we have

$$\frac{\partial}{\partial t}\underline{u}_j - d_j k_j * \underline{u}_j + d_j \underline{u}_j - f_j(\underline{U}) \leqslant 0.$$

Therefore, $\underline{U}(t,x)$ is a lower solution.

Lemma 3.2 and (30) imply that

$$W(t,x) \geqslant \underline{U}(t,x) \text{ for any } t \geqslant 0,\ x \in \mathbb{R}.$$

Let $y_{\max} \triangleq \max_{j\in J}\{y_j\}$. It follows from $y_j > z_j$ that $y_{\max} > \max_{j\in J}\{z_j\}$, which implies by (27) that

$$\nu \triangleq \min_{j\in J}\{\underline{u}_j(t, c(\lambda)t + y_{\max})\} > 0.$$

We denote $W(t,x)$ by $(w_1(t,x), \dots, w_m(t,x))$. Then it follows that

$$w_j(t, c(\lambda)t + y_{\max}) \geqslant \underline{u}_j(t, c(\lambda)t + y_{\max}) \geqslant \nu \text{ for any } t \geqslant 0 \text{ and } j \in J.$$

Since $W(t,\cdot)$ is symmetric and decreasing on $\mathbb{R}^+$, it holds that

$$w_j(t,x) \geqslant \nu \text{ for any } |x| \leqslant c(\lambda)t + y_{\max} \text{ and } j \in J.$$

By (22) we get that

$$u_j(t+1,x) \geqslant \nu \text{ for any } |x| \leqslant c(\lambda)t + y_{\max} \text{ and } j \in J,$$

which implies (14). This completes the proof of Proposition 3.3. □

4. **General case.** In this section, we give the proof of Theorem 2.2. By constructing a lower solution, we transform the proof for the general case where U_0 satisfies (6) into the special case in Section 3, where all components of the initial data have the same decay rate.

Proof of Theorem 2.2. The strictly decreasing property of $c(\lambda_0)$ with respect to $\lambda_0 \in (0, \lambda^*)$ has been obtained in Remark 2.1. By (6) and $\lambda_0 \leqslant \lambda_j$ for $j \in J$, there exists $C > 0$ such that $u_{j,0}(x) \leqslant Ce^{-\lambda_0|x|}$ for $j \in J$ and large $|x|$. Then the proof of

$$\lim_{t\to+\infty} \sup_{|x|\geqslant(c(\lambda_0)+\varepsilon)t} u_j(t,x) = 0 \quad \text{for } j \in J$$

is similar to the counterpart in the proof of Propositions 3.3, and we only need to substitute λ_0 for λ.

Now prove that for any $\varepsilon \in (0, c(\lambda_0))$, there is a constant $\nu > 0$ such that

$$\lim_{t\to+\infty} \inf_{|x|\leqslant(c(\lambda_0)-\varepsilon)t} u_j(t,x) \geqslant \nu \text{ for any } j \in J.$$

From the proof of Propositions 3.3, we only need to prove that there exist $T > 0$ and $M_0 > 0$ such that

$$u_j(T,x) \geqslant M_0 p(x), \ x \in \mathbb{R} \text{ for any } j \in J, \tag{31}$$

where

$$p(x) = e^{-\lambda_0|x|}.$$

Now we reorder the equations in the system (2), namely the components of U. Define

$$f_{j,i} = \frac{\partial}{\partial u_i} f_j(\mathbf{0}).$$

Choose the component which has the smallest decay rate as the first component u_1 of U, and then $\lambda_1 = \lambda_0 = \min\{\lambda_j, j \in J\}$. Since $F'(0)$ is irreducible, we can choose the second component u_2 such that $f_{2,1} > 0$. Similarly, we can choose the third component u_3 satisfying $f_{3,1} > 0$ or $f_{3,2} > 0$. Repeat this process, we reorder the components of U satisfying that for any $i \in \{2, 3, \ldots, m\}$ there exists $j \in \{1, 2, \ldots, i-1\}$ such that $f_{i,j} > 0$.

We give an important inequality. Since $F \in C^1[\mathbf{0}, P])$, by (A1)(b), we can find a constant $q_3 > 0$ such that

$$f_j(U) \geqslant (f_{j,j} - 1)u_j + \frac{1}{2}\sum_{i\neq j} f_{j,i}u_i \text{ for any } j \in J \text{ and } U \in [\mathbf{0}, q_3\mathbf{1}]. \tag{32}$$

In order to prove (31), we need to construct a lower solution

$$W(t,x) = (w_1(t,x), \ldots, w_m(t,x)) \in [\mathbf{0}, q_3\mathbf{1}], \quad t \geqslant 1, \ x \in \mathbb{R}.$$

The form of $W(t,x)$ will be given for every component. First, we construct the first component $w_1(t,x)$ of $W(t,x)$. By (20) and (6) with $\lambda_1 = \lambda_0$, there is a constant $C_0 \in (0, q_3]$ such that

$$u_1(1,x) \geqslant C_0 p(x) \text{ for } x \in \mathbb{R}.$$

Let

$$w_1(t,x) = M_1 e^{-\alpha(t-1)} p(x) \text{ for } t \geqslant 1, \ x \in \mathbb{R},$$

where M_1 is a constant in $(0, C_0]$ and

$$\alpha \geqslant \max_{j\in J}\{d_j + |f_{j,j}|\} + 2.$$

Note that M_1 will be reselected as a smaller constant later. It is easy to check that

$$w_1(t,x) \leqslant M_1 \leqslant C_0 \leqslant q_3 \quad \text{for } t \geqslant 1,\ x \in \mathbb{R}$$

and

$$w_1(1,x) \leqslant M_1 p(x) \leqslant u_1(1,x) \text{ for } x \in \mathbb{R}. \tag{33}$$

From $p(x) \geqslant 0$, it follows that $k_1 * w_1 \geqslant 0$. By the cooperation of F and (32), we have that $f_1(W) \geqslant (f_{1,1}-1)w_1$ for $W \in [\mathbf{0}, q_3\mathbf{1}]$. Then some calculations show that

$$\begin{aligned}&\frac{\partial}{\partial t}w_1 - d_1 k_1 * w_1 + d_1 w_1 - f_1(W)\\ \leqslant\ & M_1(-\alpha + d_1 - f_{1,1} + 1)e^{-\alpha(t-1)}p(x) \leqslant 0.\end{aligned}$$

Second, we construct the second component $w_2(t,x)$ of $W(t,x)$ under the condition $f_{2,1} > 0$. Define

$$w_2(t,x) = M_2\left(e^{-\beta_2(t-1)} - e^{-\alpha(t-1)}\right)p(x) \quad \text{for } t \geqslant 1,\ x \in \mathbb{R},$$

where

$$\beta_2 = d_2 + |f_{2,2}| + 1, \quad M_2 \triangleq \frac{f_{2,1}M_1}{2(\alpha - d_2 + f_{2,2} - 1)}.$$

By $\alpha \geqslant \beta_2$, we get that $w_2 \geqslant 0$ for $t \geqslant 1$, which implies that $k_2 * w_2 \geqslant 0$. Let M_1 be smaller (if necessary) satisfying $M_1 \leqslant 2q_3/f_{2,1}$. From $\alpha \geqslant d_2 - f_{2,2} + 2$, it follows that

$$w_2(t,x) \leqslant M_2 \leqslant \frac{1}{2}f_{2,1}M_1 \leqslant q_3 \text{ for } t \geqslant 1,\ x \in \mathbb{R}.$$

Assumption (A1)(b) and (32) show that $f_2(W) \geqslant (f_{2,2}-1)w_2 + \frac{1}{2}f_{2,1}w_1$ for $W \in [\mathbf{0}, q_3\mathbf{1}]$. Then we have that

$$\begin{aligned}&\frac{\partial}{\partial t}w_2 - d_2k_2 * w_2 + d_2w_2 - f_2(W)\\ &\leqslant \frac{\partial}{\partial t}w_2 - d_2k_2 * w_2 + d_2w_2 - (f_{2,2}-1)w_2 - \frac{1}{2}f_{2,1}w_1\\ &\leqslant M_2\left[(-\beta_2 + d_2 - f_{2,2} + 1)e^{-\beta_2(t-1)} + (\alpha - d_2 + f_{2,2} - 1)e^{-\alpha(t-1)}\right]p(x)\\ &\quad - \frac{1}{2}M_1 f_{2,1}e^{-\alpha(t-1)}p(x)\\ &\leqslant \left[M_2(\alpha - d_2 + f_{2,2} - 1) - \frac{1}{2}M_1 f_{2,1}\right]e^{-\alpha(t-1)}p(x) = 0.\end{aligned}$$

Moreover, it is easy to check that

$$w_2(1,x) = 0 \text{ for } x \in \mathbb{R}.$$

Note that $e^{-\beta_2 s} \geqslant 2e^{-\alpha s}$ for $s \geqslant \tau \triangleq \ln 2$ and then

$$w_2(t,x) \geqslant M_2 e^{-\alpha(t-1)}p(x) \quad \text{for } t \geqslant 1 + \tau,\ x \in \mathbb{R}, \tag{34}$$

which is a key inequality for the construction of w_j with $j > 2$ when $f_{j,2} > 0$.

Third, we construct the third component $w_3(t,x)$ of $W(t,x)$ under the condition $f_{3,1} > 0$ or $f_{3,2} > 0$. For the case $f_{3,1} > 0$, we can construct $w_3(t,x)$ by the same method as $w_2(t,x)$. For the case $f_{3,2} > 0$, we define

$$w_3(t,x) = \begin{cases} 0, & 1 \leqslant t \leqslant 1+\tau, \\ M_3\Big(e^{-\beta_3(t-1-\tau)} - e^{-\alpha(t-1-\tau)}\Big)p(x), & t \geqslant 1+\tau, \end{cases}$$

where

$$\beta_3 = d_3 + |f_{3,3}| + 1, \quad M_3 \triangleq \frac{f_{3,2}M_2}{2e^{\alpha\tau}(\alpha - d_3 + f_{3,3} - 1)}.$$

Let M_1 be smaller (if necessary) such that $M_3 \leqslant q_3$, and then $0 \leqslant w_3(t,x) \leqslant q_3$ for $t \geqslant 1$, $x \in \mathbb{R}$. By (32) and (34), we have that

$$f_3(W) \geqslant (f_{3,3}-1)w_3 + \frac{f_{3,2}}{2}w_2 \geqslant (f_{3,3}-1)w_3 + \frac{f_{3,2}}{2}M_2e^{-\alpha(t-1)}p(x) \quad \text{for } W \in [\mathbf{0}, q_3\mathbf{1}].$$

Following similar calculations to these for w_2, we can prove that

$$\frac{\partial}{\partial t}w_3 - d_3k_3 * w_3 + d_3w_3 - f_3(W) \leqslant 0.$$

We also have that

$$w_3(t,x) \geqslant M_3e^{-\alpha(t-1-\tau)}p(x) \quad \text{for } t \geqslant 1+2\tau,\ x \in \mathbb{R},$$

which provides the key inequality for the construction of w_j with $j > 3$ when $f_{j,3} > 0$.

To construct w_j for $j \in \{4,5,\dots,m\}$, when $f_{j,1} > 0$, we apply the construction method for w_2, and when $f_{j,i} > 0$ for some $i \in \{2,\dots,j-1\}$, we use the construction method for w_3 in the case $f_{3,2} > 0$. Then we can define every component of $W(t,x)$ satisfying

$$\frac{\partial}{\partial t}W - DK * W + DW - F(W) \leqslant \mathbf{0} \text{ for } t \geqslant 1,\ x \in \mathbb{R},$$

and

$$w_i(t,x) \geqslant M_ie^{-\alpha[t-1-(i-2)\tau]}p(x) \quad \text{for } t \geqslant 1+(i-1)\tau,\ x \in \mathbb{R},\ i = 2,\dots,m.$$

We obtain two constants

$$T = 1 + (m-1)\tau \quad \text{and } M_0 = \left\{M_1e^{-\alpha(T-1)}, \min_{i\in\{2,\dots,m\}}\left\{M_ie^{-\alpha[T-1-(i-2)\tau]}\right\}\right\}$$

such that

$$w_i(T,x) \geqslant M_0p(x),\ x \in \mathbb{R} \ \text{ for any } i = 1,\dots,m. \tag{35}$$

The definition of W also shows that $w_i(1,x) = 0$ for any $i = 2,\dots,m$. We get from (33) that

$$U(1,x) \geqslant W(1,x),\ x \in \mathbb{R}.$$

It follows from Lemma 3.2 that

$$U(t,x) \geqslant W(t,x) \quad \text{for } t \geqslant 1,\ x \in \mathbb{R}.$$

Then we have $U(T,x) \geqslant W(T,x)$ for $x \in \mathbb{R}$, which implies (31) by (35). It completes the proof of Theorem 2.2. $\square$

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