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Global dynamics of a three-species spatial food chain model

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Abstract

In this paper, we study the following initial-boundary value problem of a three-species spatial food chain model

$$\begin{cases} u_t = d_1 \Delta u + u(1-u) - b_1 uv, & x \in \Omega, t > 0 \\ v_t = d_2 \Delta v - \nabla \cdot (\xi v \nabla u) + uv - b_2 vw - \theta_1 v, & x \in \Omega, t > 0 \\ w_t = \Delta w - \nabla \cdot (\chi w \nabla v) + vw - \theta_2 w, & x \in \Omega, t > 0 \end{cases}$$

in a bounded domain $\Omega \subset \mathbb{R}^2$ with smooth boundary and homogeneous Neumann boundary conditions, where all parameters are positive constants. By the delicate coupling energy estimates, we first establish the global existence of classical solutions in two dimensional spaces for appropriate initial data. Moreover by constructing Lyapunov functionals and using LaSalle's invariance principle, we establish the global stability of the prey-only steady state, semi-coexistence and coexistence steady states.

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1. Introduction and main results

A food chain is a linearly linked network in a food web starting from producer species (such as grass or trees which use radiation from the Sun to make their food via photosynthesis) and ending at an apex predator species (like grizzly bears or killer whales), see Wikipedia. The length of a food chain is the number of links between a trophic consumer and the base of the web, and it is commonly used as a metric to quantify food web trophic structure. While food chains are often used in ecological modeling, most of theoretical attentions are focused on the three-species continuous-time food chain models although they are simplified abstractions of real food webs, but complex in their dynamics and mathematical implications [33]. The prototype of three-species food chain models was first proposed by Hasting and Powell in [12], reading as

$$\begin{cases} u_t = u(1-u) - f_1(u,v)v, & t>0, \\ v_t = f_1(u,v)v - f_2(v,w)w - \theta_1 v, & t>0, \\ w_t = f_2(v,w)w - \theta_2 w, & t>0, \end{cases} \tag{1.1}$$

where $(u,v,w) := (u,v,w)(t)$ represent the densities of the prey species, intermediate and top predators, respectively, at time $t>0$. The functions $f_i (i=1,2)$ denote the trophic functions (i.e. functional response functions). When $f_i (i=1,2)$ are Holling type II trophic functions (i.e. $f_i(y,z) = \frac{a_i}{b_i+y}$ with constants $a_i, b_i > 0$), the food chain model (1.1) exhibits complex dynamics, like chaos [12,22,24,26], periodic orbits [25], bistability [32]. If $f_i (i=1,2)$ are ratio-dependent type trophic functions (i.e. $f_i(y,z) = \frac{y}{y+m_i z}$ with constant $m_i > 0$), a complete classification of the asymptotic behavior of solutions to (1.1) with the uniqueness of limit cycles was provided in [13]. If $f_i (i=1,2)$ are Beddington-DeAngelis type trophic functions (i.e. $f_i(y,z) = \frac{y}{1+A_i y + B_i z}$ with constants $A_i, B_i > 0$), the chaotic behavior of the model (1.1) was investigated in [30,46]. Recently the food chain model with fear effect was analytically studied in [8,31].

Although the temporal food chain model (1.1) has been extensively studied in the literature for different trophic functions and rich dynamics have been revealed as recalled above, the study of spatial food chain models taking into account the spatial movement of species seems not being touched yet as far as we know. The spatial movement is an indispensable factor for most of species (if not all) to survive and thrive. The goal of this paper is to develop a food chain model with spatial movements and investigate its global dynamics. In the model, apart from the random motions (diffusions), we shall also include the directional movement of predators toward their preys based upon the prey-taxis mechanism (i.e. the predators move upward the prey density gradient). It appears that the Holling type I trophic function has not been considered for (1.1) in literatures. Therefore in the present work, we shall complement this case by assuming the trophic functions are of Holling Type I (i.e. Lotka-Volterra type), that is

$$f_1(u,v) = u, \;\; f_2(v,w) = v. \tag{1.2}$$

Then the model we consider takes the following form

$$\begin{cases} u_t = d_1\Delta u + u(1-u) - b_1 uv, & x \in \Omega, t > 0 \\ v_t = d_2\Delta v - \nabla \cdot (\xi v \nabla u) + uv - b_2 vw - \theta_1 v, & x \in \Omega, t > 0 \\ w_t = \Delta w - \nabla \cdot (\chi w \nabla v) + vw - \theta_2 w, & x \in \Omega, t > 0, \\ \partial_\nu u = \partial_\nu v = \partial_\nu w = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), v(x,0) = v_0(x), w(x,0) = w_0(x), & x \in \Omega, \end{cases} \tag{1.3}$$

where the variables, functions and parameters have the following biological meanings:

- Ω - a bounded domain in $\mathbb{R}^2$ denoting the habitat that the species reside;
- u, v, w - density of the prey, intermediate predator and top predator, respectively;
- d_1, d_2 - diffusion rates of the prey and intermediate predator, respectively;
- ξ, χ - prey-taxis coefficients;
- b_1, b_2 - consumption rates of the prey and the intermediate predator, respectively;
- θ_1, θ_2 - mortality rates of the intermediate and top predators, respectively.

In the above, all parameters are positive. $\partial_\nu = \frac{\partial}{\partial \nu}$ and ν is the outward unit normal vector on $\partial\Omega$ - the boundary of Ω. Here the homogeneous Neumann boundary conditions are imposed on $\partial\Omega$ to ensure that no individuals can cross the boundary, so that the system is closed. The global existence and large-time behavior of solutions to (1.3) in two dimensions will be established in this paper. We remark that from the mathematical point of view, the analysis of the global existence of solutions for the Holling type I trophic functions are more difficult than other types of trophic functions like Holling type II, ratio-dependent or Beddington-DeAngelis type which have priori bounds for any $u, v > 0$, while the Holling type I does not possess such a nice property directly useful for the global posedness of solutions.

Our first result regarding the global existence of classical solutions with uniform-in-time bound is stated below.

Theorem 1.1 *(Global boundedness). Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary. Assume $u_0 \in W^{2,\infty}(\Omega)$ and $(v_0, w_0) \in [W^{1,\infty}(\Omega)]^2$ with $u_0, v_0, w_0 \gneqq 0$. Then the problem (1.3) has a unique global classical solution $(u, v, w) \in [C^0(\bar{\Omega} \times [0,\infty)) \cap C^{2,1}(\bar{\Omega} \times (0,\infty))]^3$ satisfying $u, v, w > 0$ for all $t > 0$. Moreover there exists a constant $C > 0$ independent of t such that*

$$\|u(\cdot,t)\|_{W^{1,\infty}} + \|v(\cdot,t)\|_{W^{1,\infty}} + \|w(\cdot,t)\|_{L^\infty} \le C.$$

Our next results are concerned with the large time behavior of constant steady states of (1.3), denoted by (u_s, v_s, w_s), which satisfy

$$\begin{cases} u_s(1-u_s-b_1 v_s) = 0, \\ v_s(u_s - b_2 w_s - \theta_1) = 0, \\ w_s(v_s - \theta_2) = 0. \end{cases}$$

After some calculations, we can find that

$$
\begin{aligned}
&(u_s, v_s, w_s)\\
&= \begin{cases}
(0,0,0) \text{ or } (1,0,0), & \text{if } \theta_1 \geq 1,\\
(0,0,0) \text{ or } (1,0,0) \text{ or } (\theta_1, \frac{1-\theta_1}{b_1}, 0), & \text{if } \theta_1 < 1 \text{ and } \theta_1 + b_1\theta_2 \geq 1,\\
(0,0,0) \text{ or } (1,0,0) \text{ or } (\theta_1, \frac{1-\theta_1}{b_1}, 0) \text{ or } (u_*, v_*, w_*) & \text{if } \theta_1 < 1 \text{ and } \theta_1 + b_1\theta_2 < 1,
\end{cases}
\end{aligned}
$$

where

$$
(u_*, v_*, w_*) = \left(1 - b_1\theta_2, \theta_2, \frac{1 - b_1\theta_2 - \theta_1}{b_2}\right). \tag{1.4}
$$

We call $(0, 0, 0)$ the extinction steady state, $(1, 0, 0)$ the prey-only steady state, $(\theta_1, \frac{1-\theta_1}{b_1}, 0)$ the semi-coexistence steady state and (u_*, v_*, w_*) the coexistence steady state. We shall prove that the latter three steady states may be globally asymptotically stable under certain conditions. We also remark that the boundedness of $\|v\|_{L^\infty}$ shown in Theorem 1.1 is independent of the prey-taxis coefficient $\chi > 0$ (see Lemma 3.5). Then our stability results are stated in the following theorem.

Theorem 1.2 *(Global stabilization). Assume the conditions in Theorem 1.1 hold. Let (u, v, w) be the solution of (1.3) obtained in Theorem 1.1 and let $K = \max\{1, \|u_0\|_{L^\infty}\}$. Then the following results hold true.*

- *If $\theta_1 > 1$, the steady state $(1, 0, 0)$ is globally asymptotically stable;*
- *If $\theta_1 < 1$ and $\theta_1 + b_1\theta_2 > 1$, the steady state $(\theta_1, \frac{1-\theta_1}{b_1}, 0)$ is globally asymptotically stable provided*

$$
\xi^2 < \frac{4d_1d_2\theta_1}{(1-\theta_1)K^2}.
$$

- *If $\theta_1 < 1$ and $\theta_1 + b_1\theta_2 < 1$, the steady state (u_*, v_*, w_*) defined by (1.4) is globally asymptotically stable provided*

$$
\xi^2 < \frac{4d_1d_2u_*}{b_1K^2v_*} \quad \text{and} \quad \chi^2 < \frac{4d_1d_2u_*v_* - \xi^2 b_1 K^2 v_*^2}{b_2 d_1 u_* w_* \|v\|_{L^\infty}^2}, \tag{1.5}
$$

where $\|v\|_{L^\infty}$ depends on b_1, b_2, θ_1 but is independent of χ.

The spatial food chain model (1.3) essentially uses the prey-taxis mechanism to describe the directed movements of predators toward the prey. It can be regarded as an extension of the two-species predator-prey system with prey-taxis (called the prey-taxis system) originally proposed in [21]. In recent years, the global dynamics of numerous prey-taxis systems have been widely studied (cf. [1,2,6,10,17,19,27,36,38–41,44,45] and references therein). Compared to the various prey-taxis systems studied in these works, the three-species spatial food chain model (1.3) has more complex coupling structures. To derive the L^∞-bound of w, we require a priori bound for $\|\nabla v\|_{L^\infty}$ whose estimate, however, depends on w itself and u. This intertwined estimate was not encountered in existing literatures for the prey-taxis systems where the L^∞-estimates of ∇v is unneeded for global boundedness. In this paper, we shall start with a coupling entropy estimate $\|v \ln v\|_{L^1} + \|\nabla u\|_{L^2}$ for the energy estimates to derive the priori bound of $\|v\|_{L^\infty}$ with the help of semigroup theory. This idea was first developed in [37] for the classical chemotaxis system and then for prey-taxis system (cf. [17]) as well as some other type chemotaxis models [16,18]. To derive the priori bound of $\|\nabla v\|_{L^\infty}$, apart from the above-mentioned entropy estimates, we shall capture the model structure to use some essential estimates derived in [7,15] on the second-order derivative of u (see Lemma 3.6) to carry out delicate energy estimates. However, in order to use the second-order estimate as in [15], we need higher-order regularity of the initial value u_0 up to the second-order derivative (i.e., $u_0 \in W^{2,\infty}(\Omega)$). The proofs of global stability results in Theorem 1.2 are routine based upon the Lyapunov functionals alongside LaSalle's invariance principle. We remark recently an alarm taxis system was proposed in [11] and its global boundedness and asymptotics of solutions in two dimensions was established in [20]. The alarm taxis system studied in [11,20] shares some similar structures as the spatial food chain model (1.3) but has quadratic decay terms (i.e. intra-specific competitions) for v and w. The work [20] fully uses these quadratic decay terms to obtain the local-in-time integrability of L^2-norms of v and w, based on which the global boundedness of solutions was derived. Since the food chain model (1.3) has no quadratic decay terms for v and w, the methods developed in [20] are inapplicable to (1.3).

The paper is organized as follows. In section 2, we state the local existence theorem with some preliminary results. Then in section 3, we conduct delicate energy estimates to derive the global boundedness of solutions and prove Theorem 1.1. In section 4, we use Lyapunov functionals and LaSalle' invariance principle to prove the global stability results stated in Theorem 1.2.

2. Local existence and preliminaries

In what follows, we shall abbreviate $\int_\Omega f dx$ as $\int_\Omega f$ for simplicity without confusion. Moreover, we will use c_i and K_i $(i = 1, 2, \ldots)$ to denote generic positive constants independent of χ and t. The local existence of solutions to (1.3) can be readily proved by the Amann's theorem (see [3,4]). The positivity of solutions can be shown by the strong maximum principle. We omit the details for brevity and state the results below.

Lemma 2.1 *(Local existence). Let the assumptions in Theorem 1.1 hold. Then there exists a constant $T_{max} \in (0, \infty]$ such that the problem (1.3) has a unique classical solution*

$$(u, v, w) \in [C^0(\bar{\Omega} \times [0, T_{max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{max}))]^3$$

satisfying $u, v, w > 0$ for all $t > 0$. Moreover,

$$if \;\; T_{max} < \infty, \; then \; \limsup_{t \nearrow T_{max}} \left(\|u(\cdot, t)\|_{W^{1,\infty}} + \|v(\cdot, t)\|_{W^{1,\infty}} + \|w(\cdot, t)\|_{L^\infty} \right) = \infty.$$

Lemma 2.2. *Let the assumptions in Lemma 2.1 hold. Then the solution of (1.3) satisfies*

$$\|u(\cdot, t)\|_{L^\infty} \le K, \; \text{for all } t \in (0, T_{max}), \tag{2.1}$$

where $K = \max\{1, \|u_0\|_{L^\infty}\}$.

Proof. The inequality of (2.1) is a consequence of [17, Lemma 2.2]. □

Lemma 2.3. *Suppose the assumptions in Lemma 2.1 hold. Then the solution of* (1.3) *satisfies*

$$\|v(\cdot,t)\|_{L^1} + \|w(\cdot,t)\|_{L^1} \le K_1, \quad \text{for all } \ t \in (0, T_{max}), \tag{2.2}$$

where $K_1 > 0$ *is a constant independent of* t, ξ *and* χ.

Proof. Using the equations of (1.3), we can derive that

$$\frac{d}{dt}\int_\Omega (u + b_1 v + b_1 b_2 w) = \int_\Omega u(1-u) - b_1\theta_1 \int_\Omega v - b_1 b_2 \theta_2 \int_\Omega w. \tag{2.3}$$

Young's inequality entails that

$$2\int_\Omega u \le \int_\Omega u^2 + |\Omega|,$$

which substituted into (2.3) gives

$$\frac{d}{dt}\int_\Omega (u + b_1 v + b_1 b_2 w) + \int_\Omega u + b_1\theta_1 \int_\Omega v + b_1 b_2 \theta_2 \int_\Omega w \le |\Omega|. \tag{2.4}$$

Letting $\sigma_1 = \min\{1, \theta_1, \theta_2\}$, from (2.4), we have

$$\frac{d}{dt}\int_\Omega (u + b_1 v + b_1 b_2 w) + \sigma_1 \int_\Omega (u + b_1 v + b_1 b_2 w) \le |\Omega|,$$

which together with Grönwall's inequality gives (2.2). □

The following Lemma can be proved in the same way as in [35, Lemma 3.4].

Lemma 2.4. *Let* $T > 0$, $\tau \in (0, T)$, $a > 0$ *and* $b > 0$. *Suppose that* $y : [0, T) \to [0, \infty)$ *is absolutely continuous and fulfills*

$$y'(t) + ay(t) \le h(t) \quad \textit{for all } t \in (0, T),$$

with some nonnegative function $h \in L^1_{loc}([0, T))$ *satisfying* $\int_t^{t+\tau} h(t) \le b$ *for all* $t \in [0, T - \tau)$. *Then*

$$y(t) \le \max\left\{ y(0) + b, \frac{b}{a\tau} + 2b \right\} \quad \textit{for all } t \in (0, T).$$

Lemma 2.5 *([29]). Let* Ω *be a bounded domain in* $\mathbb{R}^2$ *with smooth boundary and* $f \in W^{1,2}(\Omega)$. *Then for any* $\varepsilon > 0$, *there exists a constant* $C_\varepsilon > 0$ *such that*

$$\|f\|_{L^3}^3 \le \varepsilon \|\nabla f\|_{L^2}^2 \|f \ln|f|\|_{L^1} + C_\varepsilon (\|f\|_{L^1}^2 \|f \ln|f|\|_{L^1} + \|f\|_{L^1}).$$

Lemma 2.6 *([28]). Assume that Ω is a bounded domain, and let $g \in C^2(\bar{\Omega})$ satisfy $\frac{\partial g}{\partial \nu} = 0$ on $\partial\Omega$. Then we have*

$$\frac{\partial |\nabla g|^2}{\partial \nu} \le 2\kappa |\nabla g|^2,$$

where $\kappa = \kappa(\Omega)$ is an upper bound of the curvatures of $\partial\Omega$.

Lemma 2.7. *Let $\phi \in C^2(\bar{\Omega})$ be a positive function satisfying $\frac{\partial \phi}{\partial \nu} = 0$ on $\partial\Omega$. Then there exists a constant $\kappa_1 > 0$ such that*

$$\kappa_1 \left(\int_\Omega \frac{|D^2\phi|^2}{\phi} + \int_\Omega \frac{|\nabla\phi|^4}{\phi^3} \right) \le \int_\Omega \phi |D^2 \ln\phi|^2. \tag{2.5}$$

Proof. Motivated by some ideas in [43], we first show that

$$\int_\Omega \frac{|\nabla\phi|^4}{\phi^3} \le (2+\sqrt{2})^2 \int_\Omega \phi |D^2 \ln\phi|^2. \tag{2.6}$$

In fact, using integration by parts alongside the Neumann boundary condition $\nabla\phi \cdot \nu = 0$ on $\partial\Omega$, Hölder's inequality, and noting the fact $\nabla|\nabla\phi|^2 = 2D^2\phi \cdot \nabla\phi$, we can derive that

$$\begin{aligned}
\int_\Omega \frac{|\nabla\phi|^4}{\phi^3} &= \int_\Omega |\nabla \ln\phi|^2 \nabla \ln\phi \cdot \nabla\phi \\
&= -\int_\Omega \phi \nabla |\nabla \ln\phi|^2 \cdot \nabla \ln\phi - \int_\Omega \phi |\nabla \ln\phi|^2 \Delta \ln\phi \\
&= -2\int_\Omega \frac{(D^2 \ln\phi \cdot \nabla\phi)\cdot\nabla\phi}{\phi} - \int_\Omega \frac{|\nabla\phi|^2 \Delta \ln\phi}{\phi} \\
&\le 2\left(\int_\Omega \phi|D^2\ln\phi|^2\right)^{\frac{1}{2}} \left(\int_\Omega \frac{|\nabla\phi|^4}{\phi^3}\right)^{\frac{1}{2}} + \left(\int_\Omega \phi|\Delta\ln\phi|^2\right)^{\frac{1}{2}} \left(\int_\Omega \frac{|\nabla\phi|^4}{\phi^3}\right)^{\frac{1}{2}} \\
&\le (2+\sqrt{2}) \left(\int_\Omega \phi|D^2\ln\phi|^2\right)^{\frac{1}{2}} \left(\int_\Omega \frac{|\nabla\phi|^4}{\phi^3}\right)^{\frac{1}{2}},
\end{aligned}$$

which gives (2.6).

On the other hand, using the fact $(a-b)^2 \ge \frac{1}{2}a^2 - b^2$ for all $a, b \in \mathbb{R}$, one has

$$\int_\Omega \phi |D^2\ln\phi|^2 = \int_\Omega \phi \sum_{k,l=1}^{2} \left| \frac{1}{\phi} \cdot \frac{\partial^2\phi}{\partial x_k \partial x_l} - \frac{1}{\phi^2} \cdot \frac{\partial\phi}{\partial x_k} \cdot \frac{\partial\phi}{\partial x_l} \right|^2$$

$$\geq \frac{1}{2}\int_\Omega \phi \sum_{k,l=1}^{2}\left|\frac{1}{\phi}\cdot\frac{\partial^2\phi}{\partial x_k\partial x_l}\right|^2 - \int_\Omega \phi\left|\frac{1}{\phi^2}\cdot\frac{\partial\phi}{\partial x_k}\cdot\frac{\partial\phi}{\partial x_l}\right|^2$$
$$= \frac{1}{2}\int_\Omega \frac{|D^2\phi|^2}{\phi} - \int_\Omega \frac{|\nabla\phi|^4}{\phi^3},$$

which, together with (2.6), gives

$$\int_\Omega \frac{|D^2\phi|^2}{\phi} \leq 2\int_\Omega \frac{|\nabla\phi|^4}{\phi^3} + 2\int_\Omega \phi|D^2\ln\phi|^2 \leq (14+8\sqrt{2})\int_\Omega \phi|D^2\ln\phi|^2. \tag{2.7}$$

Then combining (2.6) and (2.7), we obtain

$$\frac{1}{20+12\sqrt{2}}\left(\int_\Omega \frac{|D^2\phi|^2}{\phi} + \int_\Omega \frac{|\nabla\phi|^4}{\phi^3}\right) \leq \int_\Omega \phi|D^2\ln\phi|^2,$$

which gives (2.5) by letting $\kappa_1 := \frac{1}{20+12\sqrt{2}}$. □

Next we collect some well-known smoothing estimates for the Neumann heat semigroup, which will be used later.

Lemma 2.8 *([42]). Let $(e^{t\Delta})_{t\geq 0}$ be the Neumann heat semigroup in Ω, and let $\lambda_1 > 0$ denote the first nonzero eigenvalue of $-\Delta$ in Ω under Neumann boundary conditions. Then for all $t > 0$, there exist some constants γ_i $(i = 1, 2, 3, 4)$ depending only on Ω such that*

(i) *If $2 \leq p < \infty$, then*

$$\|\nabla e^{td\Delta} z\|_{L^p} \leq \gamma_1 e^{-\lambda_1 dt}\|\nabla z\|_{L^p} \tag{2.8}$$

for all $z \in W^{1,p}(\Omega)$.

(ii) *If $1 \leq q \leq p \leq \infty$, then*

$$\|\nabla e^{td\Delta} z\|_{L^p} \leq \gamma_2\left(1 + t^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}\right)e^{-\lambda_1 dt}\|z\|_{L^q} \tag{2.9}$$

for all $z \in L^q(\Omega)$.

(iii) *If $1 \leq q \leq p \leq \infty$, then*

$$\|e^{td\Delta} z\|_{L^p} \leq \gamma_3\left(1 + t^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}\right)\|z\|_{L^q} \tag{2.10}$$

for all $z \in L^q(\Omega)$.

(iv) *If $1 < q \leq p \leq \infty$, then*

$$\|e^{td\Delta}\nabla\cdot z\|_{L^p} \leq \gamma_4\left(1 + t^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}\right)e^{-\lambda_1 dt}\|z\|_{L^q} \tag{2.11}$$

for all $z \in (C_0^\infty(\Omega))^n$.

We note that the result in Lemma 2.8 (iv) also holds true for any $z \in L^q(\Omega)$ with $1 \le q < \infty$ since $C_0^\infty(\Omega)$ is dense in $L^q(\Omega)(1 \le q < \infty)$ (see also [42]).

3. Proof of Theorem 1.1

In this section, we shall establish the boundedness of solution in two dimensional spaces.

3.1. Boundedness of $\|v(\cdot,t)\|_{L^\infty}$

We first establish the boundedness of $\|v(\cdot,t)\|_{L^\infty}$ based on the energy estimates.

Lemma 3.1. *Let (u, v, w) be the solution obtained in Lemma 2.1. Then it holds that*

$$\begin{aligned} &\frac{d}{dt}\left(\frac{1}{2}\int_\Omega \frac{|\nabla u|^2}{u} + \frac{b_1}{\xi}\int_\Omega v\ln v\right) + d_1\int_\Omega u|D^2\ln u|^2 + \frac{b_1 d_2}{\xi}\int_\Omega \frac{|\nabla v|^2}{v} \\ &\le \frac{d_1}{2}\int_{\partial\Omega} \frac{\partial|\nabla u|^2}{\partial\nu}\frac{1}{u}dS + \frac{1}{2}\int_\Omega \frac{|\nabla u|^2}{u} + \frac{b_1(K+\theta_1)}{\xi}\int_\Omega |v\ln v| + \frac{K_1 b_1}{\xi}\left(\frac{b_2}{e}+K\right), \end{aligned} \tag{3.1}$$

for all $t \in (0, T_{max})$, where K and K_1 are presented in (2.1) and (2.2), respectively.

Proof. We multiply the first equation of (1.3) by $-\frac{\Delta u}{u}$ and integrate the resulting equation by parts to obtain

$$\begin{aligned} -\int_\Omega \frac{u_t}{u}\Delta u + d_1\int_\Omega \frac{|\Delta u|^2}{u} &= \int_\Omega (u-1)\Delta u + b_1\int_\Omega v\Delta u \\ &= -\int_\Omega |\nabla u|^2 - b_1\int_\Omega \nabla u\cdot\nabla v, \text{ for all } t \in (0, T_{max}). \end{aligned} \tag{3.2}$$

Noting that

$$\begin{aligned} -\int_\Omega \frac{u_t}{u}\Delta u &= \int_\Omega \nabla u\cdot\nabla(\ln u)_t \\ &= \int_\Omega \nabla u\cdot\left(\frac{\nabla u}{u}\right)_t \\ &= \frac{d}{dt}\int_\Omega \frac{|\nabla u|^2}{u} - \frac{1}{2}\int_\Omega \frac{(|\nabla u|^2)_t}{u} \\ &= \frac{1}{2}\frac{d}{dt}\int_\Omega \frac{|\nabla u|^2}{u} - \frac{1}{2}\int_\Omega \frac{|\nabla u|^2}{u^2}u_t, \quad \text{for all } t \in (0, T_{max}). \end{aligned} \tag{3.3}$$

Then substituting (3.3) into (3.2), for all $t \in (0, T_{max})$ one has

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \frac{|\nabla u|^2}{u} + d_1\int_\Omega \frac{|\Delta u|^2}{u} + \int_\Omega |\nabla u|^2 = \frac{1}{2}\int_\Omega \frac{|\nabla u|^2}{u^2}u_t - b_1\int_\Omega \nabla u\cdot\nabla v. \tag{3.4}$$

Using the first equation of (1.3) again, for all $t \in (0, T_{max})$ we can derive that

$$\begin{aligned}\frac{1}{2}\int_\Omega \frac{|\nabla u|^2}{u^2}u_t &= \frac{d_1}{2}\int_\Omega \frac{|\nabla u|^2}{u^2}\Delta u + \frac{1}{2}\int_\Omega \frac{|\nabla u|^2(1-u)}{u} - \frac{b_1}{2}\int_\Omega \frac{|\nabla u|^2 v}{u}\\ &= \frac{d_1}{2}\int_\Omega \frac{|\nabla u|^2}{u^2}\Delta u + \frac{1}{2}\int_\Omega \frac{|\nabla u|^2}{u} - \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{b_1}{2}\int_\Omega \frac{|\nabla u|^2 v}{u},\end{aligned}$$

which substituted into (3.4) gives

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega \frac{|\nabla u|^2}{u} + d_1\int_\Omega \frac{|\Delta u|^2}{u} + \frac{3}{2}\int_\Omega |\nabla u|^2 + \frac{b_1}{2}\int_\Omega \frac{|\nabla u|^2 v}{u}\\ &= \frac{d_1}{2}\int_\Omega \frac{|\nabla u|^2}{u^2}\Delta u - b_1\int_\Omega \nabla u\cdot\nabla v + \frac{1}{2}\int_\Omega \frac{|\nabla u|^2}{u} \quad \text{for all } t \in (0, T_{max}).\end{aligned} \tag{3.5}$$

Using the integration by parts and the fact that $\nabla u\cdot\nabla\Delta u = \frac{1}{2}\Delta|\nabla u|^2 - |D^2u|^2$, we have

$$\begin{aligned}\int_\Omega \frac{|\nabla u|^2}{u^2}\Delta u &= \int_\Omega \frac{|\Delta u|^2}{u} - \int_\Omega \frac{|D^2u|^2}{u} + \frac{1}{2}\int_\Omega \frac{\Delta|\nabla u|^2}{u}\\ &= \int_\Omega \frac{|\Delta u|^2}{u} - \int_\Omega \frac{|D^2u|^2}{u} + \frac{1}{2}\int_{\partial\Omega} \frac{\partial|\nabla u|^2}{\partial\nu}\frac{1}{u}dS + \frac{1}{2}\int_\Omega \frac{\nabla|\nabla u|^2\cdot\nabla u}{u^2}\\ &= \int_\Omega \frac{|\Delta u|^2}{u} - \int_\Omega \frac{|D^2u|^2}{u} + \frac{1}{2}\int_{\partial\Omega} \frac{\partial|\nabla u|^2}{\partial\nu}\frac{1}{u}dS + \int_\Omega \frac{|\nabla u|^4}{u^3} - \frac{1}{2}\int_\Omega \frac{|\nabla u|^2}{u^2}\Delta u\end{aligned}$$

for all $t \in (0, T_{max})$, which entails that

$$\begin{aligned}\frac{d_1}{2}\int_\Omega \frac{|\nabla u|^2}{u^2}\Delta u =& d_1\int_\Omega \frac{|\Delta u|^2}{u} + \frac{d_1}{2}\int_{\partial\Omega} \frac{\partial|\nabla u|^2}{\partial\nu}\frac{1}{u}dS - d_1\int_\Omega \frac{|D^2u|^2}{u}\\ &+ d_1\int_\Omega \frac{|\nabla u|^4}{u^3} - d_1\int_\Omega \frac{|\nabla u|^2}{u^2}\Delta u \quad \text{for all } t \in (0, T_{max}).\end{aligned} \tag{3.6}$$

On the other hand, using the integration by parts, one has

$$
\begin{aligned}
\int_\Omega u|D^2\ln u|^2 &= \int_\Omega \frac{|D^2u|^2}{u} + \int_\Omega \frac{|\nabla u|^4}{u^3} - 2\int_\Omega \frac{1}{u^2}\left(D^2u\cdot\nabla u\right)\cdot\nabla u \\
&= \int_\Omega \frac{|D^2u|^2}{u} + \int_\Omega \frac{|\nabla u|^4}{u^3} - \int_\Omega \frac{1}{u^2}\nabla(|\nabla u|^2)\cdot\nabla u \\
&= \int_\Omega \frac{|D^2u|^2}{u} + \int_\Omega \frac{|\nabla u|^4}{u^3} + \left(\int_\Omega \frac{|\nabla u|^2}{u^2}\Delta u - 2\int_\Omega \frac{|\nabla u|^4}{u^3}\right) \\
&= \int_\Omega \frac{|D^2u|^2}{u} - \int_\Omega \frac{|\nabla u|^4}{u^3} + \int_\Omega \frac{|\nabla u|^2}{u^2}\Delta u \quad \text{for all } t\in(0,T_{max}).
\end{aligned}
\tag{3.7}
$$

Then the combination of (3.6) and (3.7) gives

$$
\frac{d_1}{2}\int_\Omega \frac{|\nabla u|^2}{u^2}\Delta u = d_1\int_\Omega \frac{|\Delta u|^2}{u} + \frac{d_1}{2}\int_{\partial\Omega}\frac{\partial|\nabla u|^2}{\partial\nu}\frac{1}{u}dS - d_1\int_\Omega u|D^2\ln u|^2 \tag{3.8}
$$

for all $t\in(0,T_{max})$, which substituted into (3.5) gives

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega \frac{|\nabla u|^2}{u} + \frac{3}{2}\int_\Omega |\nabla u|^2 + d_1\int_\Omega u|D^2\ln u|^2 + \frac{b_1}{2}\int_\Omega \frac{|\nabla u|^2 v}{u} \\
&= \frac{d_1}{2}\int_{\partial\Omega}\frac{\partial|\nabla u|^2}{\partial\nu}\frac{1}{u}dS + \frac{1}{2}\int_\Omega \frac{|\nabla u|^2}{u} - b_1\int_\Omega \nabla u\cdot\nabla v \quad \text{for all } t\in(0,T_{max}).
\end{aligned}
\tag{3.9}
$$

Multiplying the second equation of (1.3) by $\frac{b_1(\ln v+1)}{\xi}$ and integrating it by parts, for all $t\in(0,T_{max})$ we obtain

$$
\begin{aligned}
&\frac{b_1}{\xi}\frac{d}{dt}\int_\Omega v\ln v + \frac{b_1 d_2}{\xi}\int_\Omega \frac{|\nabla v|^2}{v} \\
=&b_1\int_\Omega \nabla u\cdot\nabla v + \frac{b_1}{\xi}\int_\Omega uv(\ln v+1) - \frac{b_1\theta_1}{\xi}\int_\Omega v(\ln v+1) - \frac{b_1 b_2}{\xi}\int_\Omega wv(\ln v+1).
\end{aligned}
\tag{3.10}
$$

Then adding (3.9) and (3.10), and using the facts $0<u\le K$, $\|v(\cdot,t)\|_{L^1}\le K_1$, $\|w(\cdot,t)\|_{L^1}\le K_1$ and $v\ln v\ge -\frac{1}{e}$ for all $v\ge 0$, we obtain

$$
\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{2}\int_\Omega \frac{|\nabla u|^2}{u} + \frac{b_1}{\xi}\int_\Omega v\ln v\right) + \frac{3}{2}\int_\Omega |\nabla u|^2 \\
&+ d_1\int_\Omega u|D^2\ln u|^2 + \frac{b_1}{2}\int_\Omega \frac{|\nabla u|^2 v}{u} + \frac{b_1 d_2}{\xi}\int_\Omega \frac{|\nabla v|^2}{v}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{d_1}{2} \int_{\partial\Omega} \frac{\partial |\nabla u|^2}{\partial \nu} \frac{1}{u} dS + \frac{1}{2} \int_\Omega \frac{|\nabla u|^2}{u} + \frac{b_1}{\xi} \int_\Omega uv(\ln v + 1) \\
&\quad - \frac{b_1\theta_1}{\xi} \int_\Omega v(\ln v + 1) - \frac{b_1 b_2}{\xi} \int_\Omega wv(\ln v + 1) \\
&\le \frac{d_1}{2} \int_{\partial\Omega} \frac{\partial |\nabla u|^2}{\partial \nu} \frac{1}{u} dS + \frac{1}{2} \int_\Omega \frac{|\nabla u|^2}{u} + \frac{b_1(K+\theta_1)}{\xi} \int_\Omega |v \ln v| + \frac{K_1 b_1 b_2}{e\xi} + \frac{b_1 K K_1}{\xi}
\end{aligned}
$$

for all $t \in (0, T_{max})$, which gives (3.1). Then we complete the proof of Lemma 3.1. □

Lemma 3.2. *Let the assumptions in Lemma 2.1 hold and (u, v, w) be the solution of (1.3). Then one has*

$$
\|v \ln v(\cdot,t)\|_{L^1} + \|\nabla u(\cdot,t)\|_{L^2} \le K_2, \text{ for all } t \in (0, T_{max}) \tag{3.11}
$$

and

$$
\int_t^{t+\tau} \int_\Omega \frac{|\nabla v|^2}{v} + \int_t^{t+\tau} \int_\Omega |D^2 u|^2 \le K_3 \quad \text{for all } t \in (0, \widetilde{T}_{max}), \tag{3.12}
$$

where K_2 and K_3 are positive constants independent of χ and

$$
\tau := \min\left\{1, \frac{1}{2} T_{max}\right\} \text{ and } \widetilde{T}_{max} := \begin{cases} T_{max} - \tau, & \text{if } T_{max} < \infty, \\ \infty, & \text{if } T_{max} = \infty. \end{cases} \tag{3.13}
$$

Proof. Using Lemma 2.7, one can find a constant $c_1 = d_1 \kappa_1$ such that

$$
d_1 \int_\Omega u |D^2 \ln u|^2 \ge c_1 \left(\int_\Omega \frac{|D^2 u|^2}{u} + \int_\Omega \frac{|\nabla u|^4}{u^3} \right) \text{ for all } t \in (0, T_{max}),
$$

which substituted into (3.1) gives

$$
\begin{aligned}
&\frac{d}{dt} \left(\frac{1}{2} \int_\Omega \frac{|\nabla u|^2}{u} + \frac{b_1}{\xi} \int_\Omega v \ln v \right) + c_1 \int_\Omega \left(\frac{|D^2 u|^2}{u} + \frac{|\nabla u|^4}{u^3} \right) + \frac{b_1 d_2}{\xi} \int_\Omega \frac{|\nabla v|^2}{v} \\
\le &\frac{d_1}{2} \int_{\partial\Omega} \frac{\partial |\nabla u|^2}{\partial \nu} \frac{1}{u} dS + \frac{1}{2} \int_\Omega \frac{|\nabla u|^2}{u} + \frac{b_1(K+\theta_1)}{\xi} \int_\Omega |v \ln v| + \frac{b_1 K_1}{\xi} \left(\frac{b_2}{e} + K \right)
\end{aligned} \tag{3.14}
$$

for all $t \in (0, T_{max})$. Using Lemma 2.6 and the following trace inequality ([34, Remark 52.9])

$$
\|z\|_{L^2(\partial\Omega)} \le \varepsilon \|\nabla z\|_{L^2(\Omega)} + C_\varepsilon \|z\|_{L^2(\Omega)} \text{ for any } \varepsilon > 0, \tag{3.15}
$$

we can derive by the Cauchy-Schwarz inequality that

$$\begin{aligned}\frac{d_1}{2}\int_{\partial\Omega}\frac{\partial|\nabla u|^2}{\partial\nu}\frac{1}{u}dS &\le \kappa d_1\int_{\partial\Omega}\frac{|\nabla u|^2}{u}dS = 4\kappa d_1\|\nabla u^{\frac{1}{2}}\|^2_{L^2(\partial\Omega)}\\ &\le \frac{c_1}{2}\int_\Omega\left(\frac{|D^2u|^2}{u}+\frac{|\nabla u|^4}{u^3}\right)+c_2\int_\Omega\frac{|\nabla u|^2}{u} \quad \text{for all } t\in(0,T_{max}),\end{aligned}$$

which, alongside (3.14), gives

$$\begin{aligned}&\frac{d}{dt}\left(\frac{1}{2}\int_\Omega\frac{|\nabla u|^2}{u}+\frac{b_1}{\xi}\int_\Omega v\ln v\right)+\frac{c_1}{2}\int_\Omega\left(\frac{|D^2u|^2}{u}+\frac{|\nabla u|^4}{u^3}\right)+\frac{b_1d_2}{\xi}\int_\Omega\frac{|\nabla v|^2}{v}\\ &\le\frac{1+2c_2}{2}\int_\Omega\frac{|\nabla u|^2}{u}+\frac{b_1(K+\theta_1)}{\xi}\int_\Omega|v\ln v|+\frac{b_1K_1}{\xi}\left(\frac{b_2}{e}+K\right) \quad \text{for all } t\in(0,T_{max}).\end{aligned}\tag{3.16}$$

Using Young's inequality, Hölder's inequality and the fact $\|u(\cdot,t)\|_{L^\infty}\le K$, one has

$$\begin{aligned}\left(\frac{1}{2}+\frac{1+2c_2}{2}\right)\int_\Omega\frac{|\nabla u|^2}{u} &\le (1+c_2)\left(\int_\Omega\frac{|\nabla u|^4}{u^3}\right)^{\frac{1}{2}}\left(\int_\Omega u\right)^{\frac{1}{2}}\\ &\le\frac{c_1}{4}\int_\Omega\frac{|\nabla u|^4}{u^3}+\frac{(1+c_2)^2K|\Omega|}{c_1} \quad \text{for all } t\in(0,T_{max}).\end{aligned}\tag{3.17}$$

On the other hand, using the Gagliardo-Nirenberg inequality, and the fact $\|v^{\frac{1}{2}}\|_{L^2}=\|v\|^{\frac{1}{2}}_{L^1}\le K_1^{\frac{1}{2}}$ (see Lemma 2.3), we obtain

$$\begin{aligned}\frac{b_1}{\xi}\int_\Omega v\ln v+\frac{b_1(K+\theta_1)}{\xi}\int_\Omega|v\ln v| &\le\frac{b_1(1+K+\theta_1)}{\xi}\int_\Omega|v\ln v|\\ &\le\frac{c_3}{\xi}\|v^{\frac{1}{2}}\|^3_{L^3}+\frac{c_3}{\xi}\\ &\le\frac{c_4}{\xi}(\|\nabla v^{\frac{1}{2}}\|_{L^2}\|v^{\frac{1}{2}}\|^2_{L^2}+\|v^{\frac{1}{2}}\|^3_{L^2})\\ &\le\frac{c_4K_1}{\xi}\|\nabla v^{\frac{1}{2}}\|_{L^2}+\frac{c_4K_1^{\frac{3}{2}}}{\xi}\\ &\le\frac{b_1d_2}{\xi}\|\nabla v^{\frac{1}{2}}\|^2_{L^2}+\frac{c_5}{\xi} \quad \text{for all } t\in(0,T_{max}).\end{aligned}\tag{3.18}$$

Then substituting (3.17) and (3.18) into (3.16), we obtain for all $t\in(0,T_{max})$ that

$$\begin{aligned}&\frac{d}{dt}\left(\frac{1}{2}\int_\Omega \frac{|\nabla u|^2}{u}+\frac{b_1}{\xi}\int_\Omega v\ln v\right)+\left(\frac{1}{2}\int_\Omega \frac{|\nabla u|^2}{u}+\frac{b_1}{\xi}\int_\Omega v\ln v\right)\\&\quad+\frac{c_1}{4}\int_\Omega\left(\frac{|D^2u|^2}{u}+\frac{|\nabla u|^4}{u^3}\right)+\frac{3b_1d_2}{4\xi}\int_\Omega\frac{|\nabla v|^2}{v}\le c_6,\end{aligned}\tag{3.19}$$

with $c_6 := \frac{(1+c_2)^2K|\Omega|}{c_1}+\frac{c_5}{\xi}+\frac{b_1K_1}{\xi}(\frac{b_2}{e}+K)$. Then applying Grönwall's inequality to (3.19), one has

$$\frac{1}{2}\int_\Omega \frac{|\nabla u|^2}{u}+\frac{b_1}{\xi}\int_\Omega v\ln v\le c_7\left(1+\frac{1}{\xi}\right)\quad\text{for all } t\in(0,T_{max})$$

which gives (3.11) by the facts $0<u\le K$ and $v\ln v\ge -\frac{1}{e}$ for all $v\ge 0$.

Finally, with the fact $0<u\le K$, we integrate (3.19) over $(t,t+\tau)$ and use (3.11) to get

$$\frac{c_1}{4}\int_t^{t+\tau}\int_\Omega\left(\frac{|D^2u|^2}{u}+\frac{|\nabla u|^4}{u^3}\right)+\frac{3b_1d_2}{4\xi}\int_t^{t+\tau}\int_\Omega\frac{|\nabla v|^2}{v}\le c_8\left(1+\frac{1}{\xi}\right)\quad\text{for all } t\in(0,T_{max}),$$

which gives (3.12). □

With Lemma 3.2 in hand, we use coupling energy estimates to obtain the following results.

Lemma 3.3. *Let the assumptions in Lemma 2.1 hold and (u,v,w) be the solution of (1.3). Then it holds that*

$$\|v(\cdot,t)\|_{L^2}+\|\nabla u(\cdot,t)\|_{L^4}\le K_4\quad\text{for all } t\in(0,T_{max}),\tag{3.20}$$

where $K_4>0$ is a constant independent of χ and t.

Proof. Differentiating the first equation of (1.3) and multiplying the result by $2\nabla u$, then we can use the identity $\Delta|\nabla u|^2 = 2\nabla u\cdot\nabla\Delta u+2|D^2u|^2$ to obtain

$$\begin{aligned}(|\nabla u|^2)_t &= 2d_1\nabla u\cdot\nabla\Delta u-2b_1\nabla u\cdot\nabla(uv)+2|\nabla u|^2(1-2u)\\&= d_1\Delta|\nabla u|^2-2d_1|D^2u|^2-2b_1\nabla u\cdot\nabla(uv)+2|\nabla u|^2(1-2u)\end{aligned}\tag{3.21}$$

for all $t\in(0,T_{max})$. Then multiplying (3.21) by $2|\nabla u|^2$ and integrating the result by parts, along with the fact $0<u\le K$, we end up with

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega |\nabla u|^4 + 2d_1\int_\Omega |\nabla|\nabla u|^2|^2 + 4d_1\int_\Omega |\nabla u|^2|D^2u|^2 \\
&= 2d_1\int_{\partial\Omega} |\nabla u|^2\frac{\partial|\nabla u|^2}{\partial\nu}dS - 4b_1\int_\Omega |\nabla u|^2\nabla u\cdot\nabla(uv) + 4\int_\Omega |\nabla u|^4(1-2u) \\
&= 2d_1\int_{\partial\Omega} |\nabla u|^2\frac{\partial|\nabla u|^2}{\partial\nu}dS + 4\int_\Omega |\nabla u|^4(1-2u) \\
&\quad + 4b_1\int_\Omega uv\Delta u|\nabla u|^2 + 4b_1\int uv\nabla(|\nabla u|^2)\cdot\nabla u \\
&\le 2d_1\int_{\partial\Omega} |\nabla u|^2\frac{\partial|\nabla u|^2}{\partial\nu}dS + 4\int_\Omega |\nabla u|^4 + 4b_1K\int_\Omega v\left(|\Delta u||\nabla u|^2 + |\nabla|\nabla u|^2||\nabla u|\right)
\end{aligned} \tag{3.22}
$$

for all $t \in (0, T_{max})$. Using Lemma 2.6 and trace inequality (3.15) again, for all $t \in (0, T_{max})$ we can derive

$$
2d_1\int_{\partial\Omega} |\nabla u|^2\frac{\partial|\nabla u|^2}{\partial\nu}dS \le 4\kappa d_1\||\nabla u|^2\|^2_{L^2(\partial\Omega)} \le \frac{d_1}{2}\int_\Omega |\nabla|\nabla u|^2|^2 + c_1\int_\Omega |\nabla u|^4. \tag{3.23}
$$

Furthermore, the Hölder inequality, the Gagliardo-Nirenberg inequality alongside the fact $\||\nabla u|^2\|_{L^1} = \|\nabla u\|^2_{L^2} \le K_2^2$ in (3.11) yields that

$$
\begin{aligned}
(5+c_1)\int_\Omega |\nabla u|^4 = (5+c_1)\||\nabla u|^2\|^2_{L^2} &\le c_2\|\nabla|\nabla u|^2\|_{L^2}\||\nabla u|^2\|_{L^1} + c_2\||\nabla u|^2\|^2_{L^1} \\
&\le c_2K_2^2\|\nabla|\nabla u|^2\|_{L^2} + c_2K_2^4 \\
&\le \frac{d_1}{2}\int_\Omega |\nabla|\nabla u|^2|^2 + c_3 \quad \text{for all } t \in (0, T_{max}).
\end{aligned} \tag{3.24}
$$

Hence, the combination of (3.23) and (3.24) gives

$$
2d_1\int_{\partial\Omega} |\nabla u|^2\frac{\partial|\nabla u|^2}{\partial\nu}dS + 5\int_\Omega |\nabla u|^4 \le d_1\int_\Omega |\nabla|\nabla u|^2|^2 + c_3 \text{ for all } t \in (0, T_{max}). \tag{3.25}
$$

With the facts $|\Delta u| \le \sqrt{2}|D^2u|$ and $\nabla|\nabla u|^2 = 2D^2u\cdot\nabla u$, we use Young's inequality to obtain

$$
\begin{aligned}
&4b_1K\int_\Omega v\left(|\Delta u||\nabla u|^2 + |\nabla|\nabla u|^2||\nabla u|\right) \\
&\le 4b_1K\sqrt{2}\int_\Omega v|\nabla u|^2|D^2u| + 8b_1K\int_\Omega v|\nabla u|^2|D^2u|
\end{aligned} \tag{3.26}
$$

$$
\begin{aligned}
&= 4(\sqrt{2}+2)b_1 K \int_\Omega v|\nabla u|^2|D^2u| \\
&\le 2d_1 \int_\Omega |\nabla u|^2|D^2u|^2 + \frac{2(2+\sqrt{2})^2 b_1^2 K^2}{d_1}\int_\Omega v^2|\nabla u|^2 \text{ for all } t\in(0,T_{max}).
\end{aligned}
$$

Then substituting (3.25) and (3.26) into (3.22), we obtain

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega |\nabla u|^4 + \int_\Omega |\nabla u|^4 + d_1\int_\Omega |\nabla|\nabla u|^2|^2 + 2d_1\int_\Omega |\nabla u|^2|D^2u|^2 \\
&\le \frac{2(2+\sqrt{2})^2 b_1^2 K^2}{d_1}\int_\Omega v^2|\nabla u|^2 + c_3 \text{ for all } t\in(0,T_{max}).
\end{aligned} \tag{3.27}
$$

Multiplying the second equation of (1.3) by v, integrating the result by parts, and applying Young's inequality and the fact $\|u(\cdot,t)\|_{L^\infty}\le K$, we obtain for all $t\in(0,T_{max})$ that

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega v^2 + d_2\int_\Omega |\nabla v|^2 + b_2\int_\Omega v^2 w + \theta_1\int_\Omega v^2 &= \xi\int_\Omega v\nabla u\cdot\nabla v + \int_\Omega uv^2 \\
&\le \frac{d_2}{2}\int_\Omega |\nabla v|^2 + \frac{\xi^2}{2d_2}\int_\Omega v^2|\nabla u|^2 + K\int_\Omega v^2,
\end{aligned}
$$

which gives

$$
\frac{d}{dt}\int_\Omega v^2 + \int_\Omega v^2 + d_2\int_\Omega |\nabla v|^2 \le \frac{\xi^2}{d_2}\int_\Omega v^2|\nabla u|^2 + (2K+1)\int_\Omega v^2 \text{ for all } t\in(0,T_{max}). \tag{3.28}
$$

Letting $y(t) := \int_\Omega v^2 + \int_\Omega |\nabla u|^4$, and combining (3.27) and (3.28), one has

$$
\begin{aligned}
&y'(t) + y(t) + d_2\int_\Omega |\nabla v|^2 + d_1\int_\Omega |\nabla|\nabla u|^2|^2 + 2d_1\int_\Omega |\nabla u|^2|D^2u|^2 \\
&\le \left(\frac{2(2+\sqrt{2})^2 b_1^2 K^2 d_2 + \xi^2 d_1}{d_1 d_2}\right)\int_\Omega v^2|\nabla u|^2 + (2K+1)\int_\Omega v^2 + c_3 \\
&\le \left(\frac{2(2+\sqrt{2})^2 b_1^2 K^2 d_2 + \xi^2 d_1}{d_1 d_2}\right)\left(\int_\Omega v^3\right)^{\frac{2}{3}}\left(\int_\Omega |\nabla u|^6\right)^{\frac{1}{3}} + (2K+1)|\Omega|^{\frac{1}{3}}\left(\int_\Omega v^3\right)^{\frac{2}{3}} + c_3 \\
&\le c_4\|v\|_{L^3}^2(\|\nabla u\|_{L^6}^2 + 1) + c_3 \text{ for all } t\in(0,T_{max}),
\end{aligned} \tag{3.29}
$$

where $c_4 = \frac{2(2+\sqrt{2})^2 b_1^2 K^2 d_2 + \xi^2 d_1}{d_1 d_2} + (2K+1)|\Omega|^{\frac{1}{3}}$.

Using the Gagliardo-Nirenberg inequality and (3.11), we obtain

$$\begin{aligned}\|\nabla u\|_{L^6}^2 = \||\nabla u|^2\|_{L^3} &\le c_5\|\nabla|\nabla u|^2\|_{L^2}^{\frac{2}{3}}\||\nabla u|^2\|_{L^1}^{\frac{1}{3}} + c_5\||\nabla u|^2\|_{L^1} \\ &\le c_5K_2^{\frac{2}{3}}\|\nabla|\nabla u|^2\|_{L^2}^{\frac{2}{3}} + c_5K_2^2 \ \text{ for all } t \in (0, T_{max}).\end{aligned} \tag{3.30}$$

Then by Young's inequality and (3.30), one gets

$$\begin{aligned}c_4\|v\|_{L^3}^2(\|\nabla u\|_{L^6}^2 + 1) &\le c_4c_5K_2^{\frac{2}{3}}\|v\|_{L^3}^2\|\nabla|\nabla u|^2\|_{L^2}^{\frac{2}{3}} + c_4(c_5K_2^2 + 1)\|v\|_{L^3}^2 \\ &\le d_1\|\nabla|\nabla u|^2\|_{L^2}^2 + c_6\|v\|_{L^3}^3 + c_7 \ \text{ for all } t \in (0, T_{max}),\end{aligned} \tag{3.31}$$

where c_6 and c_7 are positive constants independent of χ and t.

On the other hand, using the fact $\|v \ln v\|_{L^1} \le K_2$ (see Lemma 3.2), Lemma 2.3 and Lemma 2.5, we have

$$c_6\|v\|_{L^3}^3 \le d_2\|\nabla v\|_{L^2}^2 + c_8 \ \text{ for all } t \in (0, T_{max}). \tag{3.32}$$

Then substituting (3.31) and (3.32) into (3.29), one has

$$y'(t) + y(t) \le c_3 + c_7 + c_8 \ \text{ for all } t \in (0, T_{max}),$$

which together with Grönwall's inequality gives (3.20) and completes the proof of Lemma 3.3. □

Lemma 3.4. *Let the assumptions in Lemma 2.1 hold and (u, v, w) be the solution of (1.3). Then there exists a constant $K_5 > 0$ independent of χ and t such that*

$$\|v(\cdot, t)\|_{L^3} \le K_5 \quad \text{for all } t \in (0, T_{max}). \tag{3.33}$$

Proof. We multiply the second equation of (1.3) by v^2, and integrate the result over Ω. Then with the facts (2.1) and (3.20), we obtain

$$\begin{aligned}&\frac{1}{3}\frac{d}{dt}\int_\Omega v^3 + 2d_2\int_\Omega v|\nabla v|^2 + \theta_1\int_\Omega v^3 \\ &= 2\xi\int_\Omega v^2\nabla u\cdot\nabla v + \int_\Omega uv^3 - b_2\int_\Omega v^3 w \\ &\le d_2\int_\Omega v|\nabla v|^2 + \frac{\xi^2}{d_2}\int_\Omega v^3|\nabla u|^2 + K\int_\Omega v^3 \\ &\le d_2\int_\Omega v|\nabla v|^2 + \frac{\xi^2}{d_2}\left(\int_\Omega v^6\right)^{\frac{1}{2}}\left(\int_\Omega |\nabla u|^4\right)^{\frac{1}{2}} + K\int_\Omega v^3\end{aligned}$$

$$\leq d_2 \int_\Omega v|\nabla v|^2 + \frac{K_4^2 \xi^2}{d_2} \|v\|_{L^6}^3 + K\|v\|_{L^3}^3 \text{ for all } t \in (0, T_{max}),$$

which entails that

$$\begin{aligned}\frac{d}{dt}\|v\|_{L^3}^3 + \frac{4d_2}{3}\|\nabla v^{\frac{3}{2}}\|_{L^2}^2 + 3\theta_1\|v\|_{L^3}^3 &\leq \frac{3K_4^2\xi^2}{d_2}\|v\|_{L^6}^3 + 3K\|v\|_{L^3}^3 \\ &\leq \left(\frac{3K_4^2\xi^2}{d_2} + |\Omega|^{\frac{1}{2}}\right)\|v\|_{L^6}^3 \text{ for all } t \in (0, T_{max}).\end{aligned} \tag{3.34}$$

Noting the fact $\|v^{\frac{3}{2}}(\cdot,t)\|_{L^{\frac{4}{3}}} = \|v(\cdot,t)\|_{L^2}^{\frac{3}{2}} \leq K_4^{\frac{3}{2}}$, and applying the Gagliardo-Nirenberg inequality and Young's inequality, we derive that

$$\begin{aligned}\left(\frac{3K_4^2\xi^2}{d_2} + |\Omega|^{\frac{1}{2}}\right)\|v\|_{L^6}^3 &= \left(\frac{3K_4^2\xi^2}{d_2} + |\Omega|^{\frac{1}{2}}\right)\|v^{\frac{3}{2}}\|_{L^4}^2 \\ &\leq c_1\left(\|\nabla v^{\frac{3}{2}}\|_{L^2}^{\frac{4}{3}}\|v^{\frac{3}{2}}\|_{L^{\frac{4}{3}}}^{\frac{2}{3}} + \|v^{\frac{3}{2}}\|_{L^{\frac{4}{3}}}^2\right) \\ &\leq c_2\|\nabla v^{\frac{3}{2}}\|_{L^2}^{\frac{4}{3}} + c_3 \\ &\leq \frac{4d_2}{3}\|\nabla v^{\frac{3}{2}}\|_{L^2}^2 + c_4 \text{ for all } t \in (0, T_{max}).\end{aligned} \tag{3.35}$$

Then substituting (3.35) into (3.34), we obtain

$$\frac{d}{dt}\|v\|_{L^3}^3 + 3\theta_1\|v\|_{L^3}^3 \leq c_4 \text{ for all } t \in (0, T_{max}),$$

which gives (3.33) by Grönwall's inequality. □

Lemma 3.5. *Let the assumptions in Lemma 2.1 hold and (u, v, w) be the solution of (1.3). Then it holds that*

$$\|v(\cdot,t)\|_{L^\infty} \leq K_6 \quad \text{for all } t \in (0, T_{max}), \tag{3.36}$$

where K_6 is a positive constant independent of χ and t.

Proof. Applying the variation-of-constants formula to the first equation of (1.3), we have

$$u(\cdot,t) = e^{d_1 t\Delta}u_0 + \int_0^t e^{d_1(t-s)\Delta}[u(1-u) - b_1 uv]ds \text{ for all } t \in (0, T_{max}),$$

and hence

$$\nabla u(\cdot,t)=\nabla e^{d_1t\Delta}u_0+\int_0^t \nabla e^{d_1(t-s)\Delta}[u(1-u)-b_1uv]ds \ \text{ for all } t\in(0,T_{max}). \tag{3.37}$$

Then noting the facts $0<u\le K$ and $\|v(\cdot,t)\|_{L^3}\le K_5$, one has for all $t\in(0,T_{max})$

$$\|u(1-u)-b_1uv\|_{L^3}\le \|u(1-u)\|_{L^3}+b_1\|uv\|_{L^3}\le K(1+K)|\Omega|^{\frac{1}{3}}+b_1KK_5. \tag{3.38}$$

Applying the semigroup estimates (2.9) and using (3.38), from (3.37) we have

$$\begin{aligned}\|\nabla u(\cdot,t)\|_{L^\infty}&\le \|\nabla e^{d_1t\Delta}u_0\|_{L^\infty}+\int_0^t \|\nabla e^{d_1(t-s)\Delta}[u(1-u)-b_1uv]\|_{L^\infty}ds\\
&\le 2\gamma_2e^{-d_1\lambda_1t}\|u_0\|_{L^\infty}+\gamma_2\int_0^t\left(1+(t-s)^{-\frac12-\frac13}\right)e^{-d_1\lambda_1t}\|u(1-u)-b_1uv\|_{L^3}ds\\
&\le 2\gamma_2\|u_0\|_{L^\infty}+\gamma_2[K(1+K)|\Omega|^{\frac13}+b_1KK_5]\int_0^\infty\left(1+(t-s)^{-\frac56}\right)e^{-d_1\lambda_1t}ds\\
&\le 2\gamma_2\|u_0\|_{L^\infty}+\frac{\gamma_2}{d_1\lambda_1}[K(1+K)|\Omega|^{\frac13}+b_1KK_5](1+(d_1\lambda_1)^{\frac56}\Gamma(1/6))\end{aligned}$$

for all $t\in(0,T_{max})$, which yields

$$\|\nabla u(\cdot,t)\|_{L^\infty}\le c_1 \ \text{ for all } t\in(0,T_{max}), \tag{3.39}$$

where $c_1:=2\gamma_2\|u_0\|_{L^\infty}+\frac{\gamma_2}{d_1\lambda_1}[K(1+K)|\Omega|^{\frac13}+b_1KK_5](1+(d_1\lambda_1)^{\frac56}\Gamma(1/6))$, and Γ denotes the Gamma function defined by $\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}dt$.

Then using (3.39) and the facts $\|v\|_{L^3}\le K_5$ and $\|u\|_{L^\infty}\le K$ again, we have

$$\|v\nabla u\|_{L^3}+\|uv\|_{L^3}\le(\|\nabla u\|_{L^\infty}+\|u\|_{L^\infty})\|v\|_{L^3}\le(c_1+K)K_5 \ \text{ for all } t\in(0,T_{max}). \tag{3.40}$$

We rewrite the second equation of (1.3) as follows:

$$v_t-d_2\Delta v+\theta_1v=-\xi\nabla\cdot(v\nabla u)+uv-b_2vw. \tag{3.41}$$

Then applying the variation-of-constants formula to (3.41), for all $t\in(0,T_{max})$ one has

$$\begin{aligned}v(\cdot,t)=&e^{t(d_2\Delta-\theta_1)}v_0-\xi\int_0^t e^{(t-s)(d_2\Delta-\theta_1)}\nabla\cdot(v\nabla u)ds+\int_0^t e^{(t-s)(d_2\Delta-\theta_1)}(uv-b_2vw)ds\\
\le&e^{t(d_2\Delta-\theta_1)}v_0-\xi\int_0^t e^{(t-s)(d_2\Delta-\theta_1)}\nabla\cdot(v\nabla u)ds+\int_0^t e^{(t-s)(d_2\Delta-\theta_1)}uvds.\end{aligned} \tag{3.42}$$

Applying the L^p-L^q estimate (2.10), (2.11) and using (3.40), from (3.42) we derive

$$\begin{aligned}\|v(\cdot,t)\|_{L^\infty} \le& \|e^{t(d_2\Delta-\theta_1)}v_0\|_{L^\infty} + \xi\int_0^t \|e^{(t-s)(d_2\Delta-\theta_1)}\nabla\cdot(v\nabla u)\|_{L^\infty}ds \\ &+ \int_0^t \|e^{(t-s)(d_2\Delta-\theta_1)}uv\|_{L^\infty}ds \\ \le& \|v_0\|_{L^\infty} + \gamma_4\xi\int_0^t (1+(t-s)^{-\frac{5}{6}})e^{-(\lambda_1 d_2+\theta_1)(t-s)}\|v\nabla u\|_{L^3}ds \\ &+ \gamma_3\int_0^t (1+(t-s)^{-\frac{1}{3}})e^{-\theta_1(t-s)}\|uv\|_{L^3}ds \\ \le& \|v_0\|_{L^\infty} + \gamma_4\xi(c_1+K)K_5\int_0^\infty (1+(t-s)^{-\frac{5}{6}})e^{-\theta_1(t-s)}ds \\ &+ \gamma_3(c_1+K)K_5\int_0^\infty (1+(t-s)^{-\frac{1}{3}})e^{-\theta_1(t-s)}ds \\ \le& \|v_0\|_{L^\infty} + (c_1+K)K_5\left[\frac{\gamma_4}{\theta_1}\xi(1+\theta_1^{\frac{5}{6}}\Gamma(1/6)) + \frac{\gamma_3}{\theta_1}(1+\theta_1^{\frac{1}{3}}\Gamma(2/3))\right]\end{aligned}$$

for all $t \in (0, T_{max})$, which gives (3.36). This completes the proof of Lemma 3.5. □

3.2. Boundedness of $\|w(\cdot,t)\|_{L^\infty}$

In this subsection, we shall establish the boundedness of $\|w(\cdot,t)\|_{L^\infty}$. In fact, based on some ideas in [15, Lemma 2.4] and [7, Lemma 2.5], we can obtain the following regularity results for u directly.

Lemma 3.6. *Let the assumptions in Lemma 2.1 hold and (u, v, w) be the solution of the system (1.3). Then there exists a constant $K_7 > 0$ independent of t such that and for all $p > 1$*

$$\int_t^{t+\tau} \|D^2 u\|_{L^p}^p \le K_7 \quad \text{for all } t \in (0, \widetilde{T}_{max}) \tag{3.43}$$

and

$$\int_\tau^t e^{-p(t-s)}\|\Delta u\|_{L^p}^p \le K_7 \quad \text{for all } t \in (\tau, T_{max}), \tag{3.44}$$

where τ and $\widetilde{T}_{max}$ are defined in (3.13).

Proof. We rewrite the first equation of (1.3) as follows

$$u_t - d_1\Delta u + u = F(x,t)$$

with $F(x,t) := u(2-u) - b_1 uv$. By Lemma 2.2 and Lemma 3.5, we see that $\|F(x,t)\|_{L^\infty}$ is uniformly bounded in time. With $u_0 \in W^{2,\infty}(\Omega)$, (3.43) follows from [15, Lemma 2.4] directly. Moreover $u(x,\tau) \in C^2(\bar{\Omega})$ for any $\tau \in (0, T_{max})$ due to the local existence results in Lemma 2.1. Then (3.44) is a consequence of the maximal Sobolev regularity property (see [7, Lemma 2.5]). □

Lemma 3.7. *Let the assumptions in Lemma 2.1 hold and (u, v, w) be the solution of (1.3). Then it holds that*

$$\|w \ln w(\cdot,t)\|_{L^1} \le K_8 \quad \text{for all } t \in (0, T_{max}), \tag{3.45}$$

where $K_8 > 0$ is a constant independent of t.

Proof. We multiply the second equation of (1.3) by $-\frac{\Delta v}{v}$, and integrate it by parts to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega \frac{|\nabla v|^2}{v} + d_2\int_\Omega \frac{|\Delta v|^2}{v} \\
&= \frac{1}{2}\int_\Omega \frac{|\nabla v|^2}{v^2}v_t + \xi\int_\Omega \nabla\cdot(v\nabla u)\frac{\Delta v}{v} - \int_\Omega u\Delta v - b_2\int_\Omega \nabla v\cdot\nabla w \\
&= \frac{1}{2}\int_\Omega \frac{|\nabla v|^2}{v^2}v_t + \xi\int_\Omega \frac{\nabla v\cdot\nabla u\Delta v}{v} + \xi\int_\Omega \Delta u\cdot\Delta v - \int_\Omega u\Delta v - b_2\int_\Omega \nabla v\cdot\nabla w
\end{aligned} \tag{3.46}$$

for all $t \in (0, T_{max})$. Using the second equation of (1.3) again, for all $t \in (0, T_{max})$ we have

$$\begin{aligned}
\frac{1}{2}\int_\Omega \frac{|\nabla v|^2}{v^2}v_t =& \frac{d_2}{2}\int_\Omega \frac{|\nabla v|^2}{v^2}\Delta v - \frac{\xi}{2}\int_\Omega \frac{|\nabla v|^2}{v^2}\nabla\cdot(v\nabla u) + \frac{1}{2}\int_\Omega \frac{|\nabla v|^2 u}{v} \\
&- \frac{b_2}{2}\int_\Omega \frac{|\nabla v|^2 w}{v} - \frac{\theta_1}{2}\int_\Omega \frac{|\nabla v|^2}{v} \\
=& \frac{d_2}{2}\int_\Omega \frac{|\nabla v|^2}{v^2}\Delta v - \frac{\xi}{2}\int_\Omega \frac{|\nabla v|^2\nabla v\cdot\nabla u}{v^2} - \frac{\xi}{2}\int_\Omega \frac{|\nabla v|^2\Delta u}{v} \\
&+ \frac{1}{2}\int_\Omega \frac{|\nabla v|^2 u}{v} - \frac{b_2}{2}\int_\Omega \frac{|\nabla v|^2 w}{v} - \frac{\theta_1}{2}\int_\Omega \frac{|\nabla v|^2}{v}.
\end{aligned} \tag{3.47}$$

Noting that $\nabla v\cdot\nu = 0$ on $\partial\Omega$, and using the similar arguments as deriving (3.6), (3.7) and (3.8), we can derive for all $t \in (0, T_{max})$ that

$$\frac{d_2}{2}\int_\Omega \frac{|\nabla v|^2}{v^2}\Delta v = d_2\int_\Omega \frac{|\Delta v|^2}{v} + \frac{d_2}{2}\int_{\partial\Omega} \frac{\partial|\nabla v|^2}{\partial \nu}\frac{1}{v}dS - d_2\int_\Omega v|D^2\ln v|^2. \tag{3.48}$$

Moreover, using Lemma 2.7, we can find a constant $c_1 = \kappa_1 d_2 > 0$ such that

$$c_1\left(\int_\Omega \frac{|D^2 v|^2}{v} + \int_\Omega \frac{|\nabla v|^4}{v^3}\right) \le d_2\int_\Omega v|D^2\ln v|^2 \text{ for all } t\in(0,T_{max}). \tag{3.49}$$

Then substituting (3.47), (3.48) and (3.49) into (3.46), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega \frac{|\nabla v|^2}{v} + c_1\left(\int_\Omega \frac{|D^2 v|^2}{v} + \int_\Omega \frac{|\nabla v|^4}{v^3}\right) + \frac{b_2}{2}\int_\Omega \frac{|\nabla v|^2 w}{v} + \frac{\theta_1}{2}\int_\Omega \frac{|\nabla v|^2}{v}\\
\le& \frac{d_2}{2}\int_{\partial\Omega} \frac{\partial|\nabla v|^2}{\partial\nu}\frac{1}{v}dS - \frac{\xi}{2}\int_\Omega \frac{|\nabla v|^2\nabla v\cdot\nabla u}{v^2} - \frac{\xi}{2}\int_\Omega \frac{|\nabla v|^2\Delta u}{v} + \frac{1}{2}\int_\Omega \frac{|\nabla v|^2 u}{v}\\
&+\xi\int_\Omega \frac{\nabla v\cdot\nabla u\Delta v}{v} + \xi\int_\Omega \Delta u\cdot\Delta v - \int_\Omega u\Delta v - b_2\int_\Omega \nabla v\cdot\nabla w \text{ for all } t\in(0,T_{max}).
\end{aligned} \tag{3.50}$$

Multiplying the third equation of (1.3) by $\frac{b_2}{\chi}(\ln w + 1)$, and integrating it by parts, for all $t \in (0, T_{max})$ we obtain

$$\frac{b_2}{\chi}\frac{d}{dt}\int_\Omega w\ln w + \frac{b_2}{\chi}\int_\Omega \frac{|\nabla w|^2}{w} = b_2\int_\Omega \nabla v\cdot\nabla w + \frac{b_2}{\chi}\int_\Omega vw(\ln w+1) - \frac{b_2\theta_2}{\chi}\int_\Omega w(\ln w+1). \tag{3.51}$$

Then adding (3.51) with (3.50), and using the facts $\|w(\cdot,t)\|_{L^1} \le K_1$ (see (2.2)), $\|u\|_{L^\infty} + \|\nabla u\|_{L^\infty} + \|v\|_{L^\infty} \le c_2$ (see (2.1), (3.36) and (3.39)), we can derive that

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{b_2}{\chi}\int_\Omega w\ln w + \frac{1}{2}\int_\Omega \frac{|\nabla v|^2}{v}\right) + \left(\frac{b_2}{\chi}\int_\Omega w\ln w + \frac{1}{2}\int_\Omega \frac{|\nabla v|^2}{v}\right)\\
&+c_1\left(\int_\Omega \frac{|D^2 v|^2}{v} + \int_\Omega \frac{|\nabla v|^4}{v^3}\right) + \frac{b_2}{\chi}\int_\Omega \frac{|\nabla w|^2}{w}\\
\le& \frac{d_2}{2}\int_{\partial\Omega} \frac{\partial|\nabla v|^2}{\partial\nu}\frac{1}{v}dS + \frac{\xi(c_2+1)}{2}\int_\Omega \frac{|\nabla v|^3}{v^2} + \frac{\xi}{2}\int_\Omega \frac{|\nabla v|^2|\Delta u|}{v} + \frac{c_2+1}{2}\int_\Omega \frac{|\nabla v|^2}{v}\\
&+\xi c_2\int_\Omega \frac{|\nabla v||\Delta v|}{v} + \xi\int_\Omega |\Delta u||\Delta v| + c_2\int_\Omega |\Delta v|\\
&+\frac{b_2(c_2+1+\theta_2)}{\chi}\int_\Omega |w\ln w| + \frac{b_2 c_2 K_1}{\chi} \text{ for all } t\in(0,T_{max}).
\end{aligned} \tag{3.52}$$

Using Lemma 2.6 and the trace inequality (3.15) as well as Young's inequality, for all $t \in (0, T_{max})$ one can derive that

$$\begin{aligned} \frac{d_2}{2}\int_{\partial\Omega}\frac{\partial|\nabla v|^2}{\partial\nu}\frac{1}{v}dS &\le \kappa d_2\int_{\partial\Omega}\frac{|\nabla v|^2}{v}dS = 4\kappa d_2\|\nabla v^{\frac12}\|^2_{L^2(\partial\Omega)} \\ &\le \frac{c_1}{8}\int_\Omega\left(\frac{|D^2v|^2}{v}+\frac{|\nabla v|^4}{v^3}\right)+c_3\int_\Omega\frac{|\nabla v|^2}{v} \\ &\le \frac{c_1}{6}\int_\Omega\left(\frac{|D^2v|^2}{v}+\frac{|\nabla v|^4}{v^3}\right)+c_4. \end{aligned} \tag{3.53}$$

Moreover, we can use Young's inequality and the boundedness of $\|v\|_{L^\infty}$ to derive that

$$\frac{\xi c_2}{2}\int_\Omega\frac{|\nabla v|^3}{v^2}\le\frac{c_1}{8}\int_\Omega\frac{|\nabla v|^4}{v^3}+c_5\int_\Omega\frac{|\nabla v|^2}{v}\le\frac{c_1}{6}\int_\Omega\frac{|\nabla v|^4}{v^3}+c_6 \text{ for all } t\in(0,T_{max})$$

and

$$\begin{aligned} &\frac{\xi}{2}\int_\Omega\frac{|\nabla v|^2|\Delta u|}{v}+\frac{c_2+1}{2}\int_\Omega\frac{|\nabla v|^2}{v}+\xi c_2\int_\Omega\frac{|\nabla v||\Delta v|}{v}+\xi\int_\Omega|\Delta u||\Delta v|+c_2\int_\Omega|\Delta v| \\ &\le\frac{c_1}{6}\int_\Omega\left(\frac{|D^2v|^2}{v}+\frac{|\nabla v|^4}{v^3}\right)+c_7\int_\Omega\left(v|\Delta u|^2+v\right) \\ &\le\frac{c_1}{6}\int_\Omega\left(\frac{|D^2v|^2}{v}+\frac{|\nabla v|^4}{v^3}\right)+c_8\int_\Omega|D^2u|^2+c_9 \text{ for all } t\in(0,T_{max}). \end{aligned}$$

At last, we can use the Gagliardo-Nirenberg inequality to derive that

$$\frac{b_2(c_2+1+\theta_2)}{\chi}\int_\Omega|w\ln w|\le c_{10}\int_\Omega w^{\frac32}+c_{11}\le\frac{b_2}{\chi}\int_\Omega\frac{|\nabla w|^2}{w}+c_{12} \text{ for all } t\in(0,T_{max}). \tag{3.54}$$

Then substituting (3.53)-(3.54) into (3.52), for all $t\in(0,T_{max})$ one has

$$\frac{d}{dt}\left(\frac{b_2}{\chi}\int_\Omega w\ln w+\frac12\int_\Omega\frac{|\nabla v|^2}{v}\right)+\left(\frac{b_2}{\chi}\int_\Omega w\ln w+\frac12\int_\Omega\frac{|\nabla v|^2}{v}\right)\le c_8\int_\Omega|D^2u|^2+c_{13},$$

which together with (3.12) and Lemma 2.4 gives

$$\frac{b_2}{\chi}\int_\Omega w\ln w+\frac12\int_\Omega\frac{|\nabla v|^2}{v}\le c_{14} \text{ for all } t\in(0,T_{max}),$$

and hence

$$\int_\Omega w \ln w \le c_{15} \ \text{ for all } t \in (0, T_{max}),$$

which gives (3.45) by noting the fact $w \ln w \ge -\frac{1}{e}$ for all $w \ge 0$. □

Next, we shall establish the boundedness of $\|w(\cdot,t)\|_{L^2}$ by studying the coupled energy estimate $\int_\Omega (w^2 + |\nabla v|^4)$.

Lemma 3.8. *Let the assumptions in Lemma 2.1 hold and (u, v, w) be the solution of (1.3). Then there is a constant $K_9 > 0$ such that*

$$\|w(\cdot,t)\|_{L^2} + \|\nabla v(\cdot,t)\|_{L^4} \le K_9, \quad \text{for all } t \in (0, T_{max}). \tag{3.55}$$

Proof. Multiplying the third equation of (1.3) by w, and integrating the result by parts, along with (3.36), we end up with

$$\frac{1}{2}\frac{d}{dt}\int_\Omega w^2 + \int_\Omega |\nabla w|^2 + \theta_2 \int_\Omega w^2 = \chi \int_\Omega w \nabla v \cdot \nabla w + \int_\Omega v w^2 \le \frac{1}{2}\int_\Omega |\nabla w|^2 + \frac{\chi^2}{2}\int_\Omega w^2 |\nabla v|^2 + K_6 \int_\Omega w^2 \ \text{ for all } t \in (0, T_{max}),$$

which gives

$$\frac{d}{dt}\int_\Omega w^2 + \int_\Omega |\nabla w|^2 + \int_\Omega w^2 \le \chi^2 \int_\Omega w^2 |\nabla v|^2 + (2K_6 + 1)\int_\Omega w^2 \ \text{ for all } t \in (0, T_{max}). \tag{3.56}$$

Noting the fact (3.45), (2.2) and using Lemma 2.5, one has $(2K_6 + 1)\int_\Omega w^2 \le \frac{1}{2}\int_\Omega |\nabla w|^2 + c_1$, which along with (3.56) gives

$$\frac{d}{dt}\int_\Omega w^2 + \frac{1}{2}\int_\Omega |\nabla w|^2 + \int_\Omega w^2 \le \chi^2 \int_\Omega w^2 |\nabla v|^2 + c_1 \ \text{ for all } t \in (0, T_{max}). \tag{3.57}$$

On the other hand, using the second equation of (1.3) and the fact $\nabla \Delta v \cdot \nabla v = \frac{1}{2}\Delta |\nabla v|^2 - |D^2 v|^2$, we obtain

$$\frac{1}{4}\frac{d}{dt}\int_\Omega |\nabla v|^4 = \int_\Omega |\nabla v|^2 \nabla v \cdot \nabla v_t$$
$$= d_2 \int_\Omega |\nabla v|^2 \nabla v \cdot \nabla \Delta v - \xi \int_\Omega |\nabla v|^2 \nabla v \cdot \nabla(\nabla \cdot (v \nabla u)) + \int_\Omega \nabla(uv - b_2 vw - \theta_1 v) \cdot \nabla v |\nabla v|^2$$

$$= \frac{d_2}{2}\int_{\partial\Omega} |\nabla v|^2 \frac{\partial |\nabla v|^2}{\partial \nu} dS - \frac{d_2}{2}\int_\Omega |\nabla|\nabla v|^2|^2 - d_2\int_\Omega |\nabla v|^2 |D^2 v|^2$$
$$- \xi\int_\Omega |\nabla v|^2 \nabla v\cdot\nabla(\nabla\cdot(v\nabla u)) - \int_\Omega (b_2 w+\theta_1)|\nabla v|^4 + \int_\Omega u|\nabla v|^4$$
$$+ \int_\Omega v(\nabla u - b_2\nabla w)\cdot\nabla v|\nabla v|^2 \text{ for all } t\in(0,T_{max}),$$

which, together with (3.36) and (3.39), gives

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega |\nabla v|^4 + 2d_2\int_\Omega |\nabla|\nabla v|^2|^2 + 4d_2\int_\Omega |\nabla v|^2|D^2v|^2 + 4\int_\Omega (b_2w+\theta_1)|\nabla v|^4\\
=&2d_2\int_{\partial\Omega} |\nabla v|^2\frac{\partial|\nabla v|^2}{\partial\nu}dS - 4\xi\int_\Omega |\nabla v|^2\nabla v\cdot\nabla(\nabla\cdot(v\nabla u))\\
&+4\int_\Omega v(\nabla u - b_2\nabla w)\cdot\nabla v|\nabla v|^2 + 4\int_\Omega u|\nabla v|^4\\
\le&2d_2\int_{\partial\Omega} |\nabla v|^2\frac{\partial|\nabla v|^2}{\partial\nu}dS - 4\xi\int_\Omega |\nabla v|^2\nabla v\cdot\nabla(\nabla\cdot(v\nabla u))\\
&+c_1\int_\Omega |\nabla w||\nabla v|^3 + c_1\int_\Omega |\nabla v|^3 + 4K\int_\Omega |\nabla v|^4 \text{ for all } t\in(0,T_{max}).
\end{aligned} \tag{3.58}$$

We can apply Lemma 2.6 and the trace inequality (3.15) to obtain

$$2d_2\int_{\partial\Omega} |\nabla v|^2\frac{\partial|\nabla v|^2}{\partial\nu}dS \le 4\kappa d_2\int_{\partial\Omega}|\nabla v|^4 dS \le \frac{d_2}{2}\int_\Omega |\nabla|\nabla v|^2|^2 + c_2\int_\Omega |\nabla v|^4 \tag{3.59}$$

for all $t\in(0,T_{max})$. Applying the boundedness of $\|u(\cdot,t)\|_{L^\infty}$, $\|v(\cdot,t)\|_{L^\infty}$ and $\|\nabla u(\cdot,t)\|_{L^\infty}$, Young's inequality and Hölder's inequality, for all $t\in(0,T_{max})$ we can estimate the second term on the right hand of (3.58) as follows

$$\begin{aligned}
&-4\xi\int_\Omega |\nabla v|^2\nabla v\cdot\nabla(\nabla\cdot(v\nabla u))\\
&=4\xi\int_\Omega \nabla|\nabla v|^2\cdot\nabla v\nabla\cdot(v\nabla u) + 4\xi\int_\Omega |\nabla v|^2\Delta v\nabla\cdot(v\nabla u)\\
&\le c_3\int_\Omega |\nabla v||\nabla|\nabla v|^2|(|\nabla v|+|\Delta u|) + c_3\int_\Omega |\nabla v|^2|D^2v|(|\nabla v|+|\Delta u|)
\end{aligned} \tag{3.60}$$

$$\leq d_2 \int_\Omega |\nabla|\nabla v|^2|^2 + 2d_2 \int_\Omega |\nabla v|^2|D^2 v|^2 + \frac{c_3^2}{d_2} \int_\Omega |\nabla v|^4 + \frac{c_3^2}{d_2} \int_\Omega |\nabla v|^2|\Delta u|^2$$

$$\leq d_2 \int_\Omega |\nabla|\nabla v|^2|^2 + 2d_2 \int_\Omega |\nabla v|^2|D^2 v|^2 + c_4 \int_\Omega |\nabla v|^4 + c_4 \int_\Omega |D^2 u|^4.$$

Substituting (3.59) and (3.60) into (3.58), and using Young's inequality, we end up with

$$\begin{aligned}
&\frac{d}{dt} \int_\Omega |\nabla v|^4 + \frac{d_2}{2} \int_\Omega |\nabla|\nabla v|^2|^2 + 2d_2 \int_\Omega |\nabla v|^2|D^2 v|^2 \\
&\leq (4K + c_2 + c_4) \int_\Omega |\nabla v|^4 + c_1 \int_\Omega |\nabla v|^3 + c_4 \int_\Omega |D^2 u|^4 + c_1 \int_\Omega |\nabla v|^3|\nabla w| \qquad (3.61)\\
&\leq c_5 \int_\Omega |\nabla v|^4 + c_4 \int_\Omega |D^2 u|^4 + c_1 \int_\Omega |\nabla v|^3|\nabla w| + c_6 \ \text{ for all } t \in (0, T_{max}).
\end{aligned}$$

Noting $\|v(\cdot,t)\|_{L^\infty} \leq K_6$, integrating by parts and using Young's inequality, one has

$$\begin{aligned}
(c_5+2)\int_\Omega |\nabla v|^4 &= (c_5+2)\int_\Omega |\nabla v|^2 \nabla v \cdot \nabla v \\
&= -(c_5+2)\int_\Omega v\nabla|\nabla v|^2 \cdot \nabla v - (c_5+2)\int_\Omega v|\nabla v|^2 \Delta v \\
&\leq (c_5+2)K_6 \int_\Omega |\nabla|\nabla v|^2||\nabla v| + (c_5+2)K_6\sqrt{2}\int_\Omega |\nabla v|^2|D^2 v| \\
&\leq \frac{d_2}{4}\int_\Omega |\nabla|\nabla v|^2|^2 + \frac{d_2}{2}\int_\Omega |\nabla v|^2|D^2 v|^2 + c_7 \int_\Omega |\nabla v|^2 \\
&\leq \frac{d_2}{4}\int_\Omega |\nabla|\nabla v|^2|^2 + \frac{d_2}{2}\int_\Omega |\nabla v|^2|D^2 v|^2 \\
&\quad + \int_\Omega |\nabla v|^4 + \frac{c_7^2}{4} \ \text{ for all } t \in (0, T_{max}),
\end{aligned}$$

which gives

$$(c_5+1)\int_\Omega |\nabla v|^4 \leq \frac{d_2}{4}\int_\Omega |\nabla|\nabla v|^2|^2 + \frac{d_2}{2}\int_\Omega |\nabla v|^2|D^2 v|^2 + \frac{c_7^2}{4} \ \text{ for all } t \in (0, T_{max}). \qquad (3.62)$$

Moreover, with the integration by parts, we can derive that

$$\begin{aligned}\int_\Omega |\nabla v|^6 &= \int_\Omega |\nabla v|^4 \nabla v \cdot \nabla v \\ &= -2\int_\Omega v|\nabla v|^2 \nabla|\nabla v|^2 \cdot \nabla v - \int_\Omega v|\nabla v|^4 \Delta v \\ &\le 2K_6 \int_\Omega |\nabla v|^3 |\nabla|\nabla v|^2| + \sqrt{2}K_6 \int_\Omega |\nabla v|^4 |D^2 v| \\ &\le \frac{3}{8}\int_\Omega |\nabla v|^6 + 4K_6^2 \left(\int_\Omega |\nabla|\nabla v|^2|^2 + \int_\Omega |\nabla v|^2 |D^2 v|^2 \right) \text{ for all } t \in (0, T_{max}),\end{aligned}$$

which gives

$$\int_\Omega |\nabla|\nabla v|^2|^2 + \int_\Omega |\nabla v|^2 |D^2 v|^2 \ge c_8 \int_\Omega |\nabla v|^6 \text{ for all } t \in (0, T_{max}). \tag{3.63}$$

Substituting (3.62) and (3.63) into (3.61), and using the fact $|\Delta u| \le \sqrt{2}|D^2 u|$, for all $t \in (0, T_{max})$ we obtain

$$\begin{aligned}\frac{d}{dt}\int_\Omega |\nabla v|^4 + \int_\Omega |\nabla v|^4 + \frac{c_8 d_2}{4}\int_\Omega |\nabla v|^6 &\le c_4 \int_\Omega |D^2 u|^4 + c_1 \int_\Omega |\nabla v|^3 |\nabla w| + c_9 \\ &\le c_4 \int_\Omega |D^2 u|^4 + \frac{c_8 d_2}{8}\int_\Omega |\nabla v|^6 + c_{10}\int_\Omega |\nabla w|^2 + c_{11},\end{aligned}$$

which gives for all $t \in (0, T_{max})$ that

$$\frac{d}{dt}\int_\Omega |\nabla v|^4 + \int_\Omega |\nabla v|^4 + \frac{c_8 d_2}{8}\int_\Omega |\nabla v|^6 \le c_4 \int_\Omega |D^2 u|^4 + c_{10}\int_\Omega |\nabla w|^2 + c_{11}. \tag{3.64}$$

Multiplying (3.57) by $4c_{10}$ and adding it to (3.64), we have

$$\begin{aligned}&\frac{d}{dt}\int_\Omega \left(4c_{10}w^2 + |\nabla v|^4\right) + \int_\Omega \left(4c_{10}w^2 + |\nabla v|^4\right) + c_{10}\int_\Omega |\nabla w|^2 + \frac{c_8 d_2}{8}\int_\Omega |\nabla v|^6 \\ &\le 4c_{10}\chi^2 \int_\Omega w^2 |\nabla v|^2 + c_4 \int_\Omega |D^2 u|^4 + c_{12} \\ &\le \frac{c_8 d_2}{8}\int_\Omega |\nabla v|^6 + c_{13}\int_\Omega w^3 + c_4 \int_\Omega |D^2 u|^4 + c_{12} \text{ for all } t \in (0, T_{max}),\end{aligned}$$

which gives

$$\begin{aligned}&\frac{d}{dt}\int_\Omega \left(4c_{10}w^2+|\nabla v|^4\right)+\int_\Omega \left(4c_{10}w^2+|\nabla v|^4\right)+c_{10}\int_\Omega |\nabla w|^2\\&\le c_{13}\int_\Omega w^3+c_4\int_\Omega |D^2u|^4+c_{12}\quad \text{for all } t\in(0,T_{max}).\end{aligned}\tag{3.65}$$

On the other hand, applying Lemma 2.5, (2.2) and (3.45), one can show that

$$c_{13}\int_\Omega w^3\le c_{10}\int_\Omega |\nabla w|^2+c_{13}\ \text{ for all } t\in(0,T_{max}).\tag{3.66}$$

Then substituting (3.66) into (3.65), it holds that

$$\frac{d}{dt}\int_\Omega \left(4c_{10}w^2+|\nabla v|^4\right)+\int_\Omega \left(4c_{10}w^2+|\nabla v|^4\right)\le c_4\int_\Omega |D^2u|^4+c_{14}\ \text{ for all } t\in(0,T_{max}).\tag{3.67}$$

Noting that $\int_t^{t+\tau}\int_\Omega |D^2u|^4\le c_{15}$ (see (3.43)), we obtain (3.55) directly by applying Lemma 2.4 to (3.67). Then the proof of Lemma 3.8 is completed. □

Lemma 3.9. *Let the assumptions in Lemma 2.1 hold and (u,v,w) be the solution of (1.3). Then we can find a constant $K_{10}>0$ independent of t such that*

$$\|w(\cdot,t)\|_{L^3}\le K_{10}\quad \text{for all } t\in(0,T_{max}).\tag{3.68}$$

Proof. Multiplying the third equation of (1.3) by w^2, and integrating it by parts, we have

$$\frac{1}{3}\frac{d}{dt}\int_\Omega w^3+2\int_\Omega w|\nabla w|^2+\theta_2\int_\Omega w^3=2\chi\int_\Omega w^2\nabla v\cdot\nabla w+\int_\Omega vw^3\ \text{for all } t\in(0,T_{max}).\tag{3.69}$$

Using Young's inequality and noting the fact $\|v(\cdot,t)\|_{L^\infty}\le K_6$, for all $t\in(0,T_{max})$ one has

$$\begin{aligned}2\chi\int_\Omega w^2\nabla v\cdot\nabla w+\int_\Omega vw^3&\le 2\chi\int_\Omega w^2|\nabla v||\nabla w|+K_6\int_\Omega w^3\\&\le\int_\Omega w|\nabla w|^2+\chi^2\int_\Omega w^3|\nabla v|^2+K_6\int_\Omega w^3.\end{aligned}\tag{3.70}$$

We substitute (3.70) into (3.69) to obtain

$$\frac{d}{dt}\int_\Omega w^3+3\int_\Omega w|\nabla w|^2+\int_\Omega w^3\le 3\chi^2\int_\Omega w^3|\nabla v|^2+(3K_6+1)\int_\Omega w^3\tag{3.71}$$

for all $t\in(0,T_{max})$. Then using the Gagliardo-Nirenberg inequality and Young's inequality, and noting the facts $\|w(\cdot,t)\|_{L^2}+\|\nabla v\|_{L^4}\le K_9$ (see (3.55)), one has

$$
\begin{aligned}
3\chi^2\int_\Omega w^3|\nabla v|^2 &\le 3\chi^2\left(\int_\Omega w^6\right)^{\frac{1}{2}}\left(\int_\Omega |\nabla v|^4\right)^{\frac{1}{2}}\\
&\le 3\chi^2K_9^2\|w^{\frac{3}{2}}\|_{L^4}^2\\
&\le 3\chi^2K_9^2c_1(\|\nabla w^{\frac{3}{2}}\|_{L^2}^{\frac{4}{3}}\|w^{\frac{3}{2}}\|_{L^{\frac{4}{3}}}^{\frac{2}{3}}+\|w^{\frac{3}{2}}\|_{L^{\frac{4}{3}}}^2)\\
&\le \int_\Omega w|\nabla w|^2+c_2 \ \text{ for all } t\in(0,T_{max}),
\end{aligned}
\tag{3.72}
$$

and

$$
\begin{aligned}
(3K_6+1)\int_\Omega w^3 &= (3K_6+1)\|w^{\frac{3}{2}}\|_{L^2}^2\\
&\le c_3\|\nabla w^{\frac{3}{2}}\|_{L^2}^{\frac{2}{3}}\|w^{\frac{3}{2}}\|_{L^{\frac{4}{3}}}^{\frac{4}{3}}+\|w^{\frac{3}{2}}\|_{L^{\frac{4}{3}}}^2\\
&\le \int_\Omega w|\nabla w|^2+c_4 \ \text{ for all } t\in(0,T_{max}).
\end{aligned}
\tag{3.73}
$$

Then substituting (3.72) and (3.73) into (3.71), we obtain $\frac{d}{dt}\int_\Omega w^3+\int_\Omega w^3\le c_3+c_4$, which gives (3.68) directly by Grönwall's inequality. □

Lemma 3.10. *Let the assumptions in Lemma 2.1 hold and (u,v,w) be the solution of the system (1.3). Then there exists a constant $K_{11}>0$ independent of t such that*

$$
\|\nabla v(\cdot,t)\|_{L^\infty}\le K_{11}, \quad \text{for all } t\in(0,T_{max}). \tag{3.74}
$$

Proof. Using Lemma 2.1, we see that (3.74) clearly holds true for all $t\in(0,\tau]$ with τ defined by (3.13). Then it only remains to show that (3.74) holds for all $t\in(\tau,T_{max})$.

First, we can rewrite the second equation of (1.3) as follows:

$$
v_t-d_2\Delta v+v=-\xi\nabla v\cdot\nabla u-\xi v\Delta u+uv-b_2vw+(1-\theta_1)v. \tag{3.75}
$$

Applying the variation-of-constants formula to (3.75), we obtain

$$
\begin{aligned}
\nabla v(\cdot,t)=&\nabla e^{(t-\tau)(d_2\Delta-1)}v(\cdot,\tau)-\xi\int_\tau^t\nabla e^{(t-s)(d_2\Delta-1)}(\nabla v\cdot\nabla u+v\Delta u)ds\\
&+\int_\tau^t\nabla e^{(d_2\Delta-1)(t-s)}[uv-b_2vw+(1-\theta_1)v]ds \ \text{ for all } t\in(\tau,T_{max}),
\end{aligned}
$$

which, together with the semigroup estimates in Lemma 2.8, gives

$$
\begin{aligned}
\|\nabla v(\cdot,t)\|_{L^\infty} \le& \|\nabla e^{(t-\tau)(d_2\Delta-1)}v(\cdot,\tau)\|_{L^\infty} + \xi \int_\tau^t \|\nabla e^{(t-s)(d_2\Delta-1)}(\nabla v\cdot\nabla u + v\Delta u)\|_{L^\infty} ds \\
&+ \int_\tau^t \|\nabla e^{(t-s)(d_2\Delta-1)}[uv - b_2 vw + (1-\theta_1)v]\|_{L^\infty} ds \\
\le& c_1 + \xi\gamma_2 \int_\tau^t (1+(t-s)^{-\frac{3}{4}})e^{-(d_2\lambda_1+1)(t-s)}\|\nabla v\cdot\nabla u\|_{L^4} ds \\
&+ \xi\gamma_2 \int_\tau^t (1+(t-s)^{-\frac{2}{3}})e^{-(d_2\lambda_1+1)(t-s)}\|v\Delta u\|_{L^6} ds \\
&+ \gamma_2 \int_\tau^t (1+(t-s)^{-\frac{5}{6}})e^{-(d_2\lambda_1+1)(t-s)}\|uv - b_2 vw + (1-\theta_1)v\|_{L^3} ds \\
:=& c_1 + I_1 + I_2 + I_3 \ \text{ for all } t\in(\tau, T_{max}).
\end{aligned} \tag{3.76}
$$

Noting the fact $\|\nabla u(\cdot,t)\|_{L^\infty} + \|\nabla v(\cdot,t)\|_{L^4} \le c_2$ from (3.39) and (3.55), one has

$$
\|\nabla v\cdot\nabla u\|_{L^4} \le \|\nabla u\|_{L^\infty}\|\nabla v\|_{L^4} \le c_2^2, \ \text{ for all } t\in(\tau, T_{max})
$$

and hence

$$
\begin{aligned}
I_1 =& \xi\gamma_2 \int_\tau^t (1+(t-s)^{-\frac{3}{4}})e^{-(d_2\lambda_1+1)(t-s)}\|\nabla v\cdot\nabla u)\|_{L^4} ds \\
\le& c_2^2\xi\gamma_2 \int_\tau^t (1+(t-s)^{-\frac{3}{4}})e^{-(d_2\lambda_1+1)(t-s)} ds \\
\le& c_3 \quad \text{for all } t\in(\tau, T_{max}).
\end{aligned} \tag{3.77}
$$

On the other hand, using the facts $\|v(\cdot,t)\|_{L^\infty} \le K_6$ and $\int_\tau^t e^{-6(t-s)}\|\Delta u\|_{L^6}^6 \le K_7$ (see Lemma 3.5 and Lemma 3.6), we derive that

$$
\begin{aligned}
I_2 =& \xi\gamma_2 \int_\tau^t (1+(t-s)^{-\frac{2}{3}})e^{-(d_2\lambda_1+1)(t-s)}\|v\Delta u\|_{L^6} ds \\
\le& \xi\gamma_2 K_6 \left(\int_\tau^t (1+(t-s)^{-\frac{2}{3}})^{\frac{6}{5}} e^{-\frac{6d_2\lambda_1}{5}(t-s)} ds\right)^{\frac{5}{6}} \left(\int_\tau^t e^{-6(t-s)}\|\Delta u\|_{L^6}^6 ds\right)^{\frac{1}{6}}
\end{aligned} \tag{3.78}
$$

$$\le K_7^{\frac{1}{6}}\xi\gamma_2 K_6\left(\int_\tau^t (1+(t-s)^{-\frac{2}{3}})^{\frac{6}{5}} e^{-\frac{6d_2\lambda_1}{5}(t-s)}ds\right)^{\frac{5}{6}}$$
$$\le c_4 \quad \text{for all } t\in(\tau,T_{max}).$$

At last, using the facts $\|u(\cdot,t)\|_{L^\infty}\le K$ (Lemma 2.2), $\|v(\cdot,t)\|_{L^\infty}\le K_6$ (Lemma 3.5) and $\|w(\cdot,t)\|_{L^3}\le K_{10}$ (Lemma 3.9), one has

$$\|uv-b_2vw+(1-\theta_1)v\|_{L^3}\le KK_6|\Omega|^{\frac{1}{3}}+b_2K_6K_{10}+(1+\theta_1)K_6|\Omega|^{\frac{1}{3}}\le c_5 \text{ for all } t\in(\tau,T_{max}),$$

and hence

$$\begin{aligned} I_3 &= \gamma_2\int_\tau^t (1+(t-s)^{-\frac{5}{6}})e^{-(d_2\lambda_1+1)(t-s)}\|uv-b_2vw+(1-\theta_1)v\|_{L^3}ds \\ &\le c_5\gamma_2\int_\tau^t (1+(t-s)^{-\frac{5}{6}})e^{-(d_2\lambda_1+1)(t-s)}ds \\ &\le c_6 \text{ for all } t\in(\tau,T_{max}). \end{aligned} \tag{3.79}$$

Substituting (3.77), (3.78) and (3.79) into (3.76) yields (3.74) and hence completes the proof. □

Lemma 3.11. *Let the assumptions in Lemma 2.1 hold and (u,v,w) be the solution of (1.3). Then it holds that*

$$\|w(\cdot,t)\|_{L^\infty}\le K_{12} \quad \text{for all } t\in(0,T_{max}), \tag{3.80}$$

where $K_{12}>0$ is a constant independent of t.

Proof. Using the variation of constants formula to the third equation of (1.3), we have

$$\begin{aligned} w(\cdot,t) =& e^{t(\Delta-1)}w_0-\int_0^t e^{(t-s)(\Delta-1)}\nabla\cdot(\chi w\nabla v)ds \\ &+\int_0^t e^{(t-s)(\Delta-1)}w\,(1-\theta_2+v)\,ds \text{ for all } t\in(0,T_{max}). \end{aligned}$$

Using the boundedness of $\|v(\cdot,t)\|_{W^{1,\infty}}$ and $\|w(\cdot,t)\|_{L^3}$, we can find two positive constants c_1 and c_2 such that

$$\|\chi w\nabla v\|_{L^3}\le\chi\|\nabla v\|_{L^\infty}\|w\|_{L^3}\le c_1 \text{ for all } t\in(0,T_{max}),$$

and

$$\|w(1-\theta_2+v)\|_{L^3} \le (1+\theta_2+\|v\|_{L^\infty})\|w\|_{L^3} \le c_2 \text{ for all } t \in (0, T_{max}).$$

Then using the estimates in Lemma 2.8, we have

$$\begin{aligned}
\|w(\cdot,t)\|_{L^\infty} \le & \|e^{t(\Delta-1)}w_0\|_{L^\infty} + \int_0^t \|e^{(t-s)(\Delta-1)}\nabla\cdot(\chi w \nabla v)\|_{L^\infty} ds \\
& + \int_0^t \left\| e^{(t-s)(\Delta-1)} w\,(1-\theta_2+v) \right\|_{L^\infty} \\
\le & \|w_0\|_{L^\infty} + \int_0^t (1+(t-s)^{-\frac{5}{6}})e^{-(\lambda_1+1)(t-s)}\|\chi w\nabla v\|_{L^3} ds \\
& + \int_0^t (1+(t-s)^{-\frac{1}{3}})e^{-(t-s)}\|w(1-\theta_2+v)\|_{L^3} ds \\
\le & \|w_0\|_{L^\infty} + c_1\int_0^t (1+(t-s)^{-\frac{5}{6}})e^{-(\lambda_1+1)(t-s)} ds \\
& + c_2\int_0^t (1+(t-s)^{-\frac{1}{3}})e^{-(t-s)} ds \\
\le & c_3 \quad \text{for all } t \in (0, T_{max}),
\end{aligned}$$

which yields (3.80). Then the proof of Lemma 3.11 is completed. □

Proof of Theorem 1.1. The combination of Lemma 2.2 and (3.39) gives

$$\|u(\cdot,t)\|_{W^{1,\infty}} \le c_1 \quad \text{for all } t \in (0, T_{max}).$$

From Lemma 3.5 and Lemma 3.10, we can find a constant $c_2 > 0$ such that $\|v(\cdot,t)\|_{W^{1,\infty}} \le c_2$. At last, noting Lemma 3.11, we can summary the results to obtain that

$$\|u(\cdot,t)\|_{W^{1,\infty}} + \|v(\cdot,t)\|_{W^{1,\infty}} + \|w(\cdot,t)\|_{L^\infty} \le c_3 \quad \text{for all } t \in (0, T_{max}),$$

which together with the extension criterion in Lemma 2.1 proves Theorem 1.1. □

4. Global stability

In this section, we shall study the global stability of solutions by constructing some proper Lyapunov functionals alongside the following LaSalle's invariance principle ([23, Theorem 3]). The Lyapunov functions for the predator-prey ODE systems can be constructed in routine ways (cf. [5,14]), and they are directly extendable to the corresponding PDE models with diffusion (cf.

[9]) or prey-taxis (cf. [1,17]). Here we shall further develop those ideas to the spatial food chain models.

4.1. Case of prey-only

In this subsection, we first study the global stability of prey-only steady state ($\theta_1 \geq 1$). In this case, we know that the system (1.3) has only two possible steady states: $(0,0,0)$ and $(1,0,0)$. One can easily check that the steady state $(0,0,0)$ is linearly unstable, while the steady state $(1,0,0)$ is linearly stable. Hence it is nature to studying whether or not $(1,0,0)$ is globally asymptotically stable in the case $\theta \geq 1$. To this end, we introduce the following energy functional:

$$\mathcal{E}_1(t) := \mathcal{E}_1(u,v,w) = \int_\Omega (u - 1 - \ln u) + b_1 \int_\Omega v + b_1 b_2 \int_\Omega w.$$

Then we have the following results:

Lemma 4.1. *Let (u,v,w) be the solution of (1.3) obtained in Theorem 1.1. Then if $\theta_1 > 1$, it holds that*

$$\lim_{t\to\infty} (\|u(\cdot,t) - 1\|_{L^\infty} + \|v(\cdot,t)\|_{L^\infty} + \|w(\cdot,t)\|_{L^\infty}) = 0.$$

Proof. First, we show that $\mathcal{E}_1(u,v,w) \geq 0$ and $\mathcal{E}_1(u,v,w) = 0$ if and only if $(u,v,w) = (1,0,0)$. In fact, letting $\varphi(z) = z - f_* \ln z$ and using Taylor's expansion, for all positive f and f_* one has

$$f - f_* - f_* \ln \frac{f}{f_*} = \varphi(f) - \varphi(f_*) = \varphi'(f_*)(f - f_*) + \frac{1}{2}\varphi''(\eta)(f - f_*)^2 = \frac{f_*}{2\eta^2}(f - f_*)^2, \tag{4.1}$$

where η is between f and f_*. Then letting $f = u$ and $f_* = 1$, from (4.1) one can find η_1 between u and 1 such that

$$u - 1 - \ln u = \frac{1}{2\eta_1^2}(u-1)^2 \geq 0,$$

and here "=" holds if and only if $u = 1$. Hence $\mathcal{E}_1(1,0,0) = 0$ and $\mathcal{E}_1(u,v,w) > 0$ for any $(u,v,w) \neq (1,0,0)$ since $v, w \geq 0$.

Moreover, from the equations in (1.3), we have

$$\begin{aligned} \frac{d}{dt}\mathcal{E}_1(t) &= \int_\Omega \frac{u-1}{u} u_t + b_1 \int_\Omega v_t + b_1 b_2 \int_\Omega w_t \\ &= -d_1 \int_\Omega \frac{|\nabla u|^2}{u^2} - \int_\Omega (u-1)^2 - b_1(\theta_1 - 1)\int_\Omega v - b_1 b_2 \theta_2 \int_\Omega w, \end{aligned}$$

which together with the condition $\theta_1 > 1$ gives

$$\frac{d}{dt}\mathcal{E}_1(t) \le 0,$$

where "=" holds if and only if $(u, v, w) = (1, 0, 0)$.

Then LaSalle's invariance principle ([23, Theorem 3]) implies that the trajectory (u, v, w) converges to $(1, 0, 0)$ as $t \to \infty$ in L^∞. □

4.2. Case of semi-coexistence

In this subsection, we shall study the global stabilization of steady state in the case of $\theta_1 < 1$ and $\theta_1 + b_1\theta_2 \ge 1$. To this end, we define the following Lyapunov functional:

$$\mathcal{E}_2(t) := \mathcal{E}_2(u, v, w) = \int_\Omega \left(u - \theta_1 - \theta_1 \ln \frac{u}{\theta_1}\right) + b_1 \int_\Omega \left(v - v^* - v^* \ln \frac{v}{v^*}\right) + b_1 b_2 \int_\Omega w,$$

where $v^* = \frac{1-\theta_1}{b_1} > 0$.

Lemma 4.2. *Let* (u, v, w) *be the solution of* (1.3) *obtained in Theorem 1.1. Then if* $\theta_1 < 1$, $\theta_1 + b_1\theta_2 > 1$ *and*

$$4d_1 d_2 \theta_1 > \xi^2 (1 - \theta_1)\|u\|^2_{L^\infty}, \tag{4.2}$$

we have

$$\lim_{t\to\infty} \left(\|u(\cdot, t) - \theta_1\|_{L^\infty} + \|v(\cdot, t) - \frac{1-\theta_1}{b_1}\|_{L^\infty} + \|w(\cdot, t)\|_{L^\infty} \right) = 0.$$

Proof. First, we can use Taylor's expansion as in (4.1) to find a η_2 between u and θ_1 such that

$$u - \theta_1 - \theta_1 \ln \frac{u}{\theta_1} = \frac{\theta_1}{2\eta_2^2}(u - \theta_1)^2 \ge 0$$

and "=" holds if and only if $u = \theta_1$. Similarly, $v - v^* - v^* \ln \frac{v}{v^*} \ge 0$ and "=" holds iff $v = v^*$. Hence $\mathcal{E}_2(\theta_1, \frac{1-\theta_1}{b_1}, 0) = 0$ and $\mathcal{E}_2(u, v, w) > 0$ for all $(u, v, w) \ne (\theta_1, \frac{1-\theta_1}{b_1}, 0)$.

On the other hand, using the definition of $\mathcal{E}_2(t)$ and the equations of (1.3), we have

$$\begin{aligned}\frac{d}{dt}\mathcal{E}_2(t) &= \int_\Omega \frac{u - \theta_1}{u} u_t + b_1 \int_\Omega \frac{v - \frac{1-\theta_1}{b_1}}{v} v_t + b_1 b_2 \int_\Omega w_t \\ &= \underbrace{-d_1\theta_1 \int_\Omega \frac{|\nabla u|^2}{u^2} - d_2(1-\theta_1)\int_\Omega \frac{|\nabla v|^2}{v^2} + \xi(1-\theta_1)\int_\Omega \frac{\nabla u \cdot \nabla v}{v}}_{I_1}\end{aligned} \tag{4.3}$$

$$\underbrace{+\int_\Omega (u-\theta_1)(1-u-b_1v)+b_1\int_\Omega \left(v-\frac{1-\theta_1}{b_1}\right)(u-\theta_1)+b_2(1-\theta_1-b_1\theta_2)\int_\Omega w}_{I_2}.$$

Letting $X_1=(\frac{\nabla u}{u},\frac{\nabla v}{v})$, then we can rewrite I_1 as follows

$$I_1=-\int_\Omega X_1A_1X_1^T,\quad \text{with}\ \ A_1=\begin{pmatrix} d_1\theta_1 & \frac{-\xi(1-\theta_1)u}{2}\\ \frac{-\xi(1-\theta_1)u}{2} & d_2(1-\theta_1)\end{pmatrix}.$$

After some calculations, one can check that if (4.2) holds, the matrix A_1 is positive definite and hence there exists a positive constant $\alpha_1>0$ such that

$$I_1=-\int_\Omega X_1A_1X_1^T\le -\alpha_1\int_\Omega\left(\frac{|\nabla u|^2}{u^2}+\frac{|\nabla v|^2}{v^2}\right). \tag{4.4}$$

On the other hand, we can rewrite the term I_2 as follows

$$\begin{aligned} I_2=&\int_\Omega (u-\theta_1)(\theta_1-u+1-\theta_1-b_1v)\\ &+b_1\int_\Omega\left(v-\frac{1-\theta_1}{b_1}\right)(u-\theta_1)+b_2(1-\theta_1-b_1\theta_2)\int_\Omega w\\ =&-\int_\Omega (u-\theta_1)^2+b_2(1-\theta_1-b_1\theta_2)\int_\Omega w.\end{aligned} \tag{4.5}$$

Substituting (4.4) and (4.5) into (4.3), with the fact $\theta_1+b_1\theta_2>1$, one has

$$\frac{d}{dt}\mathcal{E}_2(t)=-\int_\Omega X_1A_1X_1^T-\int_\Omega (u-\theta_1)^2+b_2(1-\theta_1-b_1\theta_2)\int_\Omega w\le 0,$$

and "=" if and only if $u=\theta_1, w=0$ and $\nabla v=0$. Noting that $\nabla v=0$ implies $v=\bar v$ for some constant $\bar v>0$. Due to $(u,v,w)=(\theta_1,\bar v,0)$ is a solution of (1.3), and hence it follows that

$$\theta_1(1-\theta_1-b_1\bar v)=0,$$

which implies $\bar v=\frac{1-\theta_1}{b_1}$. Hence $\frac{d}{dt}\mathcal{E}_2(t)=0$ implies $(u,v,w)=(\theta_1,\frac{1-\theta_1}{b_1},0)$.

Then applying the LaSalle's invariance principle ([23, Theorem 3]), we see that the solution (u,v,w) converges to $(\theta_1,\frac{1-\theta_1}{b_1},0)$ as $t\to\infty$ in L^∞. □

4.3. Case of coexistence

In this subsection, we shall show that the coexistence steady state (u_*, v_*, w_*) defined by (1.4) is globally stable in the case of $\theta_1 < 1$ and $\theta_1 + b_2\theta_2 < 1$. To this end, we construct the following energy functional:

$$\mathcal{E}_3(t) := \mathcal{E}_3(u, v, w) = \mathcal{J}_u(t) + b_1\mathcal{J}_v(t) + b_1 b_2 \mathcal{J}_w(t),$$

where

$$\mathcal{J}_\ell(t) = \int_\Omega \left(\ell - \ell_* - \ell_* \ln \frac{\ell}{\ell_*}\right),\ \ell = u, v, w.$$

Lemma 4.3. *Let (u, v, w) be the solution of (1.3) obtained in Theorem 1.1. Then if the parameters satisfy $\theta_1 < 1$ and $\theta_1 + b_1\theta_2 < 1$ as well as*

$$4d_1 d_2 u_* v_* > \xi^2 b_1 v_*^2 \|u\|_{L^\infty}^2 + \chi^2 b_2 d_1 u_* w_* \|v\|_{L^\infty}^2. \tag{4.6}$$

Then it holds that

$$\lim_{t\to\infty} (\|u - u_*\|_{L^\infty} + \|v - v_*\|_{L^\infty} + \|w - w_*\|_{L^\infty}) = 0.$$

Proof. By the same arguments in Lemma 4.1 and Lemma 4.2, we can use (4.1) to show that $\mathcal{E}_3(u_*, v_*, w_*) = 0$ and $\mathcal{E}_3(u, v, w) > 0$ for all $(u, v, w) \neq (u_*, v_*, w_*)$.

Moreover, using the fact that $1 - u_* - b_1 v_* = 0$, from the first equation of (1.3) we derive

$$\begin{aligned}\frac{d}{dt}\mathcal{J}_u(t) &= \int_\Omega \left(1 - \frac{u_*}{u}\right) u_t \\ &= -d_1 u_* \int_\Omega \frac{|\nabla u|^2}{u^2} + \int_\Omega (u - u_*)(1 - u - b_1 v) \\ &= -d_1 u_* \int_\Omega \frac{|\nabla u|^2}{u^2} - \int_\Omega (u - u_*)^2 - b_1 \int_\Omega (u - u_*)(v - v_*).\end{aligned} \tag{4.7}$$

Similarly, using the second equation of (1.3) alongside the fact $\theta_1 = u_* - b_2 w_*$ to obtain

$$\begin{aligned} b_1 \frac{d}{dt}\mathcal{J}_v(t) &= b_1 \int_\Omega \left(1 - \frac{v_*}{v}\right) v_t \\ &= -d_2 b_1 v_* \int_\Omega \frac{|\nabla v|^2}{v^2} + \xi v_* b_1 \int_\Omega \frac{\nabla u \cdot \nabla v}{v} + b_1 \int_\Omega (v - v_*)(u - b_2 w - \theta_1) \\ &= -d_2 b_1 v_* \int_\Omega \frac{|\nabla v|^2}{v^2} + \xi v_* b_1 \int_\Omega \frac{\nabla u \cdot \nabla v}{v} + b_1 \int_\Omega (u - u_*)(v - v_*)\end{aligned} \tag{4.8}$$

$$- b_1 b_2 \int_\Omega (v - v_*)(w - w_*).$$

With the fact $v_* = \theta_2$ and the third equation of (1.3), we have

$$\begin{aligned}
b_1 b_2 \frac{d}{dt}\mathcal{J}_w(t) &= b_1 b_2 \int_\Omega \left(1 - \frac{w_*}{w}\right) w_t \\
&= -w_* b_1 b_2 \int_\Omega \frac{|\nabla w|^2}{w^2} + w_* b_1 b_2 \chi \int_\Omega \frac{\nabla v \cdot \nabla w}{w} + \int_\Omega (w - w_*)(v - \theta_2) \\
&= -w_* b_1 b_2 \int_\Omega \frac{|\nabla w|^2}{w^2} + \chi w_* b_1 b_2 \int_\Omega \frac{\nabla v \cdot \nabla w}{w} + b_1 b_2 \int_\Omega (v - v_*)(w - w_*).
\end{aligned} \tag{4.9}$$

Combining (4.7), (4.8) and (4.9), we have

$$\frac{d}{dt}\mathcal{E}_3(t) = -\int_\Omega X_2 A_2 X_2^T - \int_\Omega (u - u_*)^2, \tag{4.10}$$

where $X_2 = \left(\frac{\nabla u}{u}, \frac{\nabla v}{v}, \frac{\nabla w}{w}\right)$ and A_2 is a symmetric matrix defined by

$$A_2 := \begin{pmatrix} d_1 u_* & -\frac{\xi b_1 v_* u}{2} & 0 \\ -\frac{\xi b_1 v_* u}{2} & d_2 b_1 v_* & -\frac{\chi b_1 b_2 w_* v}{2} \\ 0 & -\frac{\chi b_1 b_2 w_* v}{2} & b_1 b_2 w_* \end{pmatrix}.$$

If (4.6) holds, we can check that

$$\begin{vmatrix} d_1 u_* & -\frac{\xi b_1 v_* u}{2} \\ -\frac{\xi b_1 v_* u}{2} & d_2 b_1 v_* \end{vmatrix} = \frac{b_1 v_* (4 d_1 d_2 u_* - \xi^2 b_1 v_* u^2)}{4} \geq \frac{b_1 v_* (4 d_1 d_2 u_* - \xi^2 b_1 v_* \|u\|_{L^\infty}^2)}{4} > 0$$

and

$$\begin{aligned}
|A_2| =& d_1 d_2 b_1^2 b_2 u_* v_* w_* - \frac{\xi^2 b_1^3 b_2 w_* v_*^2 u^2}{4} - \frac{\chi^2 b_1^2 b_2^2 d_1 u_* w_*^2 v^2}{4} \\
=& \frac{b_1^2 b_2 w_*}{4}\left(4 d_1 d_2 u_* v_* - \xi^2 b_1 v_*^2 u^2 - \chi^2 b_2 d_1 u_* w_* v^2\right) \\
\geq& \frac{b_1^2 b_2 w_*}{4}\left(4 d_1 d_2 u_* v_* - \xi^2 b_1 v_*^2 \|u\|_{L^\infty}^2 - \chi^2 b_2 d_1 u_* w_* \|v\|_{L^\infty}^2\right) > 0,
\end{aligned}$$

which implies the matrix A_2 is positive definite and hence there exists a positive constant α_2 such that

$$-\int_\Omega X_2 A_2 X_2^T \leq -\alpha_2 \int_\Omega \left(\frac{|\nabla u|^2}{u^2} + \frac{|\nabla v|^2}{v^2} + \frac{|\nabla w|^2}{v^2}\right). \tag{4.11}$$

Substituting (4.11) into (4.10), we obtain

$$\frac{d}{dt}\mathcal{E}_3(t) \le -\alpha_2 \int_\Omega \left(\frac{|\nabla u|^2}{u^2} + \frac{|\nabla v|^2}{v^2} + \frac{|\nabla w|^2}{v^2} \right) - \int_\Omega (u-u_*)^2,$$

which implies $\frac{d}{dt}\mathcal{E}_3(t) \le 0$ for all u, v, w and $u = u_*$, $\nabla v = 0$ and $\nabla w = 0$ if $\frac{d}{dt}\mathcal{E}_3(t) = 0$.

Noting that $\nabla v = 0$ and $\nabla w = 0$ imply that

$$v = \tilde{v}_* \text{ and } w = \tilde{w}_*,$$

where $\tilde{v}_*$ and $\tilde{w}_*$ are positive constants. Noting that $(u, v, w) = (u_*, \tilde{v}_*, \tilde{w}_*)$ is the solution of (1.3), then it has that

$$\begin{cases} u_*(1 - u_* - b_1\tilde{v}_*) = 0, \\ \tilde{v}_*(u_* - b_2\tilde{w}_* - \theta_1) = 0, \\ \tilde{w}_*(\tilde{v}_* - \theta_2) = 0. \end{cases} \tag{4.12}$$

Solving the system (4.12) with $u_* = 1 - b_1\theta_2$, we obtain

$$\tilde{v}_* = \theta_2 = v_* \text{ and } \tilde{w}_* = \frac{1 - b_1\theta_2 - \theta_1}{b_2} = w_*.$$

Hence if $\frac{d}{dt}\mathcal{E}_3(t) = 0$, then $(u, v, w) = (u_*, v_*, w_*)$. Using LaSalle's invariance principle ([23, Theorem 3]), we know that the coexistence steady state (u_*, v_*, w_*) is globally asymptotically stable. □

Proof of Theorem 1.2. Theorem 1.2 is a consequence of Lemma 4.1, Lemma 4.2 and Lemma 4.3. □

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