SIAM J. MATH. ANAL.
Vol. 55, No. 4, pp. 2838–2876


# GLOBAL SOLVABILITY AND STABILITY OF AN ALARM-TAXIS SYSTEM*

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**Abstract.** This paper is concerned with the global boundedness and stability of classical solutions to an alarm-taxis system describing the burglar alarm hypothesis as an important mechanism of antipredation behavior when prey species are threatened by predators. Compared to the existing prey-taxis systems, the alarm-taxis system has more complicated coupling structure and additionally requires the gradient estimate of the primary predator density to attain the global boundedness of solutions. By the sophisticated coupling energy estimates based on the Neumann semigroup smoothing properties, we establish the existence of globally bounded solutions in two dimensions with Neumann boundary conditions and furthermore prove the global stability of coexistence homogeneous steady states under certain conditions on the system parameters.

**Key words.** alarm-taxis, global boundedness, global stability, coexistence steady states

**MSC codes.** 35A01, 35B40, 35B44, 35K57, 35Q92, 92C17

**DOI.** 10.1137/22M1477143

**1. Introduction and main results.** Alarm calls are an important mechanism of antipredation behavior when species are approached by predators, where alarm call signals may be chemical, acoustic, sound, visible movement, or any other changes that are detectable by the receiver (cf. [16, 33]). There are numerous hypotheses on the structure and function of alarm calls, among which is the "burglar alarm" hypothesis (cf. [12]): a prey species renders itself dangerous to a primary predator by generating an alarm call to attract a second predator at higher trophic levels in the food chain that prey on the primary predator. This attraction of a secondary predator has been observed in the marine environment where dinoflagellates bioluminesce when stimulated by disturbances from copepod feeding currents may attract a secondary predator like fish (cf. [1]), and in many other species (like plants [18], birds [26], and primates [20]). To test this hypothesis, a mathematical model was recently proposed in [23], which reads in its multidimensional form as

$$(1.1)\qquad \begin{cases} u_t = d_1\Delta u + f(u,v,w), \\ v_t = d_2\Delta v - \nabla\cdot(\xi v\nabla u) + g(u,v,w), \\ w_t = \Delta w - \nabla\cdot(\chi w\nabla\phi(u,v)) + h(u,v,w), \end{cases}$$

*Received by the editors February 10, 2022; accepted for publication (in revised form) January 17, 2023; published electronically July 14, 2023.
https://doi.org/10.1137/22M1477143

**Funding:** The work of the first author was supported by the National Natural Science Foundation of China grant 11871226, Guangdong Basic and Applied Basic Research Foundation grant 2022B1515020032 and the Fundamental Research Funds for the Central Universities. The work of the second author was partially supported by the Hong Kong RGC GRF grant 15306121 and the 2020 Hong Kong Scholars Program. The work of the third author was supported by the 2020 Hong Kong Scholars Program project P0031250 and Primary Work Programme YZ3S.

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where $u, v$, and $w$ represent resource or prey (e.g., dinoflagellate), primary predator (e.g., copepod), and secondary predator (e.g., fish), respectively; $d_1, d_2$ are positive constants representing the random dispersal rates, the positive constants $\xi$ and $\chi$ are referred to as the prey-taxis and alarm-taxis coefficients, respectively; the reaction functions $f, g, h$ describe the interspecific and/or intraspecific interactions among the prey, the primary predator, and the second predator, and $\phi(u,v)$ is a signalling function describing the intensity of the alarm signal which is produced as a result of interaction between the prey and primary predator to act as a burglar alarm attracting the secondary predator. While there are many ways one could postulate the signal intensity function $\phi(u,v)$, a simple but plausible assumption is that the signal intensity is proportional to the encounter rate between the prey and the primary predator, that is (cf. [23]),

$$\phi(u,v) \propto uv.$$

Generally the reaction functions $f, g, h$ have the prototypical forms

$$\begin{aligned} f(u,v,w) &= \phi_1(u) - b_1 v F_1(u,v) - b_3 w F_2(u,w),\\ g(u,v,w) &= \phi_2(v) + c_1 v F_1(u,v) - b_2 w F_3(v,w),\\ h(u,v,w) &= \phi_3(w) + c_2 w F_3(v,w) + c_3 w F_2(u,w), \end{aligned} \tag{1.2}$$

where $b_1, b_2, c_1, c_2 > 0$ and $b_3, c_3 \geq 0$ are constants, and $\phi_i\,(i = 1,2,3)$ describes the intraspecific interactions of species. For the prey species, the population dynamics $\phi_1(u)$ in the absence of predators is usually described by the logistic growth as prey species usually are prolific breeders and, if left alone, their populations would rise to the ability of their ecosystem to feed them, that is, $\phi_1(u) = \mu_1 u \left(1 - \frac{u}{K}\right)$, where $\mu_1 > 0$ denotes the intrinsic growth rate and $K > 0$ is the carrying capacity. The population dynamics of predators in the absence of prey species are usually described by the function

$$\phi_i(s) = \mu_i s - \theta_i s^2, \quad i \in \{2,3\},$$

where $s = v$ if $i = 2$ and $s = w$ if $i = 3$, $\mu_i > 0$ (resp., $< 0$) denotes the intrinsic growth (resp., death) rate of species and $\theta_i \geq 0$ denotes the intraspecific competition strength ($\theta_i = 0$ means there is no intraspecific competition between species). In particular, if $\mu_i (i = 2,3) > 0$ (resp., $< 0$), then the corresponding predator is called a generalist (resp., specialist) predator. $F_i\,(i = 1,2,3)$ are called the functional response (or trophic) functions describing the consumption rate of a prey by a predator, which may have various forms such as Holling type I, II, and III [27], ratio-dependent [7, 22], Beddington–DeAngelis type [9, 17], and so on (cf. [40, 48]). We remark that without prey-taxis and alarm-taxis, the model (1.1) is generally called a food chain model (cf. [24]) if $b_3 = c_3 = 0$ (i.e., the second predator $w$ does not utilize the resource), and an intraguild predation model (cf. [28]) if $b_3, c_3 > 0$ (i.e., the second predator $w$ can utilize the resource of its prey $v$). For the alarm-taxis model, the first qualitative result was obtained in [23] for the following one dimensional form with Neumann boundary conditions:

$$\begin{cases} u_t = d_1 u_{xx} + \mu_1 u(1-u) - b_1 uv - b_3 uw, & x \in (0,L),\ t > 0,\\ v_t = d_2 v_{xx} - (\xi v u_x)_x + \mu_2 v(1-v) + c_1 uv - b_2 vw, & x \in (0,L),\ t > 0,\\ w_t = w_{xx} - (\chi w(v u_x + u v_x))_x + \mu_3 w(1-w) + c_2 vw + c_3 uw, & x \in (0,L),\ t > 0,\\ u_x = v_x = w_x = 0, & x = 0, L,\ t > 0,\\ (u,v,w)(x,0) = (u_0, v_0, w_0)(x), & x \in (0,L), \end{cases} \tag{1.3}$$

where $L > 0$, $d_1, d_2, \mu_1, \mu_2, \mu_3, b_1, b_2, \xi, \chi > 0$, and $b_3, c_3 \geq 0$ are constants. The existence of global bounded solutions of (1.3) was established in [23], and the global stability of coexistence steady state for the case $b_3 = c_3 = 0$ was further proved under certain conditions (which will be mentioned later).

The system (1.1), as the first mathematical model for alarm-taxis proposed in [23], provides basic theoretical framework to understand the mechanism of antipredation behavior of the prey by releasing alarm call signals. The mathematical studies of the alarm-taxis model (1.1) was initiated in [23] for the specialized form (1.3) in one dimension only. Hence there are many interesting questions worthwhile to explore so as to gain more insights into the understanding of the alarm-taxis mechanism, for instance the global dynamics of alarm-taxis models in a more realistic multidimensional spatial domain with different functional response functions $F_i$ and so on. This motivates us, among other things, to consider the following alarm-taxis system

$$(1.4)\quad \begin{cases} u_t = d_1\Delta u + \mu_1 u(1-u) - b_1 uv - b_3\frac{uw}{u+w}, & x\in\Omega, t>0\\ v_t = d_2\Delta v - \nabla\cdot(\xi v\nabla u) + \mu_2 v(1-v) + uv - b_2 vw, & x\in\Omega, t>0\\ w_t = \Delta w - \nabla\cdot[\chi w(v\nabla u + u\nabla v)] + \mu_3 w(1-w) + vw + c_3\frac{uw}{u+w}, & x\in\Omega, t>0,\\ \frac{\partial u}{\partial\nu} = \frac{\partial v}{\partial\nu} = \frac{\partial w}{\partial\nu} = 0, & x\in\partial\Omega,\ t>0,\\ (u,v,w)(x,0) = (u_0,v_0,w_0)(x), & x\in\Omega \end{cases}$$

in a bounded smooth domain $\Omega \subset \mathbb{R}^N (N \geq 2)$ with parameters $d_1, d_2, \mu_1, \mu_2, \mu_3, b_1, b_2, \xi, \chi > 0$ and $b_3, c_3 \geq 0$, where $\nu$ denotes the outward normal vector of $\partial\Omega$. Particularly, when $b_3 = c_3 = 0$ (case of food chain), the model (1.4) is nothing but the multidimensional version of (1.3). The difference is that when $b_3, c_3 > 0$ (case of intraguild predation), the model (1.4) employs the ratio-dependent functional response while (1.3) uses the Lotka–Volterra functional response. The food chain model with spatial movements has not been investigated in the literature to the best of our knowledge though its ODE counterpart (i.e., the temporal model) has been extensively studied (cf. [32, 37, 41] and references therein). Without prey-taxis and alarm-taxis, the intraguild predation models with some particular functional response functions have been analytically studied in [14, 43]. Same as [23], here we have assumed both $v$ and $w$ are generalist predators with intraspecific competitions and other possible cases will not be considered in this paper. The main goal of this paper is to investigate the global dynamics of the alarm-taxis model (1.4) by establishing the global boundedness of solutions in multidimensions and the global stability of coexistence steady states for both $b_3 = c_3 = 0$ and $b_3, c_3 > 0$. To compare, we recall that the work [23] obtains the global boundedness of solutions for the one dimensional model (1.3) and establishes the global stability of coexistence steady states for the case $b_3 = c_3 = 0$ only. The global boundedness of solutions to (1.3) in multidimensions still remains open and our results show that the global boundedness of classical solutions can be ensured if the interaction between $u$ and $w$ is described by the ratio-dependent functional response.

From mathematical point of view, the structure of (1.1) with (1.2) is analogous to the following prey-taxis system:

$$(1.5)\quad \begin{cases} u_t = d_1\Delta u - vF(u,v) + f(u), & x\in\Omega,\ t>0,\\ v_t = d_2\Delta v - \nabla\cdot(\chi v\nabla u) + \gamma vF(u,v) - vh(v), & x\in\Omega,\ t>0,\\ \frac{\partial u}{\partial\nu} = \frac{\partial v}{\partial\nu} = 0, & x\in\partial\Omega,\ t>0, \end{cases}$$

where $F(u,v)$ is the functional response function which may depend on $u$ only like Holling type I, II, and III or on both $u$ and $v$ like the ratio-dependent response (cf. [40, 48]). The system (1.5) is a simplified version of the original prey-taxis system proposed in [31], and has been studied from different analytical perspectives for different functional response functions in the literature (cf. [3, 45, 25, 35, 51, 49]). Among other things, if $F(u,v)$ is prey-dependent only (i.e., depends on $u$ only), small-data global solutions in three or higher dimensions were attained in [49, 51]. However, if $F(u,v)$ is ratio-dependent, large-data global solutions can be attained in any dimensions [13]. For results of other classes of taxis models in the predator-prey system such as indirect prey-taxis or predator-taxis systems, we refer to [2, 21, 38, 47, 52] and references therein. Compared to the prey-taxis system (1.5), the alarm-taxis system (1.4) has more intricate coupling structures where, in particular, a priori $\|\nabla v\|_{L^\infty}$ estimate is required to derive $\|w\|_{L^\infty}$ which, however, in turn affects the gradient estimates of $u$. Hence, how to untie these tangled coupling estimates to deduce the a priori estimates of $\|\nabla u\|_{L^\infty}$ and $\|\nabla v\|_{L^\infty}$ is the key to obtaining the global boundedness of solutions for (1.4), where the estimate of $\|\nabla v\|_{L^\infty}$ is the main new challenge arising in the model to overcome. We have not found existing works addressing how to obtain the $L^\infty$-estimates for the gradient of predator densities. In this paper, we shall first fully capture the ratio-dependent functional response structure to get the global estimate of $\|\nabla u\|_{L^\infty}$ and hence $\|v\|_{L^\infty}$. Then we start from some elegant estimates on the second-order derivative estimate of $u$ based on works [15, 30] to derive the global estimate of $\|\nabla v\|_{L^\infty}$ from which the estimate of $\|w\|_{L^\infty}$ follows alongside the application of Neumann semigroup smoothing properties. We remark that a conventional method used to study the global boundedness of solutions to taxis equations (cf. [46]) by resorting to the entropy estimates is also applicable to establish the boundedness of $u$ and $v$, but with more complicated estimates. In this paper, we provide a simpler approach by utilizing the local-in-time integrability of $L^2$-norm of $v, w$ (see Lemmas 2.3 and 2.4) resulting from the quadratic decay in the kinetics terms and by using a second-order estimate (see Lemma 3.7).

Our first result concerning the global existence and boundedness of classical solutions of (1.4) is stated in the following theorem.

THEOREM 1.1 (global boundedness). *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary. Assume $(u_0, v_0, w_0) \in [W^{2,\infty}(\Omega)]^3$ with $u_0, v_0, w_0 \gneqq 0$. Then the problem* (1.4) *has a unique global classical solution $(u,v,w) \in [C^0(\bar\Omega \times [0,\infty)) \cap C^{2,1}(\bar\Omega \times (0,\infty))]^3$ satisfying $u,v,w > 0$ for all $t > 0$. Furthermore, there exists a constant $C > 0$ independent of $t$ such that*

$$\|u(\cdot,t)\|_{W^{1,\infty}} + \|v(\cdot,t)\|_{W^{1,\infty}} + \|w(\cdot,t)\|_{L^\infty} \le C,$$

*where, in addition, $\|u\|_{L^\infty}$ is independent of $\xi$ and $\chi$ while $\|v\|_{L^\infty}$ is independent of $\chi$.*

Our next results are concerned with the asymptotical behavior of solutions to the system (1.4). In particular, we shall explore under what conditions the positive coexistence steady state can be asymptotically achieved. In our analysis, we just need the positivity of parameters $\mu_i\,(i = 1,2,3)$ to ensure the global boundedness of solutions and the specific values of $\mu_1, \mu_2$, and $\mu_3$ are not of importance. Hence for simplicity, we assume that $\mu_1 = \mu_2 = \mu_3 = 1$ without loss of generality for the stability analysis. Then the system (1.4) can be rewritten as

$$(1.6)\quad \begin{cases} u_t = d_1\Delta u + u(1 - u - b_1 v - b_3\frac{w}{u+w}), & x \in \Omega, t > 0,\\ v_t = d_2\Delta v - \nabla\cdot(\xi v\nabla u) + v(1 - v + u - b_2 w), & x \in \Omega, t > 0,\\ w_t = \Delta w - \nabla\cdot[\chi w(v\nabla u + u\nabla v)] + w(1 - w + v + c_3\frac{u}{u+w}), & x \in \Omega, t > 0,\\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, & x \in \partial\Omega,\, t > 0,\\ (u,v,w)(x,0) = (u_0, v_0, w_0)(x), & x \in \Omega. \end{cases}$$

Depending on whether or not the secondary predator $w$ consumes the resource $u$, we divide our analysis into two cases:

(1) $b_3 = c_3 = 0$: the secondary predator $w$ does not consume the (prey) resource $u$; that is, the temporal dynamics is a case of food chain.

(2) $b_3, c_3 > 0$: the secondary predator $w$ consumes the resource $u$; that is, the temporal dynamics is a case of intraguild predation.

We first consider the case $b_3 = c_3 = 0$, for which one can check that (1.6) has three types of homogeneous (constant) steady states as follows:

1. Trivial steady states: $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$.
2. Semitrivial steady states: $(1,0,1)$, $(\frac{1-b_1}{1+b_1}, \frac{2}{1+b_1}, 0)$, $(0, \frac{1-b_2}{1+b_2}, \frac{2}{1+b_2})$.
3. Coexistence steady state: $(u^*, v^*, w^*)$ with

$$u^* = \frac{1+b_1b_2+b_2-b_1}{1+b_1+b_2}, \quad v^* = \frac{2-b_2}{1+b_1+b_2}, \quad w^* = \frac{3+b_1}{1+b_1+b_2}, \tag{1.7}$$

where $u^*, v^*, w^*$ are all positive if

$$0 < b_2 < 2 \quad \text{and} \quad 0 < b_1 < 1 + b_1b_2 + b_2. \tag{1.8}$$

For the positive coexistence steady state $(u^*, v^*, w^*)$ defined in (1.8), we have the following global stability result.

Theorem 1.2 (global stability for the case of food chain). *Assume the assumptions in Theorem 1.1 hold, and let $(u, v, w)$ be the solution of (1.6) with $b_3 = c_3 = 0$. If the parameters $b_1, b_2$ satisfy (1.8) with*

$$(b_1 - 1)^2 + (b_2 - 1)^2 < 4, \tag{1.9}$$

*then there exist $\xi_1 > 0$ and $\chi_1 > 0$ such that whenever $\xi \in (0, \xi_1)$ and $\chi \in (0, \chi_1)$ it holds that*

$$\|u(\cdot,t) - u^*\|_{L^\infty} + \|v(\cdot,t) - v^*\|_{L^\infty} + \|w(\cdot,t) - w^*\|_{L^\infty} \le C_1 e^{-\sigma_1 t} \quad \text{for all } t > t_0,$$

*with some $t_0 > 0$, where $C_1$ and $\sigma_1$ are positive constants independent of $t$.*

*Remark* 1.3. We underline that the admissible regime for the parameters $b_1, b_2 > 0$ satisfying (1.8)–(1.9) is nonempty and can be explicitly identified; see Figure 1.

Next, we explore the global stability of solutions in the case $b_3, c_3 > 0$. For simplicity, we further assume that $c_3 = 1$ without loss of generality. Then there are also three types of homogenous steady states as follows:

1. Trivial steady states: $(0,0,1)$ and $(1,0,0)$.
2. Semitrivial steady states: $(0, \frac{1-b_2}{1+b_2}, \frac{2}{1+b_2})$, $(\frac{1-b_1}{1+b_1}, \frac{2}{1+b_1}, 0)$ and $(1 - 2b_3 + \frac{2b_3^2 + b_3\sqrt{2(1-b_3)}}{1+b_3}, 0, \frac{2b_3 + \sqrt{2(1-b_3)}}{1+b_3})$.
3. Coexistence steady state: $(u_*, v_*, w_*)$, where $u_*, v_*, w_*$ satisfy the following equations:

$$\begin{cases} 1 - u_* - b_1 v_* - b_3 \frac{w_*}{u_*+w_*} = 0, \\ 1 - v_* + u_* - b_2 w_* = 0, \\ 1 - w_* + v_* + \frac{u_*}{u_*+w_*} = 0. \end{cases} \tag{1.10}$$

We note that $(u_*, v_*, w_*)$ can be explicitly solved and, furthermore, if $b_1$ and $b_3$ are sufficiently small, (1.10) has a unique positive solution $(u_*, v_*, w_*)$ (see the appendix for details), for which we have the following global stability result.

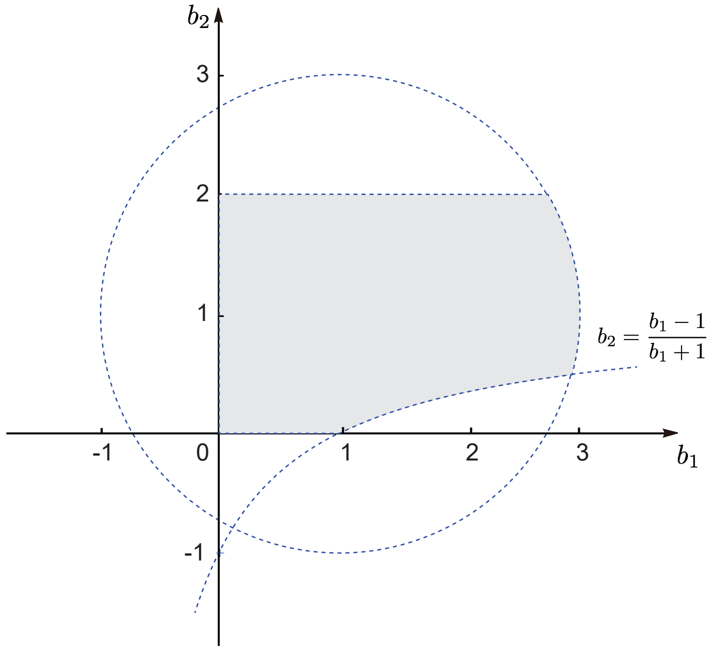


FIG. 1. *Illustration of the admissible regime (shaded region) for parameters* $b_1, b_2 > 0$ *satisfying* (1.8)–(1.9).

THEOREM 1.4 (global stability for the case of intraguild predation). *Let the assumptions in Theorem* 1.1 *hold, and let* $(u, v, w)$ *be the solution of* (1.6) *with* $b_3 > 0$ *and* $c_3 = 1$. *Assume that* $(u_*, v_*, w_*)$ *is the positive coexistence steady state satisfying* (1.10). *If*

$$b_1 \text{ and } b_3 \text{ are sufficiently small, } \frac{1}{10} \le b_2 < \sqrt{2}, \tag{1.11}$$

*then there exist* $\xi_2 > 0$ *and* $\chi_2 > 0$ *such that whenever* $\xi \in (0, \xi_2)$ *and* $\chi \in (0, \chi_2)$ *it holds that*

$$\|u(\cdot,t) - u_*\|_{L^\infty} + \|v(\cdot,t) - v_*\|_{L^\infty} + \|w(\cdot,t) - w_*\|_{L^\infty} \le C_2 e^{-\sigma_2 t} \quad \text{for all } t > T_0,$$

*with some* $T_0 > 0$, *where* $C_2$ *and* $\sigma_2$ *are positive constants independent of* $t$.

The rest of this paper is arranged as follows. In section 2, we show the local existence of solutions and prove some basic properties of solutions. In section 3, we demonstrate the details of obtaining the necessary a priori estimates of solutions and prove Theorem 1.1. Then in section 4, we prove the global stability of coexistence steady states under certain conditions stated in Theorems 1.2 and 1.4 by employing the Lyapunov functional method alongside Barbălat's lemma. The appendix shows the existence of positive coexistence steady state under conditions imposed in Theorem 1.4.

**2. Local existence and preliminaries.** In what follows, we shall use $C_i$ $(i = 1, 2, \ldots)$ to denote a generic positive constant which may vary in the context. Without confusion, the integration variables $x$ and $t$ will be omitted, for instance, $\int_0^t \int_\Omega f(x,s)dxds$ will be abbreviated as $\int_0^t \int_\Omega f(x,s)$. The existence and uniqueness of local solutions of (1.4) can be readily proved by Amann's theorem [4, 6].

LEMMA 2.1 (local existence). *Let the assumptions in Theorem* 1.1 *hold. Then there exists* $T_{max} \in (0,\infty]$ *such that the problem* (1.4) *has a unique classical solution*

$$(u,v,w) \in [C^0(\bar{\Omega}\times[0,T_{max}))\cap C^{2,1}(\bar{\Omega}\times(0,T_{max}))]^3$$

*satisfying* $u,v,w>0$ *for all* $t>0$. *Moreover,*

$$\text{(2.1)} \quad \text{if } T_{max}<\infty, \text{ then } \lim_{t\nearrow T_{max}} (\|u(\cdot,t)\|_{W^{1,\infty}} + \|v(\cdot,t)\|_{W^{1,\infty}} + \|w(\cdot,t)\|_{L^\infty}) = \infty.$$

*Proof.* Denote $z=(u,v,w)$. Then the system (1.4) can be written as

$$\text{(2.2)} \qquad \begin{cases} z_t = \nabla\cdot(P(z)\nabla z) + Q(z), & x\in\Omega, t>0,\\ \frac{\partial z}{\partial\nu} = 0, & x\in\partial\Omega, t>0,\\ z(\cdot,0) = (u_0,v_0,w_0), & x\in\Omega, \end{cases}$$

where

$$P(z) = \begin{pmatrix} d_1 & 0 & 0\\ -\xi v & d_2 & 0\\ -\chi vw & -\chi uw & 1 \end{pmatrix}, \quad Q(z) = \begin{pmatrix} u(\mu_1-\mu_1 u - b_1 v - \frac{b_3 w}{u+w})\\ v(\mu_2-\mu_2 v + u - b_2 w)\\ w(\mu_3-\mu_3 w + v + \frac{c_3 u}{u+w}) \end{pmatrix}.$$

The matrix $P(z)$ is positive definite for the given initial data, which means the system (2.2) is uniformly parabolic. Then the application of [4, Theorem 7.3] yields a $T_{max}>0$ such that the system (2.2) possesses a unique solution $(u,v,w)\in[C^0(\bar{\Omega}\times[0,T_{max}))\cap C^{2,1}(\bar{\Omega}\times(0,T_{max}))]^3$.

Next, we prove the positivity of $u,v$, and $w$. Applying the strong maximum principle to the first equation in (1.4), we have $u(x,t)>0$ for all $(x,t)\in\Omega\times(0,T_{max})$, due to the fact $u_0 \gneqq 0$. Moreover, we can rewrite the equation of $v$ as follows:

$$\text{(2.3)} \qquad \begin{cases} v_t - d_2\Delta v + \xi\nabla u\cdot\nabla v + \Psi(x,t) = 0, & x\in\Omega, t\in(0,T_{\max}),\\ \frac{\partial v}{\partial\nu} = 0, & x\in\partial\Omega, t\in(0,T_{\max}),\\ v(x,0) = v_0(x)\geq 0, & x\in\Omega, \end{cases}$$

where $\Psi(x,t) = \xi v\Delta u - v(\mu_2-\mu_2 v + u - b_2 w)$. Then the strong maximum principle applied to (2.3) yields $v(x,t)>0$ for all $(x,t)\in\Omega\times(0,T_{max})$. Similarly, we can derive that $w>0$ for all $(x,t)\in\Omega\times(0,T_{max})$. In addition, since $P(z)$ is a lower triangular matrix, the blow-up criterion (2.1) follows from [5, Theorem 5.2] directly. Then the proof of Lemma 2.1 is completed. ∎

LEMMA 2.2. *Let the assumptions in Theorem* 1.1 *hold. Then the solution of* (1.4) *satisfies*

$$\text{(2.4)} \qquad \|u(\cdot,t)\|_{L^\infty} \leq K$$

*for all* $t>0$, *where* $K := \max\{1, \|u_0\|_{L^\infty}\}$, *and*

$$\text{(2.5)} \qquad \limsup_{t\to\infty} u(\cdot,t) \leq 1 \quad \textit{for all } x\in\bar{\Omega}.$$

*Proof.* The results can be easily obtained based on a comparison principle applied to the first equation of (1.4) along with the nonnegativity of $u,v$, and $w$, but we omit the details for brevity. ∎

LEMMA 2.3. *Suppose the assumptions in Theorem* 1.1 *hold. Then the solution of* (1.4) *satisfies*

$$\|v(\cdot,t)\|_{L^1} \le K_1 \quad \text{for all } \ t\in(0,T_{max}), \tag{2.6}$$

*and*

$$\int_t^{t+\tau}\int_\Omega v^2 \le \frac{4K_1}{\mu_2} \quad \text{for all } \ t\in(0,\widetilde{T}_{max}), \tag{2.7}$$

*where* $K_1=\|v_0\|_{L^1}+\frac{(\mu_2+K+1)^2|\Omega|}{2\mu_2}$, $\tau$ *is a constant such that*

$$0<\tau<\min\{1,T_{max}\} \ \text{ and } \ \widetilde{T}_{max}:=\begin{cases} T_{max}-\tau & \text{if } \ T_{max}<\infty,\\ \infty & \text{if } \ T_{max}=\infty. \end{cases} \tag{2.8}$$

*Proof.* Integrating the second equation of (1.4) by parts with respect to $x\in\Omega$, and using the fact (2.4) as well as the positivity of $(u,v,w)$, we end up with

$$\begin{aligned} \frac{d}{dt}\int_\Omega v+\int_\Omega v+\mu_2\int_\Omega v^2 &= (\mu_2+1)\int_\Omega v+\int_\Omega uv-b_2\int_\Omega vw\\ &\le (\mu_2+K+1)\int_\Omega v\\ &\le \frac{\mu_2}{2}\int_\Omega v^2+\frac{(\mu_2+K+1)^2|\Omega|}{2\mu_2}, \end{aligned}$$

which gives

$$\frac{d}{dt}\int_\Omega v+\int_\Omega v+\frac{\mu_2}{2}\int_\Omega v^2 \le \frac{(\mu_2+K+1)^2|\Omega|}{2\mu_2}. \tag{2.9}$$

Applying the Grönwall inequality to (2.9), we derive

$$\int_\Omega v \le \|v_0\|_{L^1}+\frac{(\mu_2+K+1)^2|\Omega|}{2\mu_2} := K_1, \tag{2.10}$$

which yields (2.6). Integrating (2.9) over $(t,t+\tau)$ and using (2.10), one has

$$\int_t^{t+\tau}\int_\Omega v^2 \le \frac{2}{\mu_2}\int_\Omega v+\frac{(\mu_2+K+1)^2|\Omega|\tau}{\mu_2^2} \le \frac{2\|v_0\|_{L^1}}{\mu_2}+\frac{2(\mu_2+K+1)^2|\Omega|}{\mu_2^2} \le \frac{4K_1}{\mu_2},$$

which gives (2.7). ∎

LEMMA 2.4. *Suppose the assumptions in Theorem* 1.1 *hold. Then there exist two constants* $K_2>0$ *and* $K_3>0$ *which are independent of* $\xi$ *and* $\chi$ *such that the solution of* (1.4) *satisfies*

$$\|w(\cdot,t)\|_{L^1} \le K_2 \quad \text{for all } \ t\in(0,T_{max}), \tag{2.11}$$

*and*

$$\int_t^{t+\tau}\int_\Omega w^2 \le K_3 \quad \text{for all } \ t\in(0,\widetilde{T}_{max}), \tag{2.12}$$

*where* $\tau$ *and* $\widetilde{T}_{max}$ *are defined by* (2.8).

*Proof.* Multiplying the third equation in (1.4) by $b_2$, adding the result to the second equation in (1.4), followed by an integration, we have

$$\begin{aligned}\frac{d}{dt}\int_\Omega (v+b_2w)+\mu_2\int_\Omega v^2+b_2\mu_3\int_\Omega w^2\\ =\mu_2\int_\Omega v+\int_\Omega uv+b_2\mu_3\int_\Omega w+b_2c_3\int_\Omega \frac{uw}{u+w}.\end{aligned}\tag{2.13}$$

Adding $\int_\Omega (v+b_2w)$ to both sides of (2.13), along with (2.4) and (2.6), one obtains

$$\begin{aligned}&\frac{d}{dt}\int_\Omega (v+b_2w)+\int_\Omega (v+b_2w)+\mu_2\int_\Omega v^2+b_2\mu_3\int_\Omega w^2\\ &=(\mu_2+1)\int_\Omega v+\int_\Omega uv+b_2(\mu_3+1)\int_\Omega w+b_2c_3\int_\Omega \frac{uw}{u+w}\\ &\le (\mu_2+1+K)\int_\Omega v+b_2(\mu_3+1+c_3)\int_\Omega w\\ &\le (\mu_2+1+K)K_1+b_2(\mu_3+1+c_3)\int_\Omega w\\ &\le (\mu_2+1+K)K_1+\frac{b_2\mu_3}{2}\int_\Omega w^2+\frac{b_2(\mu_3+1+c_3)^2|\Omega|}{2\mu_3},\end{aligned}$$

which gives

$$\frac{d}{dt}\int_\Omega (v+b_2w)+\int_\Omega (v+b_2w)+\frac{b_2\mu_3}{2}\int_\Omega w^2\le (\mu_2+1+K)K_1+\frac{b_2(\mu_3+1+c_3)^2|\Omega|}{2\mu_3}.\tag{2.14}$$

With Grönwall's inequality applied to (2.14), one has

$$\int_\Omega (v+b_2w)\le \int_\Omega (v_0+b_2w_0)+(\mu_2+1+K)K_1+\frac{b_2(\mu_3+1+c_3)^2|\Omega|}{2\mu_3},\tag{2.15}$$

which gives (2.11) by defining

$$K_2:=\frac{\|v_0\|_{L^1}+b_2\|w_0\|_{L^1}+(\mu_2+1+K)K_1+\frac{b_2(\mu_3+1+c_3)^2|\Omega|}{2\mu_3}}{b_2}.\tag{2.16}$$

Then integrating (2.14) over $(t,t+\tau)$, and using (2.15) and (2.16), we derive that

$$\begin{aligned}\frac{b_2\mu_3}{2}\int_t^{t+\tau}\int_\Omega w^2&\le \int_\Omega (v+b_2w)+\left[(\mu_2+1+K)K_1+\frac{b_2(\mu_3+1+c_3)^2|\Omega|}{2\mu_3}\right]\tau\\ &\le b_2K_2+K_1+(\mu_2+1+K)K_1+\frac{b_2(\mu_3+1+c_3)^2|\Omega|}{2\mu_3},\end{aligned}$$

which gives (2.12) by letting

$$K_3:=\frac{2}{b_2\mu_3}\cdot\left(b_2K_2+(\mu_2+2+K)K_1+\frac{b_2(\mu_3+1+c_3)^2|\Omega|}{2\mu_3}\right).\tag{2.17}$$

The proof of Lemma 2.4 is now completed. ∎

For later use, we list some well-known $L^p$-$L^q$ estimates for the Neumann heat semigroup (cf. [50]).

LEMMA 2.5. *Let $(e^{td\Delta})_{t\geq 0}$ be the Neumann heat semigroup in $\Omega$, and let $\lambda_1 > 0$ denote the first nonzero eigenvalue of $-\Delta$ in $\Omega$ under Neumann boundary conditions, where $d$ is a positive constant. Then for all $t > 0$, there exist some constants $\gamma_i$ $(i = 1, 2, 3, 4)$ depending only on $\Omega$ such that*

(i) *If $2 \leq p < \infty$, then*

$$\|\nabla e^{td\Delta} z\|_{L^p} \leq \gamma_1 (1 + t^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}) e^{-d\lambda_1 t} \|\nabla z\|_{L^q} \tag{2.18}$$

*for all $z \in W^{1,q}(\Omega)$.*

(ii) *If $1 \leq q \leq p \leq \infty$, then*

$$\|\nabla e^{td\Delta} z\|_{L^p} \leq \gamma_2 \left(1 + t^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}\right) e^{-d\lambda_1 t} \|z\|_{L^q} \tag{2.19}$$

*for all $z \in L^q(\Omega)$.*

(iii) *If $1 \leq q \leq p \leq \infty$, then*

$$\|e^{td\Delta} z\|_{L^p} \leq \gamma_3 \left(1 + t^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}\right) \|z\|_{L^q} \tag{2.20}$$

*for all $z \in L^q(\Omega)$.*

(iv) *If $1 < q \leq p \leq \infty$, then*

$$\|e^{td\Delta} \nabla \cdot z\|_{L^p} \leq \gamma_4 \left(1 + t^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}\right) e^{-d\lambda_1 t} \|z\|_{L^q} \tag{2.21}$$

*for all $z \in (C_0^\infty(\Omega))^n$.*

## 3. Proof of Theorem 1.1.

In this section, we shall derive the global a priori estimates of solutions to (1.4) which enable us to extend local solutions to global ones. Since the global stability analysis in the next section requires us to elucidate how the upper bounds of $\|v\|_{L^\infty}$ and $\|w\|_{L^\infty}$ depend on the system parameters, we shall keep the dependencies of relevant estimates on the system parameters for later use although estimates uniformly in time will suffice to obtain the global existence/boundedness of solutions.

### 3.1. Boundedness of $\|v(\cdot,t)\|_{L^\infty}$.

Since the conditions imposed for the global stability of coexistence steady states shown in section 4 depend on $\|v(\cdot,t)\|_{L^\infty}$, we shall detail the dependencies of the upper bound of $\|v(\cdot,t)\|_{L^\infty}$ on the system parameters as transparent as possible for later use.

LEMMA 3.1. *Let the assumptions in Theorem* 1.1 *hold, and let $(u, v, w)$ be the solution of* (1.4). *Then there exist two positive constants $K_4$ and $K_5$ which are independent of $\xi$ and $\chi$ such that*

$$\|\nabla u\|_{L^2} \leq K_4 \quad \text{for all } t \in (0, T_{max}), \tag{3.1}$$

*and*

$$\int_t^{t+\tau} \int_\Omega |\Delta u|^2 \leq K_5 \quad \text{for all } t \in (0, \widetilde{T}_{max}), \tag{3.2}$$

*where $\tau$ and $\widetilde{T}_{max}$ are defined in* (2.8).

*Proof.* Multiplying the first equation of (1.4) by $-\Delta u$ and using the fact that $\|u(\cdot,t)\|_{L^\infty} \le K$ in Lemma 2.2, we end up with

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u|^2 + d_1\int_\Omega |\Delta u|^2 &= -\mu_1\int_\Omega u(1-u)\Delta u + b_1\int_\Omega uv\Delta u + b_3\int_\Omega \frac{uw}{u+w}\Delta u\\ &\le K[\mu_1(1+K)+b_3]\int_\Omega |\Delta u| + b_1K\int_\Omega v|\Delta u|\\ &\le \frac{d_1}{2}\int_\Omega |\Delta u|^2 + \frac{K^2[\mu_1(1+K)+b_3]^2|\Omega|}{d_1} + \frac{b_1^2K^2}{d_1}\int_\Omega v^2,\end{aligned}$$

which gives

$$\frac{d}{dt}\int_\Omega |\nabla u|^2 + d_1\int_\Omega |\Delta u|^2 \le \frac{2b_1^2K^2}{d_1}\int_\Omega v^2 + \frac{2K^2[\mu_1(1+K)+b_3]^2|\Omega|}{d_1}. \tag{3.3}$$

On the other hand, using (2.4) and Young's inequality, we have

$$\int_\Omega |\nabla u|^2 = \int_\Omega \nabla u\cdot\nabla u = -\int_\Omega u\Delta u \le K\int_\Omega |\Delta u| \le \frac{d_1}{4}\int_\Omega |\Delta u|^2 + \frac{K^2}{d_1}|\Omega|,$$

which, substituted into (3.3), gives

$$\begin{aligned}&\frac{d}{dt}\int_\Omega |\nabla u|^2 + \int_\Omega |\nabla u|^2 + \frac{3d_1}{4}\int_\Omega |\Delta u|^2\\ &\quad\le \frac{2b_1^2K^2}{d_1}\int_\Omega v^2 + \frac{K^2\left(2[\mu_1(1+K)+b_3]^2|\Omega| + |\Omega|\right)}{d_1}.\end{aligned} \tag{3.4}$$

Then applying Grönwall's type inequality (e.g., see [44, Lemma 3.4]) and using the fact (2.7), from (3.4), we obtain (3.1) with

$$K_4 := \left[\|\nabla u_0\|_{L^2}^2 + \frac{2\tau+1}{\tau}\cdot\left(\frac{4b_1^2K^2K_1}{d_1\mu_2} + \frac{K^2\left(2[\mu_1(1+K)+b_3]^2|\Omega| + |\Omega|\right)}{d_1}\right)\right]^{\frac{1}{2}}. \tag{3.5}$$

Now we integrate (3.4) over $(t,t+\tau)$, and use (2.7) as well as (3.1) to obtain for all $t\in(0,\widetilde{T}_{max})$ that

$$\begin{aligned}\frac{3d_1}{4}\int_t^{t+\tau}\int_\Omega |\Delta u|^2 &\le \int_\Omega |\nabla u|^2 + \frac{2b_1^2K^2}{d_1}\int_t^{t+\tau}\int_\Omega v^2 + \frac{K^2\left(2[\mu_1(1+K)+b_3]^2|\Omega| + |\Omega|\right)}{d_1}\\ &\le K_4^2 + \frac{8b_1^2K^2K_1}{d_1\mu_2} + \frac{K^2\left(2[\mu_1(1+K)+b_3]^2|\Omega| + |\Omega|\right)}{d_1},\end{aligned}$$

which gives (3.2) with

$$K_5 := \frac{4}{3d_1}\cdot\left(K_4^2 + \frac{8b_1^2K^2K_1}{d_1\mu_2} + \frac{K^2\left(2[\mu_1(1+K)+b_3]^2|\Omega| + |\Omega|\right)}{d_1}\right). \tag{3.6}$$

Then we complete the proof of Lemma 3.1. ∎

LEMMA 3.2. *Let the assumptions in Theorem* 1.1 *hold, and let* $(u,v,w)$ *be the solution of* (1.4). *Then there exists a positive constant* $K_6$ *independent of* $\chi$ *such that*

$$\|v(\cdot,t)\|_{L^2} \le K_6 \quad \text{for all } t\in(0,T_{max}). \tag{3.7}$$

*Proof.* Multiplying the second equation of the system (1.4) by $v$, integrating the result by parts, and appealing to Young's inequality, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega v^2 + d_2\int_\Omega |\nabla v|^2 + \mu_2\int_\Omega v^3 + b_2\int_\Omega v^2 w \\
&= \xi\int_\Omega v\nabla u\cdot\nabla v + \mu_2\int_\Omega v^2 + \int_\Omega uv^2 \\
&\le \frac{d_2}{2}\int_\Omega |\nabla v|^2 + \frac{\xi^2}{2d_2}\int_\Omega v^2|\nabla u|^2 + (K+\mu_2)\int_\Omega v^2 \\
&\le \frac{d_2}{2}\int_\Omega |\nabla v|^2 + \frac{\xi^2}{2d_2}\left(\int_\Omega v^4\right)^{\frac{1}{2}}\left(\int_\Omega |\nabla u|^4\right)^{\frac{1}{2}} + (K+\mu_2)\int_\Omega v^2,
\end{aligned}$$

which gives

$$\frac{d}{dt}\int_\Omega v^2 + d_2\int_\Omega |\nabla v|^2 \le \frac{\xi^2}{d_2}\left(\int_\Omega v^4\right)^{\frac{1}{2}}\left(\int_\Omega |\nabla u|^4\right)^{\frac{1}{2}} + 2(K+\mu_2)\int_\Omega v^2. \tag{3.8}$$

Using the Gagliardo–Nirenberg inequality in two dimensions, we can find two positive constants $C_1$ and $C_2$ such that

$$\|v\|_{L^4}^2 \le C_1(\|\nabla v\|_{L^2}\|v\|_{L^2} + \|v\|_{L^2}^2) \tag{3.9}$$

and

$$\|\nabla u\|_{L^4}^2 \le C_2(\|\Delta u\|_{L^2}\|\nabla u\|_{L^2} + \|\nabla u\|_{L^2}^2), \tag{3.10}$$

where the boundary condition $\frac{\partial u}{\partial \nu}|_{\partial\Omega} = 0$ has been used to obtain (3.10). Then using the fact $\|\nabla u\|_{L^2} \le K_4$ (see Lemma 3.1), (3.9), and (3.10), one has

$$\begin{aligned}
&\frac{\xi^2}{d_2}\left(\int_\Omega v^4\right)^{\frac{1}{2}}\left(\int_\Omega |\nabla u|^4\right)^{\frac{1}{2}} \\
&\le \frac{\xi^2 C_1 C_2}{d_2}(\|\nabla v\|_{L^2}\|v\|_{L^2} + \|v\|_{L^2}^2)(\|\Delta u\|_{L^2}\|\nabla u\|_{L^2} + \|\nabla u\|_{L^2}^2) \\
&\le \frac{\xi^2 C_1 C_2 K_4}{d_2}\|\nabla v\|_{L^2}\|v\|_{L^2}\|\Delta u\|_{L^2} + \frac{\xi^2 C_1 C_2 K_4^2}{d_2}\|\nabla v\|_{L^2}\|\|v\|_{L^2} \\
&\quad + \frac{\xi^2 C_1 C_2 K_4}{d_2}\|v\|_{L^2}^2\|\Delta u\|_{L^2} + \frac{\xi^2 C_1 C_2 K_4^2}{d_2}\|v\|_{L^2}^2 \\
&\le d_2\|\nabla v\|_{L^2}^2 + C_3\|v\|_{L^2}^2\|\Delta u\|_{L^2}^2 + C_4\|v\|_{L^2}^2
\end{aligned} \tag{3.11}$$

with $C_3 = \frac{(\xi^2 C_1 C_2 K_4)^2(1+d_2^2)}{2d_2^3}$ and $C_4 = \frac{(\xi^2 C_1 C_2 K_4^2)^2 + d_2^2 + 2d_2^2\xi^2 C_1 C_2 K_4^2}{2d_2^3}$. Substituting (3.11) into (3.8) and using Young's inequality, we derive for all $t \in (0, T_{max})$ that

$$\frac{d}{dt}\|v\|_{L^2}^2 \le C_3\|v\|_{L^2}^2\|\Delta u\|_{L^2}^2 + [C_4 + 2(K+\mu_2)]\,\|v\|_{L^2}^2. \tag{3.12}$$

By virtue of (2.7), there exits a constant $t_0 \in [(t-\tau)_+, t)$ for any $t \in (0, T_{max})$ such that

$$\|v(\cdot, t_0)\|_{L^2}^2 \le C_5 := \max\left\{\|v_0\|_{L^2}^2, \frac{4K_1}{\mu_2\tau}\right\} \tag{3.13}$$

in both cases $t \in (0,\tau)$ and $t \geq \tau$, where $\tau$ is defined in (2.8). We can also derive from (3.2) that

$$\int_{t_0}^{t_0+\tau}\int_\Omega |\Delta u(\cdot,s)|^2 \leq K_5. \tag{3.14}$$

Noticing the fact $t_0 < t \leq t_0+\tau \leq t_0+1$, we integrate (3.12) over $(t_0,t)$ alongside (3.13)–(3.14) and get

$$\|v(\cdot,t)\|_{L^2}^2 \leq \|v(\cdot,t_0)\|_{L^2}^2 e^{C_3\int_{t_0}^t \|\Delta u(\cdot,s)\|_{L^2}^2 ds + C_4 + 2(K+\mu_2)} \leq C_5 e^{C_3K_5+C_4+2(K+\mu_2)}$$

for all $t \in (0,T_{max})$, which gives (3.7) by defining

$$K_6 := C_5^{\frac{1}{2}} e^{\frac{C_3K_5+C_4+2(K+\mu_2)}{2}}. \tag{3.15}$$

This completes the proof. ∎

LEMMA 3.3. *Let the assumptions in Theorem* 1.1 *hold, and let* $(u,v,w)$ *be the solution of the system* (1.4). *Then there exists a positive constant* $K_7$ *independent of* $\chi$ *such that*

$$\|\nabla u(\cdot,t)\|_{L^4} \leq K_7 \quad \text{for all } t \in (0,T_{max}). \tag{3.16}$$

*Proof.* To begin with, we rewrite the first equation of (1.4) as follows:

$$u_t - d_1\Delta u + u = F(x,t) \tag{3.17}$$

with $F(x,t) := \mu_1 u(1-u) - b_1uv - b_3\frac{uw}{u+w} + u$. Then by (2.4) and (3.7), for all $t \in (0,T_{max})$, one has

$$\begin{aligned} \|F(\cdot,t)\|_{L^2} &= \left\|\mu_1u(1-u) - b_1uv - b_3\frac{uw}{u+w} + u\right\|_{L^2} \\ &\leq \mu_1K(1+K)|\Omega|^{\frac{1}{2}} + b_1K\|v\|_{L^2} + b_3K|\Omega|^{\frac{1}{2}} + K|\Omega|^{\frac{1}{2}} \\ &\leq [\mu_1(1+K)+b_3+1]K|\Omega|^{\frac{1}{2}} + b_1KK_6. \end{aligned} \tag{3.18}$$

We apply the variation-of-constants formula to (3.17) and obtain

$$u(\cdot,t) = e^{(d_1\Delta-1)t}u_0 + \int_0^t e^{(d_1\Delta-1)(t-s)}F(\cdot,s)ds. \tag{3.19}$$

Then using the estimates (2.18) and (2.19), one can derive from (3.19) that

$$\begin{aligned} &\|\nabla u(\cdot,t)\|_{L^4} \\ &\leq \|\nabla e^{(d_1\Delta-1)t}u_0\|_{L^4} + \int_0^t \|\nabla e^{(d_1\Delta-1)(t-s)}F(\cdot,s)\|_{L^4}ds \\ &\leq e^{-t}\|\nabla e^{td_1\Delta}u_0\|_{L^4} + \int_0^t \|\nabla e^{(t-s)d_1\Delta}F(\cdot,s)\|_{L^4}ds \\ &\leq 2\gamma_1 e^{-d_1\lambda_1 t}\|\nabla u_0\|_{L^4} + \gamma_2\int_0^t \left(1+(t-s)^{-\frac{3}{4}}\right)e^{-d_1\lambda_1(t-s)}\|F(\cdot,s)\|_{L^2}ds \\ &\leq 2\gamma_1\|\nabla u_0\|_{L^4} + \gamma_2\left([\mu_1(1+K)+b_3+1]K|\Omega|^{\frac{1}{2}} + b_1KK_6\right)\int_0^\infty \left(1+z^{-\frac{3}{4}}\right)e^{-d_1\lambda_1 z}dz \\ &\leq 2\gamma_1\|\nabla u_0\|_{L^4} + \frac{\gamma_2}{d_1\lambda_1}\left([\mu_1(1+K)+b_3+1]K|\Omega|^{\frac{1}{2}} + b_1KK_6\right)\left(1+\Gamma(1/4)(d_1\lambda_1)^{\frac{3}{4}}\right), \end{aligned}$$

where $\Gamma$ denotes the usual Gamma function defined by $\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}dt$. This gives (3.16) with

$$K_7 := 2\gamma_1\|\nabla u_0\|_{L^4}+\frac{\gamma_2}{d_1\lambda_1}\big([\mu_1(1+K)+b_3+1]K|\Omega|^{\frac{1}{2}}+b_1KK_6\big)\big(1+\Gamma(1/4)(d_1\lambda_1)^{\frac{3}{4}}\big), \tag{3.20}$$

and completes the proof of this lemma. ∎

LEMMA 3.4. *Let the assumptions in Theorem* 1.1 *hold, and let* $(u,v,w)$ *be the solution of* (1.4). *Then there exists a positive constant* $K_8$ *independent of* $\chi$ *such that*

$$\|v(\cdot,t)\|_{L^3}\le K_8 \quad \text{for all } t\in(0,T_{max}). \tag{3.21}$$

*Proof.* Multiplying the second equation of (1.4) by $v^2$ and integrating the result over $\Omega$ along with (2.4) and (3.16), we get

$$\begin{aligned}
&\frac{1}{3}\frac{d}{dt}\int_\Omega v^3+2d_2\int_\Omega v|\nabla v|^2+\mu_2\int_\Omega v^4\\
&=2\xi\int_\Omega v^2\nabla u\cdot\nabla v+\mu_2\int_\Omega v^3+\int_\Omega uv^3-b_2\int_\Omega v^3w\\
&\le d_2\int_\Omega v|\nabla v|^2+\frac{\xi^2}{d_2}\int_\Omega v^3|\nabla u|^2+(\mu_2+K)\int_\Omega v^3\\
&\le d_2\int_\Omega v|\nabla v|^2+\frac{\xi^2}{d_2}\left(\int_\Omega v^6\right)^{\frac{1}{2}}\left(\int_\Omega|\nabla u|^4\right)^{\frac{1}{2}}+(\mu_2+K)\int_\Omega v^3\\
&\le d_2\int_\Omega v|\nabla v|^2+\frac{K_7^2\xi^2}{d_2}\|v\|^3_{L^6}+(\mu_2+K)\|v\|^3_{L^3},
\end{aligned}$$

which implies

$$\frac{d}{dt}\|v\|^3_{L^3}+\frac{4d_2}{3}\|\nabla v^{\frac{3}{2}}\|^2_{L^2}+3\mu_2\|v\|^4_{L^4}\le\frac{3K_7^2\xi^2}{d_2}\|v\|^3_{L^6}+3(\mu_2+K)\|v\|^3_{L^3}. \tag{3.22}$$

From (3.7), it follows that $\|v^{\frac{3}{2}}(\cdot,t)\|_{L^{\frac{4}{3}}}=\|v(\cdot,t)\|^{\frac{3}{2}}_{L^2}\le K_6^{\frac{3}{2}}$. Then applying the Gagliardo–Nirenberg inequality and Young's inequality, we can derive that

$$\begin{aligned}
\frac{3K_7^2\xi^2}{d_2}\|v\|^3_{L^6}&=\frac{3K_7^2\xi^2}{d_2}\|v^{\frac{3}{2}}\|^2_{L^4}\\
&\le\frac{3K_7^2\xi^2C_1}{d_2}\left(\|\nabla v^{\frac{3}{2}}\|^{\frac{4}{3}}_{L^2}\|v^{\frac{3}{2}}\|^{\frac{2}{3}}_{L^{\frac{4}{3}}}+\|v^{\frac{3}{2}}\|^2_{L^{\frac{4}{3}}}\right)\\
&\le\frac{3K_7^2\xi^2C_1K_6}{d_2}\|\nabla v^{\frac{3}{2}}\|^{\frac{4}{3}}_{L^2}+\frac{3K_7^2\xi^2C_1K_6^3}{d_2}\\
&\le\frac{4d_2}{3}\|\nabla v^{\frac{3}{2}}\|^2_{L^2}+C_2,
\end{aligned} \tag{3.23}$$

where $C_2=\frac{9(K_7^2\xi^2C_1K_6)^3}{4d_2^5}+\frac{3K_7^2\xi^2C_1K_6^3}{d_2}$. On the other hand, using Young's inequality, one has

$$[3(\mu_2+K)+1]\|v\|^3_{L^3}\le 3\mu_2\|v\|^4_{L^4}+C_3, \tag{3.24}$$

where $C_3=4^{-4}\mu_2^{-3}[3(\mu_2+K)+1]^4|\Omega|$. Then adding $\|v\|^3_{L^3}$ on both sides of (3.22) and substituting (3.23)–(3.24) into the resulting inequality, we obtain

$$\frac{d}{dt}\|v\|^3_{L^3}+\|v\|^3_{L^3}\le C_2+C_3,$$

which immediately gives (3.21) with $K_8:=\left(\|v_0\|^3_{L^3}+C_2+C_3\right)^{\frac{1}{3}}$. This completes the proof of Lemma 3.4. ∎

LEMMA 3.5. *Let the assumptions in Theorem* 1.1 *hold, and let* $(u,v,w)$ *be the solution of system* (1.4). *Then there exists a positive constant* $K_9$ *independent of* $\chi$ *such that*

$$\|v(\cdot,t)\|_{L^\infty} \le K_9 \quad \text{for all } t\in(0,T_{max}). \tag{3.25}$$

*Proof.* Using Lemma 3.4 and the fact $0<u\le K$ (see Lemma 2.2), one has

$$\begin{aligned} \|F(\cdot,t)\|_{L^3} &= \left\|\mu_1 u(1-u) - b_1 uv - b_3\frac{uw}{u+w} + u\right\|_{L^3} \\ &\le \mu_1 K(1+K)|\Omega|^{\frac13} + b_1K\|v\|_{L^3} + b_3K|\Omega|^{\frac13} + K|\Omega|^{\frac13} \\ &\le [\mu_1(1+K)+b_3+1]K|\Omega|^{\frac13} + b_1KK_8. \end{aligned} \tag{3.26}$$

Taking $\nabla$ on both sides of (3.19) and using $L^p$-$L^q$ estimates (2.19), we have from (3.26) that

$$\begin{aligned} &\|\nabla u(\cdot,t)\|_{L^\infty} \\ &\le \|\nabla e^{(d_1\Delta-1)t}u_0\|_{L^\infty} + \int_0^t \|\nabla e^{(d_1\Delta-1)(t-s)}F(\cdot,s)\|_{L^\infty}ds \\ &\le c_1\|u_0\|_{W^{1,\infty}} + \int_0^t \|\nabla e^{(t-s)d_1\Delta}F(\cdot,s)\|_{L^\infty}ds \\ &\le c_1\|u_0\|_{W^{1,\infty}} + \gamma_2\int_0^t \left(1+(t-s)^{-\frac56}\right)e^{-d_1\lambda_1(t-s)}\|F(\cdot,s)\|_{L^3}ds \\ &\le c_1\|u_0\|_{W^{1,\infty}} + \gamma_2\left([\mu_1(1+K)+b_3+1]K|\Omega|^{\frac13} + b_1KK_8\right)\int_0^\infty \left(1+z^{-\frac56}\right)e^{-d_1\lambda_1 z}dz \\ &\le c_1\|u_0\|_{W^{1,\infty}} + \frac{\gamma_2}{d_1\lambda_1}\left([\mu_1(1+K)+b_3+1]K|\Omega|^{\frac13} + b_1KK_8\right)\left(1+(d_1\lambda_1)^{\frac56}\Gamma(1/6)\right), \end{aligned}$$

which gives

$$\|\nabla u(\cdot,t)\|_{L^\infty} \le K_8^* \tag{3.27}$$

with

$$\begin{aligned} K_8^* &:= c_1\|u_0\|_{W^{1,\infty}} \\ &\quad + \frac{\gamma_2}{d_1\lambda_1}\left([\mu_1(1+K)+b_3+1]K|\Omega|^{\frac13} + b_1KK_8\right)\left(1+(d_1\lambda_1)^{\frac56}\Gamma(1/6)\right). \end{aligned}$$

We rewrite the second equation of (1.4) as follows:

$$v_t - d_2\Delta v + v = -\xi\nabla\cdot(v\nabla u) + \mu_2 v(1-v) + uv - b_2vw + v. \tag{3.28}$$

Then applying the variation-of-constants formula to (3.28), one has

$$\begin{aligned} v(\cdot,t) &= e^{(d_2\Delta-1)t}v_0 - \xi\int_0^t e^{(d_2\Delta-1)(t-s)}\nabla\cdot(v\nabla u)(\cdot,s)ds \\ &\quad + \int_0^t e^{(d_2\Delta-1)(t-s)}[v(\mu_2+1-\mu_2 v+u-b_2w)](\cdot,s)ds \\ &\le e^{(d_2\Delta-1)t}v_0 - \xi\int_0^t e^{(d_2\Delta-1)(t-s)}\nabla\cdot(v\nabla u)(\cdot,s)ds \\ &\quad + \int_0^t e^{(d_2\Delta-1)(t-s)}[v(\mu_2+1+u)](\cdot,s)ds, \end{aligned}$$

which implies

$$\|v(\cdot,t)\|_{L^\infty} \le \|e^{(d_2\Delta-1)t}v_0\|_{L^\infty} + \xi\int_0^t \|e^{(d_2\Delta-1)(t-s)}\nabla\cdot(v\nabla u)(\cdot,s)\|_{L^\infty}ds + \int_0^t \|e^{(d_2\Delta-1)(t-s)}[v(\mu_2+1+u)](\cdot,s)\|_{L^\infty}ds. \tag{3.29}$$

Using (2.20) and (3.21), we have

$$\|e^{(d_2\Delta-1)t}v_0\|_{L^\infty} \le 2\gamma_3\|v_0\|_{L^\infty} \tag{3.30}$$

and

$$\begin{aligned}&\int_0^t \|e^{(d_2\Delta-1)(t-s)}[v(\mu_2+1+u)](\cdot,s)\|_{L^\infty}ds\\ &\le(\mu_2+1+K)\gamma_3\int_0^t(1+(t-s))^{-\frac{1}{3}}e^{-(t-s)}\|v(\cdot,s)\|_{L^3}ds\\ &\le(\mu_2+1+K)K_8\gamma_3\int_0^{+\infty}(1+z^{-\frac{1}{3}})e^{-z}dz\\ &=(\mu_2+1+K)K_8\gamma_3[1+\Gamma(2/3)].\end{aligned} \tag{3.31}$$

On the other hand, using the $L^p$-$L^q$ estimate (2.21) alongside the fact that $C_0^\infty(\Omega)$ is dense in $L^p(\Omega)$ for any $1\le p<\infty$ and estimates (3.21) and (3.27), we have

$$\begin{aligned}&\xi\int_0^t \|e^{(d_2\Delta-1)(t-s)}\nabla\cdot(v\nabla u)(\cdot,s)\|_{L^\infty}ds\\ &\le\xi\gamma_4\int_0^t(1+(t-s)^{-\frac{5}{6}})e^{-d_2\lambda_1 t}\|v(\cdot,s)\|_{L^3}\|\nabla u(\cdot,s)\|_{L^\infty}ds\\ &\le\xi\gamma_4K_8K_8^*\int_0^\infty(1+z^{-\frac{5}{6}})e^{-d_2\lambda_1 z}dz\\ &=\frac{\xi\gamma_4K_8K_8^*}{d_2\lambda_1}[1+\Gamma(1/6)(d_2\lambda_1)^{\frac{5}{6}}].\end{aligned} \tag{3.32}$$

Substituting (3.30), (3.31), and (3.32) into (3.29), one has (3.25) with

$$\begin{aligned}K_9 :=\ &2\gamma_3\|v_0\|_{L^\infty}+(\mu_2+1+K)K_8\gamma_3[1+\Gamma(2/3)]\\ &+\frac{\xi\gamma_4K_8K_8^*}{d_2\lambda_1}\left(1+\Gamma(1/6)(d_2\lambda_1)^{\frac{5}{6}}\right).\end{aligned}$$

∎

*Remark* 3.6. From the definitions of $K_4, K_5$, and $K_7$, we see that $\|v(\cdot,t)\|_{L^\infty}\le K_9\le c_0(1+b_1+b_1^2)$ with some constant $c_0>0$ independent of $b_1$. This result will be used later in the stability analysis in section 4.

**3.2. Boundedness of $\|w(\cdot,t)\|_{L^\infty}$.** To obtain uniform-in-time boundedness of $\|w(\cdot,t)\|_{L^\infty}$, we first give some higher order derivative estimates of $u, v$ below.

LEMMA 3.7. *Let the assumptions in Theorem* 1.1 *hold, and let* $(u,v,w)$ *be the solution of* (1.4). *Then there exists a constant* $K_{10}>0$ *independent of* $t$ *such that for all* $p>1$

$$\int_t^{t+\tau}\|D^2u\|_{L^p}^p\le K_{10}\quad\text{for all } t\in(0,\widetilde{T}_{max}) \tag{3.33}$$

*and*

$$\int_\tau^t e^{-p(t-s)}\|\Delta u\|_{L^p}^p \le K_{10} \quad \text{for all } t\in(\tau, T_{max}), \tag{3.34}$$

*where* $\tau$ *and* $\widetilde{T}_{max}$ *are defined in* (2.8).

*Proof.* We rewrite the first equation of (1.4) as follows:

$$\begin{cases} u_t - d_1\Delta u + u = F(x,t), & x\in\Omega, t>0,\\ \frac{\partial u}{\partial \nu} = 0, & x\in\partial\Omega, t>0,\\ u(x,0)=u_0(x), & x\in\Omega, \end{cases}$$

where

$$F(x,t) := \mu_1 u(1-u) + u - b_1 uv - \frac{b_3 uw}{u+w}.$$

Then by the boundedness of $|F(x,t)|$(due to the boundedness of $u$ and $v$; see (2.4) and (3.25)), we can obtain (3.33) by [30, Lemma 2.3]. Moreover (3.34) follows from the maximal Sobolev regularity property; see [15, Lemma 2.5]. ∎

Next, we shall show the following lemma.

LEMMA 3.8. *Let the assumptions in Theorem* 1.1 *hold, and let* $(u,v,w)$ *be the solution of* (1.4). *Then there exist two positive constants* $K_{11}$ *and* $K_{12}$ *independent of* $t$ *such that*

$$\|\nabla v(\cdot,t)\|_{L^2} \le K_{11} \quad \text{for all } t\in(0,T_{max}) \tag{3.35}$$

*and*

$$\int_t^{t+\tau} \|\Delta v\|_{L^2}^2 \le K_{12} \quad \text{for all } t\in(0,\widetilde{T}_{max}), \tag{3.36}$$

*where* $\tau$ *and* $\widetilde{T}_{max}$ *are defined in* (2.8).

*Proof.* Multiplying the second equation of (1.4) by $-\Delta v$ and integrating the result by parts, we have

$$\begin{aligned} &\frac{d}{dt}\int_\Omega |\nabla v|^2 + d_2\int_\Omega |\Delta v|^2 \\ &= \xi\int_\Omega \nabla\cdot(v\nabla u)\Delta v - \mu_2\int_\Omega v\Delta v + \mu_2\int_\Omega v^2\Delta v - \int_\Omega uv\Delta v + b_2\int_\Omega vw\Delta v. \end{aligned} \tag{3.37}$$

Noting the facts (2.4), (3.25), and (3.27), and using Young's inequality, we can derive that

$$\begin{aligned} \xi\int_\Omega \nabla\cdot(v\nabla u)\Delta v &= \xi\int_\Omega \nabla v\cdot\nabla u\Delta v + \xi\int_\Omega v\Delta u\Delta v\\ &\le \xi K_8^*\int_\Omega |\nabla v||\Delta v| + \xi K_9\int_\Omega |\Delta u||\Delta v|\\ &\le \frac{d_2}{2}\int_\Omega |\Delta v|^2 + \frac{\xi^2(K_8^*)^2}{d_2}\int_\Omega |\nabla v|^2 + \frac{\xi^2 K_9^2}{d_2}\int_\Omega |\Delta u|^2 \end{aligned} \tag{3.38}$$

and

$$\begin{aligned}
&-\mu_2\int_\Omega v\Delta v+\mu_2\int_\Omega v^2\Delta v-\int_\Omega uv\Delta v+b_2\int_\Omega vw\Delta v\\
&\le(\mu_2K_9+\mu_2K_9^2+KK_9)\int_\Omega|\Delta v|+b_2K_9\int_\Omega w|\Delta v|\\
&\le\frac{d_2}{4}\int_\Omega|\Delta v|^2+\frac{2b_2^2K_9^2}{d_2}\int_\Omega w^2+C_1,
\end{aligned}\tag{3.39}$$

with $C_1:=\frac{2(\mu_2K_9+\mu_2K_9^2+KK_9)^2|\Omega|}{d_2}$. On the other hand, Young's inequality gives us

$$\begin{aligned}
\left(\frac{\xi^2(K_8^*)^2}{d_2}+1\right)\int_\Omega|\nabla v|^2&=-\left(\frac{\xi^2(K_8^*)^2}{d_2}+1\right)\int_\Omega v\Delta v\\
&\le\left(\frac{\xi^2(K_8^*)^2}{d_2}+1\right)K_9\int_\Omega|\Delta v|\\
&\le\frac{d_2}{8}\int_\Omega|\Delta v|^2+C_2
\end{aligned}\tag{3.40}$$

with $C_2:=\frac{2(\xi^2(K_8^*)^2+d_2)^2K_9^2|\Omega|}{d_2^3}$. Substituting (3.38), (3.39), and (3.40) into (3.37), one has

$$\frac{d}{dt}\int_\Omega|\nabla v|^2+\int_\Omega|\nabla v|^2+\frac{d_2}{8}\int_\Omega|\Delta v|^2\le\frac{\xi^2K_9^2}{d_2}\int_\Omega|\Delta u|^2+\frac{2b_2^2K_9^2}{d_2}\int_\Omega w^2+C_1+C_2.\tag{3.41}$$

Using (2.12) and (3.2), and applying Grönwall's type inequality (cf. [44, Lemma 3.4]) to (3.41), one gets (3.35) directly. Then integrating (3.41) over $(t,t+\tau)$, we obtain

$$\frac{d_2}{8}\int_t^{t+\tau}\int_\Omega|\Delta v|^2\le\int_\Omega|\nabla v|^2+\frac{\xi^2K_9^2}{d_2}\int_t^{t+\tau}\int_\Omega|\Delta u|^2+\frac{2b_2^2K_9^2}{d_2}\int_t^{t+\tau}\int_\Omega w^2+(C_1+C_2)\tau,$$

which gives (3.36) due to (2.12) and (3.2). Then we complete the proof of this lemma. ∎

Next, we give the estimate of $\|w(\cdot,t)\|_{L^2}$.

LEMMA 3.9. *Let the assumptions in Theorem* 1.1 *hold, and let* $(u,v,w)$ *be the solution of* (1.4). *Then one can find two positive constants* $K_{13}$ *and* $K_{14}$ *independent of* $t$ *such that*

$$\|w(\cdot,t)\|_{L^2}\le K_{13}\quad\text{for all } t\in(0,T_{max})\tag{3.42}$$

*and*

$$\int_t^{t+\tau}\|\nabla w(\cdot,s)\|_{L^2}^2+\int_t^{t+\tau}\|w(\cdot,s)\|_{L^3}^3\le K_{14}\quad\text{for all } t\in(0,\widetilde{T}_{max}),\tag{3.43}$$

*where* $\tau$ *and* $\widetilde{T}_{max}$ *are defined in* (2.8).

*Proof.* Multiplying the third equation of (1.4) by $w$ followed by an integration by parts, one has

$$\begin{aligned}
&\frac12\frac{d}{dt}\int_\Omega w^2+\int_\Omega|\nabla w|^2+\mu_3\int_\Omega w^3\\
&\quad=\chi\int_\Omega(wv\nabla u+wu\nabla v)\cdot\nabla w+\mu_3\int_\Omega w^2+\int_\Omega vw^2+c_3\int_\Omega\frac{uw^2}{w+u}.
\end{aligned}\tag{3.44}$$

Since $\|u(\cdot,t)\|_{L^\infty} \le K$ (see Lemma 2.2), $\|v(\cdot,t)\|_{L^\infty} \le K_9$ (see Lemma 3.5) and $\|\nabla u(\cdot,t)\|_{L^\infty} \le K_8^*$ (see (3.27)), one can use Young's inequality to obtain

$$\begin{aligned}\chi\int_\Omega (wv\nabla u + wu\nabla v)\cdot\nabla w &\le \chi K_8^* K_9\int_\Omega w|\nabla w| + \chi K\int_\Omega w|\nabla v||\nabla w| \\ &\le \frac{1}{2}\int_\Omega |\nabla w|^2 + \chi^2 (K_8^*)^2 K_9^2\int_\Omega w^2 + \chi^2 K^2\int_\Omega w^2|\nabla v|^2\end{aligned} \tag{3.45}$$

and

$$\mu_3\int_\Omega w^2 + \int_\Omega vw^2 + c_3\int_\Omega \frac{uw^2}{w+u} \le (\mu_3 + K_9 + c_3)\int_\Omega w^2. \tag{3.46}$$

Then substituting (3.45) and (3.46) into (3.44), we obtain

$$\frac{d}{dt}\int_\Omega w^2 + \int_\Omega |\nabla w|^2 + 2\mu_3\int_\Omega w^3 \le 2\chi^2 K^2\int_\Omega w^2|\nabla v|^2 + C_1\int_\Omega w^2, \tag{3.47}$$

where $C_1 := 2[\chi^2 (K_8^*)^2 K_9^2 + \mu_3 + K_9 + c_3]$.

Using the Gagliardo–Nirenberg inequality in two dimensional spaces, one can find two positive constants $C_2$ and $C_3$ such that

$$\|w\|_{L^4}^2 \le C_2\left(\|\nabla w\|_{L^2}\|w\|_{L^2} + \|w\|_{L^2}^2\right) \tag{3.48}$$

and

$$\|\nabla v\|_{L^4}^2 \le C_3\left(\|\Delta v\|_{L^2}\|\nabla v\|_{L^2} + \|\nabla v\|_{L^2}^2\right) \le C_3 K_{11}(\|\Delta v\|_{L^2} + K_{11}), \tag{3.49}$$

where we have used the fact $\|\nabla v(\cdot,t)\|_{L^2} \le K_{11}$ in (3.35) and a result in [10, Lemma 1] for Neumann boundary conditions. Then we can use (3.48) and (3.49) alongside Young's inequality to obtain

$$\begin{aligned}2\chi^2 K^2\int_\Omega w^2|\nabla v|^2 &\le 2\chi^2 K^2\|w\|_{L^4}^2\|\nabla v\|_{L^4}^2 \\ &\le C_4\left(\|\nabla w\|_{L^2}\|w\|_{L^2} + \|w\|_{L^2}^2\right)(\|\Delta v\|_{L^2} + K_{11}) \\ &\le C_4\|\nabla w\|_{L^2}\|w\|_{L^2}\|\Delta v\|_{L^2} + C_4 K_{11}\|\nabla w\|_{L^2}\|w\|_{L^2} \\ &\quad + C_4\|w\|_{L^2}^2\|\Delta v\|_{L^2} + C_4 K_{11}\|w\|_{L^2}^2 \\ &\le \frac{1}{2}\|\nabla w\|_{L^2}^2 + 2C_4^2\|w\|_{L^2}^2\|\Delta v\|_{L^2}^2 + \frac{1+4C_4^2K_{11}^2}{4}\|w\|_{L^2}^2,\end{aligned} \tag{3.50}$$

where $C_4 = 2\chi^2 K^2 K_{11} C_2 C_3$. Then substituting (3.50) into (3.47) yields

$$\begin{aligned}\frac{d}{dt}\|w\|_{L^2}^2 + \frac{1}{2}\|\nabla w\|_{L^2}^2 + 2\mu_3\|w\|_{L^3}^3 &\le 2C_4^2\|w\|_{L^2}^2\|\Delta v\|_{L^2}^2 + \frac{1+4C_4^2K_{11}^2+4C_1}{4}\|w\|_{L^2}^2 \\ &\le C_5\|w\|_{L^2}^2(\|\Delta v\|_{L^2}^2 + 1)\end{aligned} \tag{3.51}$$

with $C_5 := \frac{1+4C_4^2K_{11}^2+4C_1+8C_4^2}{4}$.

Since $\int_t^{t+\tau}\|w\|_{L^2}^2 \le K_3$ (see (2.12)) and $\int_t^{t+\tau}\|\Delta v\|_{L^2}^2 \le K_{12}$ (see (3.36)), by using the similar argument as in Lemma 3.2, we can obtain from (3.51) that

$$\|w(\cdot,t)\|_{L^2}^2 \le K_3 e^{C_5 K_{12}+C_5},$$

which gives (3.42) with $K_{13} = \left(K_3 e^{C_5K_{12}+C_5}\right)^{\frac{1}{2}}$.

Integrating (3.51) over $(t, t+\tau)$, using (3.42) and $\int_t^{t+\tau} \|\Delta v\|_{L^2}^2 \le K_{12}$ (see (3.36)), one has

$$\begin{aligned}\frac{1}{2}\int_t^{t+\tau} \|\nabla w\|_{L^2}^2 + 2\mu_3 \int_t^{t+\tau} \|w\|_{L^3}^3 &\le \|w\|_{L^2}^2 + C_5 \int_t^{t+\tau} \|w\|_{L^2}^2(\|\Delta v\|_{L^2}^2 + 1)\\ &\le K_{13}^2 + C_5 K_{13}^2 (K_{12}+1) := K_{14},\end{aligned}$$

which gives (3.43). □

LEMMA 3.10. *Let the assumptions in Theorem* 1.1 *hold, and let* $(u,v,w)$ *be the solution of system* (1.4). *Then there exists a constant* $K_{15} > 0$ *independent of* $t$ *such that*

$$\|\nabla v(\cdot,t)\|_{L^4} \le K_{15} \quad \text{for all } t \in (0, T_{max}). \tag{3.52}$$

*Proof.* By the second equation of (1.4), one has

$$\begin{aligned}\frac{1}{4}\frac{d}{dt}\int_\Omega |\nabla v|^4 &= \int_\Omega |\nabla v|^2 \nabla v\cdot\nabla v_t\\ &= d_2\int_\Omega |\nabla v|^2\nabla v\cdot\nabla\Delta v - \xi\int_\Omega |\nabla v|^2 \nabla v\cdot\nabla(\nabla\cdot(v\nabla u))\\ &\quad + \int_\Omega \nabla[\mu_2 v(1-v) + uv - b_2 vw]\cdot\nabla v|\nabla v|^2\\ &:= I_1 + I_2 + I_3.\end{aligned} \tag{3.53}$$

Since $\nabla\Delta v\cdot\nabla v = \frac{1}{2}\Delta|\nabla v|^2 - |D^2 v|^2$, we can estimate the term $I_1$ as follows:

$$\begin{aligned} I_1 &= d_2\int_\Omega |\nabla v|^2\nabla v\cdot\nabla\Delta v\\ &= \frac{d_2}{2}\int_\Omega |\nabla v|^2\Delta|\nabla v|^2 - d_2\int_\Omega |\nabla v|^2|D^2v|^2\\ &= \frac{d_2}{2}\int_{\partial\Omega}|\nabla v|^2\frac{\partial|\nabla v|^2}{\partial\nu}dS - \frac{d_2}{2}\int_\Omega |\nabla|\nabla v|^2|^2 - d_2\int_\Omega|\nabla v|^2|D^2 v|^2.\end{aligned} \tag{3.54}$$

Using the facts $\|u(\cdot,t)\|_{L^\infty} \le K$ (see Lemma 2.2), $\|v(\cdot,t)\|_{L^\infty} \le K_9$ (see Lemma 3.5), and $\|\nabla u(\cdot,t)\|_{L^\infty} \le K_8^*$ (see (3.27)), one can estimate terms $I_2$ and $I_3$ as follows:

(3.55)

$$\begin{aligned} I_2 &= -\xi\int_\Omega |\nabla v|^2\nabla v\cdot\nabla(\nabla\cdot(v\nabla u))\\ &= \xi\int_\Omega \nabla|\nabla v|^2\cdot\nabla v\nabla\cdot(v\nabla u) + \xi\int_\Omega |\nabla v|^2\Delta v\nabla\cdot(v\nabla u)\\ &\le \xi\int_\Omega |\nabla v||\nabla|\nabla v|^2|(K_8^*|\nabla v| + K_9|\Delta u|) + \sqrt{2}\xi\int_\Omega |\nabla v|^2|D^2 v|(K_8^*|\nabla v| + K_9|\Delta u|)\\ &\le \frac{d_2}{4}\int_\Omega|\nabla|\nabla v|^2|^2 + \frac{d_2}{2}\int_\Omega |\nabla v|^2|D^2v|^2 + \frac{4\xi^2(K_8^*)^2}{d_2}\int_\Omega|\nabla v|^4 + \frac{4\xi^2K_9^2}{d_2}\int_\Omega|\nabla v|^2|\Delta u|^2\\ &\le \frac{d_2}{4}\int_\Omega|\nabla|\nabla v|^2|^2 + \frac{d_2}{2}\int_\Omega |\nabla v|^2|D^2v|^2 + \frac{4\xi^2[(K_8^*)^2+K_9^2]}{d_2}\int_\Omega|\nabla v|^4 + \frac{\xi^2K_9^2}{d_2}\int_\Omega|\Delta u|^4\end{aligned}$$

and

$$
\begin{aligned}
I_3 &= \int_\Omega \nabla(\mu_2 v(1-v) + uv - b_2 vw)\cdot\nabla v|\nabla v|^2 \\
&= \mu_2\int_\Omega |\nabla v|^4 - 2\mu_2\int_\Omega v|\nabla v|^4 + \int_\Omega v\nabla u\cdot\nabla v|\nabla v|^2 + \int_\Omega u|\nabla v|^4 \\
&\quad - b_2\int_\Omega w|\nabla v|^4 - b_2\int_\Omega v|\nabla v|^2\nabla w\cdot\nabla v \\
&\le (\mu_2+K)\int_\Omega |\nabla v|^4 + K_8^* K_9\int_\Omega |\nabla v|^3 + b_2K_9\int_\Omega |\nabla v|^3|\nabla w|.
\end{aligned} \tag{3.56}
$$

Substituting (3.54)–(3.56) into (3.53), and using Young's inequality, we end up with

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega |\nabla v|^4 + d_2\int_\Omega |\nabla|\nabla v|^2|^2 + 2d_2\int_\Omega |\nabla v|^2|D^2 v|^2 \\
&\le 2d_2\int_{\partial\Omega} |\nabla v|^2\frac{\partial|\nabla v|^2}{\partial\nu}dS + C_1\int_\Omega |\nabla v|^4 + \frac{4\xi^2K_9^2}{d_2}\int_\Omega |\Delta u|^4 \\
&\quad + 4b_2K_9\int_\Omega |\nabla v|^3|\nabla w| + 4K_8^*K_9\int_\Omega |\nabla v|^3,
\end{aligned} \tag{3.57}
$$

where $C_1 = \frac{16\xi^2[(K_8^*)^2+K_9^2]}{d_2} + 4(\mu_2+K)$. With the inequality $\frac{\partial|\nabla v|^2}{\partial\nu} \le 2\sigma|\nabla v|^2$ on $\partial\Omega$ for some constant $\sigma > 0$ (see [39, Lemma 4.2]), and the trace inequality $\|\varphi\|_{L^2(\partial\Omega)} \le \varepsilon\|\nabla\varphi\|_{L^2(\Omega)} + C_\varepsilon\|\varphi\|_{L^2(\Omega)}$ for any $\varepsilon > 0$ (see [42, Remark 52.9]), we derive

$$
2d_2\int_{\partial\Omega} |\nabla v|^2\frac{\partial|\nabla v|^2}{\partial\nu}dS \le 4\sigma d_2\int_{\partial\Omega}|\nabla v|^4 dS \le \frac{d_2}{2}\int_\Omega |\nabla|\nabla v|^2|^2 + C_2\int_\Omega |\nabla v|^4. \tag{3.58}
$$

On the other hand, using Young's inequality, we have

$$
4K_8^*K_9\int_\Omega |\nabla v|^3 \le C_1\int_\Omega |\nabla v|^4 + C_3 \tag{3.59}
$$

with $C_3 = \frac{27(K_8^*K_9)^4|\Omega|}{C_1^3}$. We substitute (3.58) and (3.59) into (3.57) to obtain

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega |\nabla v|^4 + \frac{d_2}{2}\int_\Omega |\nabla|\nabla v|^2|^2 + 2d_2\int_\Omega |\nabla v|^2|D^2v|^2 \\
&\le (2C_1+C_2)\int_\Omega |\nabla v|^4 + \frac{4\xi^2K_9^2}{d_2}\int_\Omega |\Delta u|^4 + 4b_2K_9\int_\Omega |\nabla v|^3|\nabla w| + C_3.
\end{aligned} \tag{3.60}
$$

Moreover, integrating by parts, noting $\|v(\cdot,t)\|_{L^\infty} \le K_9$ and using Young's inequality, one has

$$
\begin{aligned}
(2C_1+C_2+2)\int_\Omega |\nabla v|^4 &= C_4\int_\Omega |\nabla v|^2\nabla v\cdot\nabla v \\
&= -C_4\int_\Omega v\nabla|\nabla v|^2\cdot\nabla v - C_4\int_\Omega v|\nabla v|^2\Delta v \\
&\le C_4K_9\int_\Omega |\nabla|\nabla v|^2||\nabla v| + C_4K_9\sqrt{2}\int_\Omega |\nabla v|^2|D^2v| \\
&\le \frac{d_2}{4}\int_\Omega |\nabla|\nabla v|^2|^2 + \frac{d_2}{2}\int_\Omega |\nabla v|^2|D^2v|^2 + \frac{2C_4^2K_9^2}{d_2}\int_\Omega |\nabla v|^2 \\
&\le \frac{d_2}{4}\int_\Omega |\nabla|\nabla v|^2|^2 + \frac{d_2}{2}\int_\Omega |\nabla v|^2|D^2v|^4 + \int_\Omega |\nabla v|^4 + \frac{C_4^4K_9^4|\Omega|}{d_2^2},
\end{aligned}
$$

which gives

$$(2C_1+C_2+1)\int_\Omega |\nabla v|^4 \le \frac{d_2}{4}\int_\Omega |\nabla|\nabla v|^2|^2 + \frac{d_2}{2}\int_\Omega |\nabla v|^2|D^2 v|^2 + \frac{C_4^4K_9^4|\Omega|}{d_2^2}. \tag{3.61}$$

Similarly, one can derive that

$$\begin{aligned}\int_\Omega |\nabla v|^6 &= \int_\Omega |\nabla v|^4\nabla v\cdot\nabla v\\ &= -2\int_\Omega v|\nabla v|^2\nabla|\nabla v|^2\cdot\nabla v - \int_\Omega v|\nabla v|^4\Delta v\\ &\le 2K_9\int_\Omega |\nabla v|^3|\nabla|\nabla v|^2| + \sqrt{2}K_9\int_\Omega |\nabla v|^4|D^2 v|\\ &\le \frac{3}{8}\int_\Omega |\nabla v|^6 + 4K_9^2\left(\int_\Omega |\nabla|\nabla v|^2|^2 + \int_\Omega |\nabla v|^2|D^2v|^2\right),\end{aligned}$$

which entails us that

$$\int_\Omega |\nabla|\nabla v|^2|^2 + \int_\Omega |\nabla v|^2|D^2v|^2 \ge \frac{5}{32K_9^2}\int_\Omega |\nabla v|^6. \tag{3.62}$$

Then substituting (3.61) and (3.62) into (3.60), we obtain

$$\begin{aligned}&\frac{d}{dt}\int_\Omega |\nabla v|^4 + \int_\Omega |\nabla v|^4 + \frac{5d_2}{128K_9^2}\int_\Omega |\nabla v|^6\\ &\quad\le \frac{4\xi^2K_9^2}{d_2}\int_\Omega |\Delta u|^4 + 4b_2K_9\int_\Omega |\nabla v|^3|\nabla w| + C_3 + \frac{C_4^4K_9^4|\Omega|}{d_2^2}\\ &\quad\le \frac{4\xi^2K_9^2}{d_2}\int_\Omega |\Delta u|^4 + \frac{5d_2}{128K_9^2}\int_\Omega |\nabla v|^6 + \frac{512b_2^2K_9^4}{5d_2}\int_\Omega |\nabla w|^2 + C_3 + \frac{C_4^4K_9^4|\Omega|}{d_2^2},\end{aligned}$$

and hence

$$\frac{d}{dt}\int_\Omega |\nabla v|^4 + \int_\Omega |\nabla v|^4 \le \frac{16\xi^2K_9^2}{d_2}\int_\Omega |D^2u|^4 + \frac{512b_2^2K_9^4}{5d_2}\int_\Omega |\nabla w|^2 + C_3 + \frac{4C_4^4K_9^4}{4d_2^2}, \tag{3.63}$$

where we use the fact $|\Delta u| \le \sqrt{2}|D^2u|$. Then applying Grönwall's type inequality (cf. [44, Lemma 3.4]), and using (3.33) and (3.43), from (3.63), we get (3.52). ∎

Lemma 3.11. *Let the assumptions in Theorem* 1.1 *hold, and let* $(u,v,w)$ *be the solution of system* (1.4). *Then it holds that*

$$\|w(\cdot,t)\|_{L^3} \le K_{16}, \tag{3.64}$$

*where* $K_{16}>0$ *is a constant independent of* $t$.

*Proof.* Multiplying the third equation of (1.4) by $w^2$, and integrating the result by parts, we obtain

$$\begin{aligned}&\frac{1}{3}\frac{d}{dt}\int_\Omega w^3 + 2\int_\Omega w|\nabla w|^2 + \mu_3\int_\Omega w^4\\ &\quad= 2\chi\int_\Omega w^2(v\nabla u + u\nabla v)\cdot\nabla w + \mu_3\int_\Omega w^3 + \int_\Omega vw^3 + c_3\int_\Omega \frac{uw^3}{w+u}.\end{aligned} \tag{3.65}$$

By the facts $\|u(\cdot,t)\|_{L^\infty} \le K$ (see Lemma 2.2), $\|v(\cdot,t)\|_{L^\infty} \le K_9$ (see Lemma 3.5), $\|\nabla u(\cdot,t)\|_{L^\infty} \le K_8^*$ (see (3.27)), and $\|\nabla v\|_{L^4} \le K_{15}$ (see Lemma 3.10), one can derive that

$$\begin{aligned} &2\chi\int_\Omega w^2(v\nabla u + u\nabla v)\cdot\nabla w \\ &\le 2\chi K_8^* K_9 \int_\Omega w^2|\nabla w| + 2\chi K\int_\Omega w^2|\nabla v||\nabla w| \\ &\le \int_\Omega w|\nabla w|^2 + 2\chi^2(K_8^*)^2K_9^2\int_\Omega w^3 + 2\chi^2K^2\int_\Omega w^3|\nabla v|^2 \end{aligned} \tag{3.66}$$

and

$$\mu_3\int_\Omega w^3 + \int_\Omega vw^3 + c_3\int_\Omega \frac{uw^3}{w+u} \le (\mu_3 + K_9 + c_3)\int_\Omega w^3. \tag{3.67}$$

Substituting (3.66) and (3.67) into (3.65) gives

$$\frac{d}{dt}\int_\Omega w^3 + 3\int_\Omega w|\nabla w|^2 + 3\mu_3\int_\Omega w^4 \le 6\chi^2K^2\int_\Omega w^3|\nabla v|^2 + C_1\int_\Omega w^3, \tag{3.68}$$

with $C_1 := 6\chi^2(K_8^*)^2K_9^2 + 3(\mu_3 + K_9 + c_3)$. Then using the Gagliardo-Nirenberg inequality and Young's inequality, and utilizing the facts $\|w(\cdot,t)\|_{L^2} \le K_{13}$ (see (3.42)) and $\|\nabla v\|_{L^4} \le K_{15}$ (see Lemma 3.10), one has

$$\begin{aligned} 6\chi^2K^2\int_\Omega w^3|\nabla v|^2 &\le 6\chi^2K^2\left(\int_\Omega w^6\right)^{\frac12}\left(\int_\Omega |\nabla v|^4\right)^{\frac12} \\ &\le 6\chi^2K^2K_{15}^2\|w^{\frac32}\|_{L^4}^2 \\ &\le 6\chi^2K^2K_{15}^2C_2(\|\nabla w^{\frac32}\|_{L^2}^{\frac43}\|w^{\frac32}\|_{L^{\frac43}}^{\frac23} + \|w^{\frac32}\|_{L^{\frac43}}^2) \\ &= 6\chi^2K^2K_{15}^2C_2K_{13}\|\nabla w^{\frac32}\|_{L^2}^{\frac43} + 6\chi^2K^2K_{15}^2C_2K_{13}^3 \\ &\le \int_\Omega w|\nabla w|^2 + C_3. \end{aligned} \tag{3.69}$$

Then substituting (3.69) into (3.68), and adding $\int_\Omega w^3$ on both sides of the resulting inequality alongside the Young's inequality: $(C_1+1)\int_\Omega w^3 \le \mu_3\int_\Omega w^4 + C_4$, we obtain

$$\frac{d}{dt}\int_\Omega w^3 + \int_\Omega w^3 \le C_3 + C_4,$$

which gives (3.64) directly upon an application of Grönwall's inequality. ∎

Next, we shall show the boundedness $\|\nabla v(\cdot,t)\|_{L^\infty}$. Precisely, we have the following lemma.

Lemma 3.12. *Let the assumptions in Theorem* 1.1 *hold, and let* $(u,v,w)$ *be the solution of* (1.4). *Then there exists a constant* $K_{17} > 0$ *independent of* $t$ *such that for all* $t \in (0, T_{max})$

$$\|\nabla v(\cdot,t)\|_{L^\infty} \le K_{17}. \tag{3.70}$$

*Proof.* From Lemma 2.1, we know that (3.70) holds for all $t \in (0,\tau]$, where $\tau$ is defined by (2.8). Hence to prove Lemma 3.12, we need only show (3.70) holds for all $t \in (\tau, T_{max})$.

To this end, we rewrite the second equation of (1.4) as follows:

$$v_t - d_2\Delta v + v = -\xi\nabla\cdot(v\nabla u) + \mu_2 v(1-v) + uv - b_2 vw + v. \tag{3.71}$$

With the variation-of-constants formula for (3.71), we obtain

$$\begin{aligned}\nabla v(\cdot,t) = &\nabla e^{(d_2\Delta-1)(t-\tau)}v(\cdot,\tau) - \xi\int_\tau^t \nabla e^{(d_2\Delta-1)(t-s)}[\nabla\cdot(v\nabla u)]ds\\ &+\int_\tau^t \nabla e^{(d_2\Delta-1)(t-s)}[\mu_2 v(1-v)+uv-b_2vw+v]ds,\end{aligned}$$

which gives

$$\begin{aligned}\|\nabla v(\cdot,t)\|_{L^\infty} \le &\|\nabla e^{(d_2\Delta-1)(t-\tau)}v(\cdot,\tau)\|_{L^\infty} + \xi\int_\tau^t \|\nabla e^{(d_2\Delta-1)(t-s)}[\nabla\cdot(v\nabla u)]\|_{L^\infty}ds\\ &+\int_\tau^t \|\nabla e^{(d_2\Delta-1)(t-s)}[\mu_2 v(1-v)+uv-b_2vw+v]\|_{L^\infty}ds\\ :=&\ell_1+\ell_2+\ell_3.\end{aligned} \tag{3.72}$$

Using the semigroup smoothing estimates in Lemma 2.5, we first estimate the term $\ell_1$ as

$$\ell_1 = \|\nabla e^{(d_2\Delta-1)(t-\tau)}v(\cdot,\tau)\|_{L^\infty} \le C_1, \tag{3.73}$$

and estimate the term $\ell_2$ as

$$\begin{aligned}\ell_2 =& \xi\int_\tau^t \|\nabla e^{(d_2\Delta-1)(t-s)}(\nabla v\cdot\nabla u + v\Delta u)\|_{L^\infty}ds\\ \le& \xi\int_\tau^t \|\nabla e^{(d_2\Delta-1)(t-s)}(\nabla v\cdot\nabla u)\|_{L^\infty}ds + \xi\int_\tau^t \|\nabla e^{(d_2\Delta-1)(t-s)}(v\Delta u)\|_{L^\infty}ds\\ :=& \ell_{21}+\ell_{22}.\end{aligned} \tag{3.74}$$

Then from Lemmas 2.5, 3.5, and 3.10, and (3.27), it follows that

$$\begin{aligned}\ell_{21} =& \xi\int_\tau^t \|\nabla e^{(d_2\Delta-1)(t-s)}(\nabla v\cdot\nabla u)\|_{L^\infty}ds\\ \le& C_2\int_\tau^t (1+(t-s)^{-\frac12-\frac14})e^{-(d_2\lambda_1+1)(t-s)}\|\nabla v\cdot\nabla u\|_{L^4}ds\\ \le& C_2\int_\tau^t (1+(t-s)^{-\frac34})e^{-(d_2\lambda_1+1)(t-s)}\|\nabla v\|_{L^4}\|\nabla u\|_{L^\infty}ds\\ \le& C_3\int_\tau^t (1+(t-s)^{-\frac34})e^{-(d_2\lambda_1+1)(t-s)}ds\\ \le& C_4\end{aligned} \tag{3.75}$$

and

$$
\begin{aligned}
\ell_{22} &= \xi \int_\tau^t \|\nabla e^{(d_2\Delta-1)(t-s)}(v\Delta u)\|_{L^\infty} ds \\
&\le C_5 \int_\tau^t (1+(t-s)^{-\frac{1}{2}-\frac{1}{p}}) e^{-(d_2\lambda_1+1)(t-s)} \|v\Delta u\|_{L^p} ds \\
&\le C_6 \int_\tau^t (1+(t-s)^{-\frac{1}{2}-\frac{1}{p}}) e^{-(d_2\lambda_1+1)(t-s)} \|\Delta u\|_{L^p} ds \\
&\le C_7 \int_\tau^t \left(1+(t-s)^{-\frac{1}{2}-\frac{1}{p}}\right)^{\frac{p}{p-1}} e^{-\frac{d_2\lambda_1 p}{p-1}(t-s)} ds + C_7 \int_\tau^t e^{-p(t-s)} \|\Delta u\|_{L^p}^p ds.
\end{aligned} \tag{3.76}
$$

Choosing $p>4$, we can check that $\frac{p+2}{2(p-1)}<1$, and hence

$$
\begin{aligned}
\int_\tau^t \left(1+(t-s)^{-\frac{1}{2}-\frac{1}{p}}\right)^{\frac{p}{p-1}} e^{-\frac{d_2\lambda_1 p}{p-1}(t-s)} ds &\le C_8 \int_\tau^t \left(1+(t-s)^{-\frac{p+2}{2(p-1)}}\right) e^{-\frac{d_2\lambda_1 p}{p-1}(t-s)} ds \\
&\le C_9,
\end{aligned}
$$

which, alongside (3.34), gives $\ell_{22} \le C_{10}$ for some constant $C_{10}>0$ from (3.76). This entails that

$$
\ell_2 \le C_4 + C_{10}. \tag{3.77}
$$

Finally, using the boundedness of $\|v(\cdot,t)\|_{L^\infty}, \|u(\cdot,t)\|_{L^\infty}$, and $\|w(\cdot,t)\|_{L^3}$, we get the estimate for $\ell_3$ as follows:

$$
\begin{aligned}
\ell_3 &= \int_\tau^t \|\nabla e^{(d_2\Delta-1)(t-s)}[\mu_2 v(1-v)+uv-b_2vw+v]\|_{L^\infty} ds \\
&\le C_{11} \int_\tau^t (1+(t-s)^{-\frac{1}{2}-\frac{1}{3}}) e^{-(d_2\lambda_1+1)(t-s)} \|\mu_2 v(1-v)+uv-b_2vw+v\|_{L^3} ds \\
&\le C_{12}.
\end{aligned} \tag{3.78}
$$

Then, substituting (3.73), (3.77), and (3.78) into (3.72), we obtain (3.70). The proof of Lemma 3.12 is now completed. ∎

LEMMA 3.13. *Let the assumptions in Theorem* 1.1 *hold, and let* $(u,v,w)$ *be the solution of system* (1.4). *Then one can find a constant* $K_{18}>0$ *independent of* $t$ *such that*

$$
\|w(\cdot,t)\|_{L^\infty} \le K_{18} \quad \text{for all } t\in(0,T_{max}). \tag{3.79}
$$

*Proof.* By the variation of constants formula, $w$ can be represented as

$$
\begin{aligned}
w(\cdot,t) &= e^{(\Delta-1)t}w_0 - \chi \int_0^t e^{(\Delta-1)(t-s)} \nabla\cdot(wv\nabla u + wu\nabla v) \\
&\quad + \int_0^t e^{(\Delta-1)(t-s)} w\left(\mu_3+1-\mu_3 w - v + c_3\frac{u}{u+w}\right) \\
&\le e^{(\Delta-1)t}w_0 - \chi \int_0^t e^{(\Delta-1)(t-s)} \nabla\cdot(wv\nabla u + wu\nabla v) \\
&\quad + \int_0^t e^{(\Delta-1)(t-s)} w\left(\mu_3+1+c_3\frac{u}{u+w}\right),
\end{aligned}
$$

and hence

$$\begin{aligned}\|w(\cdot,t)\|_{L^\infty} \leq &\|e^{(\Delta-1)t}w_0\|_{L^\infty} + \chi\int_0^t \|e^{(\Delta-1)(t-s)}\nabla\cdot(wv\nabla u + wu\nabla v)\|_{L^\infty} \\ &+ \int_0^t \left\|e^{(\Delta-1)(t-s)}w\left(\mu_3+1+c_3\frac{u}{u+w}\right)\right\|_{L^\infty} := J_1+J_2+J_3.\end{aligned} \tag{3.80}$$

Using the well-known semigroup smoothing estimates (see Lemma 2.5), we have

$$J_1 = \|e^{(\Delta-1)t}w_0\|_{L^\infty} \leq C_1\|w_0\|_{L^\infty} \tag{3.81}$$

for some constant $C > 0$. Noting the facts $\|u(\cdot,t)\|_{L^\infty} \leq K$ (see Lemma 2.2), $\|v(\cdot,t)\|_{L^\infty} \leq K_9$ (see Lemma 3.5), $\|\nabla u(\cdot,t)\|_{L^\infty} \leq K_8^*$ (see (3.27)), $\|\nabla v(\cdot,t)\|_{L^\infty} \leq K_{17}$ (see Lemma 3.12), and $\|w(\cdot,t)\|_{L^3} \leq K_{16}$ (see Lemma 3.11), one can use (2.21) with the fact that $C_0^\infty(\Omega)$ is dense in $L^p(\Omega)$ for any $1 \leq p < \infty$ to obtain

$$\begin{aligned}J_2 &\leq \chi\gamma_4\int_0^t (1+(t-s)^{-\frac{5}{6}})e^{-(\lambda_1+1)(t-s)}\|w(v\nabla u+u\nabla v)\|_{L^3}ds \\ &\leq \chi\gamma_4(K_9K_8^*+KK_{17})\int_0^t(1+(t-s)^{-\frac{5}{6}})e^{-(\lambda_1+1)(t-s)}\|w\|_{L^3}ds \\ &\leq \chi\gamma_4(K_9K_8^*+KK_{17})K_{16}\int_0^t(1+(t-s)^{-\frac{5}{6}})e^{-(\lambda_1+1)(t-s)}ds \\ &\leq \frac{\chi\gamma_4(K_9K_8^*+KK_{17})K_{16}}{\lambda_1}(1+\Gamma(1/6)\,\lambda_1^{\frac{5}{6}}).\end{aligned} \tag{3.82}$$

Moreover, we can use (2.20) to derive that

$$\begin{aligned}J_3 &\leq \gamma_3(\mu_3+1+c_3)\int_0^t(1+(t-s)^{-\frac{1}{3}})e^{-(t-s)}\|w\|_{L^3}ds \\ &\leq \gamma_3(\mu_3+1+c_3)K_{16}\int_0^t(1+(t-s)^{-\frac{1}{3}})e^{-(t-s)}ds \\ &\leq \gamma_3(\mu_3+1+c_3)K_{16}(1+\Gamma(2/3)).\end{aligned} \tag{3.83}$$

Then substituting (3.81), (3.82), and (3.83) into (3.80), one obtains (3.79). ∎

*Proof of Theorem* 1.1. The combination of Lemma 2.2, (3.27), and Lemmas 3.5, 3.12 and 3.13 yields a constant $C_1 > 0$ independent of $t$ such that

$$\|u(\cdot,t)\|_{W^{1,\infty}} + \|v(\cdot,t)\|_{W^{1,\infty}} + \|w(\cdot,t)\|_{L^\infty} \leq C_1,$$

which together with the extension criterion in Lemma 2.1 proves Theorem 1.1. ∎

**4. Global stabilization of solutions.** In this section, we are devoted to studying the global stability of coexistence steady states as asserted in Theorems 1.2 and 1.4 by the Lyapunov functional method along with Barbălat's lemma as stated below.

LEMMA 4.1 (Barbălat's lemma [8]). *If $h:[1,\infty)\to\mathbb{R}$ is a uniformly continuous function such that $\lim_{t\to\infty}\int_1^t h(s)ds$ exists, then $\lim_{t\to\infty}h(t)=0$.*

Moreover, we need higher regularity of solutions as follows.

LEMMA 4.2. *Let* $(u,v,w)$ *be the unique global bounded classical solution of* (1.4) *given by Theorem* 1.1. *Then for any given* $0<\alpha<1$, *there exists a constant* $C>0$ *such that*

$$\|u(\cdot,t)\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\bar\Omega\times[1,\infty))}+\|v(\cdot,t)\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\bar\Omega\times[1,\infty))}+\|w(\cdot,t)\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\bar\Omega\times[1,\infty))}\le C.$$

*Proof.* By the boundedness of $(u,v,w)$ (see Theorem 1.1) and the interior $L^p$ estimate [36], we find a constant $C_1>0$ such that

$$\|u\|_{W_p^{2,1}(\Omega\times[i+\frac14,i+3])}+\|v\|_{W_p^{2,1}(\Omega\times[i+\frac14,i+3])}+\|w\|_{W_p^{2,1}(\Omega\times[i+\frac14,i+3])}\le C_1 \quad \text{for all } i\ge 0. \tag{4.1}$$

Then the Sobolev embedding theorem with $p$ suitably large gives

$$\|u\|_{C^{1+\alpha,\frac{1+\alpha}{2}}(\bar\Omega\times[\frac14,\infty))}+\|v\|_{C^{1+\alpha,\frac{1+\alpha}{2}}(\bar\Omega\times[\frac14,\infty))}+\|w\|_{C^{1+\alpha,\frac{1+\alpha}{2}}(\bar\Omega\times[\frac14,\infty))}\le C_2. \tag{4.2}$$

Using (4.2) and applying the Schauder estimate [34] to the first equation of (1.4), we obtain

$$\|u\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\bar\Omega\times[i+\frac13,i+3])}\le C_3 \quad \text{for all } i\ge 0, \tag{4.3}$$

and hence

$$\|u\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\bar\Omega\times[\frac13,+\infty))}\le C_4. \tag{4.4}$$

The second equation of (1.4) can be rewritten as

$$v_t-d_2\Delta v+\xi\nabla u\cdot\nabla v=G(x,t), \quad x\in\Omega, \quad t>0, \tag{4.5}$$

where

$$G(x,t)=-\xi\Delta u\cdot v+v(\mu_2-\mu_2 v+u-b_2 w).$$

By (4.3) and (4.4), we have

$$\|G\|_{C^{\alpha,\frac{\alpha}{2}}(\bar\Omega\times[i+\frac13,i+3])}+\|\xi\nabla u\|_{C^{\alpha,\frac{\alpha}{2}}(\bar\Omega\times[i+\frac13,i+3])}\le C_5 \quad \text{for all } i\ge 0.$$

Then applying the standard parabolic Schauder estimate to (4.5), one can find a positive constant $C_6>0$ such that $\|v\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\bar\Omega\times[i+1,i+3])}\le C_6$ for all $i\ge 0$, which gives

$$\|v\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\bar\Omega\times[1,+\infty))}\le C_7. \tag{4.6}$$

Finally, similar arguments applied to the third equation of (1.4) give us a constant $C_8>0$ such that $\|w\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\bar\Omega\times[1,+\infty))}\le C_8$, which completes the proof of Lemma 4.2. ∎

**4.1. Global stability for $b_3=c_3=0$.** In this subsection, we first show the global stabilization of coexistence steady state in the case of $b_3=c_3=0$. In this case, the system (1.6) becomes

$$\begin{cases} u_t=d_1\Delta u+u(1-u-b_1 v), & x\in\Omega, t>0,\\ v_t=d_2\Delta v-\xi\nabla\cdot(v\nabla u)+v(1-v+u-b_2 w), & x\in\Omega, t>0,\\ w_t=\Delta w-\chi\nabla\cdot(wv\nabla u+wu\nabla v)+w(1-w+v), & x\in\Omega, t>0,\\ \frac{\partial u}{\partial\nu}=\frac{\partial v}{\partial\nu}=\frac{\partial w}{\partial\nu}=0, & x\in\partial\Omega,\ t>0,\\ u(x,0)=u_0(x),\ v(x,0)=v_0(x),\ w(x,0)=w_0(x), & x\in\Omega. \end{cases} \tag{4.7}$$

One can easily check that the system (4.7) has a unique constant coexistence steady state $(u^*, v^*, w^*)$ defined by (1.7) provided

$$0 < b_2 < 2 \quad \text{and} \quad 1 + b_1 b_2 + b_2 > b_1. \tag{4.8}$$

Our purpose is to study the global stability of $(u^*, v^*, w^*)$. To this end, we introduce the following energy functional:

$$\mathcal{E}_1(t) = \mathcal{I}_u(t) + \mathcal{I}_v(t) + \mathcal{I}_w(t), \tag{4.9}$$

where

$$\mathcal{I}_s(t) = \int_\Omega \left( s - s^* - s^* \ln \frac{s}{s^*} \right), \quad s \in \{u, v, w\}.$$

The energy functional like (4.9) is the Lyapunov functional to the corresponding ODE (cf. [11, 29]), and it can be extended to PDE models with diffusion (cf. [19, 53]) or prey-taxis.

Then we have the following results.

LEMMA 4.3. *Let the condition* (4.8) *hold. If*

$$(b_1 - 1)^2 + (b_2 - 1)^2 < 4, \tag{4.10}$$

*then there exist $\xi_1 > 0$ and $\chi_1 > 0$ such that whenever $\xi \in (0, \xi_1)$ and $\chi \in (0, \chi_1)$ there holds for some $T_0 > 0$ that*

$$\|u(\cdot,t) - u^*\|_{L^\infty} + \|v(\cdot,t) - v^*\|_{L^\infty} + \|w(\cdot,t) - w^*\|_{L^\infty} \le K_{19} e^{-\lambda t} \quad \text{for all } t > T_0, \tag{4.11}$$

*where $K_{19} > 0$ and $\lambda > 0$ are constants independent of $t$.*

*Proof.* The coexistence steady state $(u^*, v^*, w^*)$ satisfies equations

$$\begin{cases} 1 - u^* - b_1 v^* = 0, \\ 1 - v^* + u^* - b_2 w^* = 0, \\ 1 - w^* + v^* = 0. \end{cases} \tag{4.12}$$

Then one can check that (4.12) has a unique positive steady state $(u^*, v^*, w^*)$ defined by (1.7) under the condition (4.8).

**Step 1.** In this step, we shall show the global stability of $(u^*, v^*, w^*)$ by means of the energy functional $\mathcal{E}_1(t)$. Using the first equation of (4.7) and the fact that $1 - u^* - b_1 v^* = 0$, we find

$$\begin{aligned} \frac{d}{dt}\mathcal{I}_u(t) &= \int_\Omega \left(1 - \frac{u^*}{u}\right) u_t \\ &= -d_1 u^* \int_\Omega \frac{|\nabla u|^2}{u^2} + \int_\Omega (u - u^*)(1 - u - b_1 v) \\ &= -d_1 u^* \int_\Omega \frac{|\nabla u|^2}{u^2} - \int_\Omega (u - u^*)^2 - b_1 \int_\Omega (u - u^*)(v - v^*). \end{aligned} \tag{4.13}$$

Similarly, we can use the second equation of (4.7) and the fact $1 - v^* + u^* - b_2 w^* = 0$ to obtain

(4.14)
$$\begin{aligned}\frac{d}{dt}\mathcal{I}_v(t) &= \int_\Omega \left(1-\frac{v^*}{v}\right) v_t\\ &= -d_2 v^* \int_\Omega \frac{|\nabla v|^2}{v^2} + \xi v^* \int_\Omega \frac{\nabla u\cdot\nabla v}{v} + \int_\Omega (v-v^*)(1-v+u-b_2 w)\\ &= -d_2 v^* \int_\Omega \frac{|\nabla v|^2}{v^2} + \xi v^* \int_\Omega \frac{\nabla u\cdot\nabla v}{v} - \int_\Omega (v-v^*)^2 + \int_\Omega (u-u^*)(v-v^*)\\ &\quad - b_2 \int_\Omega (v-v^*)(w-w^*).\end{aligned}$$

Furthermore, with the fact $1-w^*+v^*=0$ and using the third equation of (4.7), we have

(4.15)
$$\begin{aligned}\frac{d}{dt}\mathcal{I}_w(t) &= \int_\Omega \left(1-\frac{w^*}{w}\right) w_t\\ &= -w^* \int_\Omega \frac{|\nabla w|^2}{w^2} + \chi w^* \int_\Omega \frac{v\nabla u\cdot\nabla w + u\nabla v\cdot\nabla w}{w} + \int_\Omega (w-w^*)(1-w+v)\\ &= -w^* \int_\Omega \frac{|\nabla w|^2}{w^2} + \chi w^* \int_\Omega \frac{v\nabla u\cdot\nabla w + u\nabla v\cdot\nabla w}{w}\\ &\quad - \int_\Omega (w-w^*)^2 + \int_\Omega (v-v^*)(w-w^*).\end{aligned}$$

Then substituting (4.13), (4.14), and (4.15) into (4.9), we end up with

$$\frac{d}{dt}\mathcal{E}_1(t) = -\int_\Omega XA_1X^T - \int_\Omega YB_1Y^T, \tag{4.16}$$

where $X=(u-u^*, v-v^*, w-w^*)$, $Y=\left(\frac{\nabla u}{u}, \frac{\nabla v}{v}, \frac{\nabla w}{w}\right)$, and $A_1, B_1$ are symmetric matrices denoted by

$$A_1 := \begin{pmatrix} 1 & \frac{b_1-1}{2} & 0\\ \frac{b_1-1}{2} & 1 & \frac{b_2-1}{2}\\ 0 & \frac{b_2-1}{2} & 1\end{pmatrix}, \quad B_1 := \begin{pmatrix} d_1u^* & -\frac{\xi v^* u}{2} & -\frac{\chi w^* uv}{2}\\ -\frac{\xi v^* u}{2} & d_2 v^* & -\frac{\chi w^* uv}{2}\\ -\frac{\chi w^* uv}{2} & -\frac{\chi w^* uv}{2} & w^*\end{pmatrix}.$$

Next, we show that the matrices $A_1$ and $B_1$ are positive definite and positive semidefinite, respectively. Notice (4.10) implies $0<b_1<3$. Then

$$\begin{vmatrix} 1 & \frac{b_1-1}{2}\\ \frac{b_1-1}{2} & 1\end{vmatrix} = \frac{(3-b_1)(1+b_1)}{4} > 0$$

and

$$|A_1| = \frac{4-(b_1-1)^2-(b_2-1)^2}{4} > 0.$$

Therefore, $A_1$ is positive definite and there exists a constant $\alpha>0$ such that

$$XA_1X^T \geq \alpha |X|^2. \tag{4.17}$$

On the other hand, after some calculations, one can derive that

$$\begin{vmatrix} d_1u^* & -\frac{\xi v^* u}{2}\\ -\frac{\xi v^* u}{2} & d_2v^*\end{vmatrix} = \frac{v^*(4d_1d_2u^* - \xi^2 v^* u^2)}{4} \geq \frac{v^*(4d_1d_2u^* - \xi^2 v^* \|u\|^2_{L^\infty})}{4}$$

and

$$\begin{aligned}|B_1| &= -\frac{w^*}{4}\left[\xi\chi^2 v^* w^* u^3 v^2 + \chi^2 w^*(d_1u^* + d_2v^*)u^2v^2 + \xi^2(v^*)^2u^2 - 4d_1d_2u^*v^*\right]\\ &\geq -\frac{w^*}{4}\left[\xi\chi^2 v^* w^*\|u\|^3_{L^\infty}\|v\|^2_{L^\infty} + \chi^2 w^*(d_1u^* + d_2v^*)\|u\|^2_{L^\infty}\|v\|^2_{L^\infty}\right]\\ &\quad -\frac{w^*}{4}\left[\xi^2(v^*)^2\|u\|^2_{L^\infty} - 4d_1d_2u^*v^*\right].\end{aligned}$$

Noticing that $\|u\|_{L^\infty}$ and $\|v\|_{L^\infty}$ are independent of parameters $\xi$ and $\chi$ (see Theorem 1.1), we can find appropriate numbers $\xi_1 > 0$ and $\chi_1 > 0$, for example,

$$\xi_1 = \sqrt{\frac{2d_1d_2u^*}{v^*\|u\|^2_{L^\infty}}} \quad \text{and}$$
$$\chi_1 = \sqrt{2d_1d_2u^*v^*/\|u\|^2_{L^\infty}\|v\|^2_{L^\infty}(\xi v^*w^*\|u\|_{L^\infty} + w^*(d_1u^* + d_2v^*))}$$

such that if $0 < \xi < \xi_1, 0 < \chi < \chi_1$, then

$$\begin{aligned}4d_1d_2u^*v^* &> \xi\chi^2 v^* w^*\|u\|^3_{L^\infty}\|v\|^2_{L^\infty}\\ &\quad + \chi^2 w^*(d_2v^* + d_1u^*)\|u\|^2_{L^\infty}\|v\|^2_{L^\infty} + \xi^2(v^*)^2\|u\|^2_{L^\infty},\end{aligned}$$

which guarantees that $B$ is a positive semidefinite matrix, and hence

$$YB_1Y^T \geq 0. \tag{4.18}$$

Substituting (4.17) and (4.18) into (4.16), we obtain

$$\frac{d}{dt}\mathcal{E}_1(t) + \alpha\mathcal{F}_1(t) \leq 0 \tag{4.19}$$

with

$$\mathcal{F}_1(t) := \int_\Omega (u-u^*)^2 + (v-v^*)^2 + (w-w^*)^2.$$

Moreover, we can show that $\mathcal{E}_1(t) \geq 0$ for all $t > 0$. In fact, letting $\varphi(z) := z - u^*\ln z$ for $z > 0$, one can check that $\varphi'(z) = 1 - \frac{u^*}{z}$ and $\varphi''(z) = \frac{u^*}{z^2}$. By Taylor's expansion, there exists a quantity $\eta = \theta u + (1-\theta)u^*$ with $\theta \in (0,1)$ such that

$$u - u^* - u^*\ln\frac{u}{u^*} = \varphi(u) - \varphi(u^*) = \frac{\varphi''(\eta)}{2}(u-u^*)^2 = \frac{u^*}{2\eta^2}(u-u^*)^2 \geq 0, \tag{4.20}$$

which implies $\mathcal{I}_u(t) \geq 0$. Similarly, we have that $\mathcal{I}_v(t) \geq 0$ and $\mathcal{I}_w(t) \geq 0$. Then it follows that $\mathcal{E}_1(t) = \mathcal{I}_u(t) + \mathcal{I}_v(t) + \mathcal{I}_w(t) \geq 0$.

Then integrating (4.19) with respect of $t$ over $(1,\infty)$ along with $\mathcal{E}_1(t) \geq 0$, gives

$$\int_1^\infty \mathcal{F}_1(t) \leq \frac{1}{\alpha}\mathcal{E}_1(1) < \infty. \tag{4.21}$$

Then using the regularity of $u, v, w$ obtained in Lemma 4.2, one can derive that $\mathcal{F}_1(t)$ is uniformly continuous in $[1,\infty)$. Then using (4.21) and applying Lemma 4.1, we obtain

$$\mathcal{F}_1(t) = \int_\Omega (u-u^*)^2 + \int_\Omega (v-v^*)^2 + \int_\Omega (w-w^*)^2 \to 0 \quad \text{as } t \to \infty,$$

which gives

$$\lim_{t\to\infty}(\|u-u^*\|_{L^2}+\|v-v^*\|_{L^2}+\|w-w^*\|_{L^2})=0. \tag{4.22}$$

By Lemma 4.2, we find

$$\|u-u^*\|_{W^{1,\infty}}+\|v-v^*\|_{W^{1,\infty}}+\|w-w^*\|_{W^{1,\infty}}\le C_1 \quad \text{for all } t>1. \tag{4.23}$$

Then applying the Gagliardo–Nirenberg inequality and using (4.23), one has

$$\|u-u^*\|_{L^\infty}\le C_2\|u-u^*\|^{\frac{1}{2}}_{W^{1,\infty}}\|u-u^*\|^{\frac{1}{2}}_{L^2}\le C_2C_1^{\frac{1}{2}}\|u-u^*\|^{\frac{1}{2}}_{L^2}, \tag{4.24}$$

which together with (4.22) implies

$$\lim_{t\to\infty}\|u-u^*\|_{L^\infty}=0. \tag{4.25}$$

The same argument gives us that

$$\lim_{t\to\infty}(\|v-v^*\|_{L^\infty}+\|w-w^*\|_{L^\infty})=0. \tag{4.26}$$

**Step 2.** In this step, we shall show that the convergence rate is exponential. In fact, (4.25) implies there exists $t_1>1$ such that

$$\frac{u^*}{2}\le\eta=[\theta u+(1-\theta)u^*]\le\frac{3u^*}{2} \quad \text{for all } t\ge t_1.$$

Then by (4.20), one has

$$\frac{2}{9u^*}(u-u^*)^2\le u-u^*-u^*\ln\frac{u}{u^*}\le\frac{2}{u^*}(u-u^*)^2. \tag{4.27}$$

Similarly, using (4.26), there exist two constants $t_2>1$ and $t_3>1$ such that

$$\frac{2}{9v^*}(u-u^*)^2\le v-v^*-v^*\ln\frac{v}{v^*}\le\frac{2}{v^*}(v-v^*)^2 \quad \text{for all } t>t_2, \tag{4.28}$$

and

$$\frac{2}{9w^*}(w-w^*)^2\le w-w^*-w^*\ln\frac{w}{w^*}\le\frac{2}{w^*}(w-w^*)^2 \quad \text{for all } t>t_3. \tag{4.29}$$

By the definition of $\mathcal{E}_1(t)$ and $\mathcal{F}_1(t)$ along with (4.27), (4.28), and (4.29), we can find two positive constants $\alpha_1$ and $\alpha_2$ such that

$$\alpha_1\mathcal{F}_1(t)\le\mathcal{E}_1(t)\le\alpha_2\mathcal{F}_1(t) \tag{4.30}$$

for some $t\ge\bar{t}=\max\{t_1,t_2,t_3\}$. Then the combination of (4.19) and (4.30) gives

$$\frac{d}{dt}\mathcal{E}_1(t)+\frac{\alpha}{\alpha_2}\mathcal{E}_1(t)\le 0 \quad \text{for all } t\ge\bar{t},$$

which implies for all $t\ge\bar{t}$

$$\mathcal{E}_1(t)\le\mathcal{E}_1(\bar{t})e^{-\frac{\alpha}{\alpha_2}(t-\bar{t})}\le C_3e^{-\frac{\alpha}{\alpha_2}t}. \tag{4.31}$$

Then it follows from (4.30)–(4.31) that

$$\mathcal{F}_1(t)\le\frac{1}{\alpha_1}\mathcal{E}_1(t)\le\frac{C_3}{\alpha_1}e^{-\frac{\alpha}{\alpha_2}t} \quad \text{for all } t\ge\bar{t},$$

which, alongside the definition of $\mathcal{F}_1(t)$, gives

$$\|u-u^*\|_{L^2}^2+\|v-v^*\|_{L^2}^2+\|w-w^*\|_{L^2}^2 \leq \frac{C_3}{\alpha_1}e^{-\frac{\alpha}{\alpha_2}t} \quad \text{for all } t\geq \bar{t}. \tag{4.32}$$

Then combining (4.24) and (4.32), one can find there exist two positive constant $C_4$ and $\alpha_3$ such that

$$\|u(\cdot,t)-u^*\|_{L^\infty} \leq C_4 e^{-\alpha_3 t} \quad \text{for all } t\geq \bar{t}. \tag{4.33}$$

Similarly, it holds that

$$\|v(\cdot,t)-v^*\|_{L^\infty}+\|w(\cdot,t)-w^*\|_{L^\infty} \leq C_5 e^{-\alpha_4 t} \quad \text{for all } \ t\geq \bar{t}. \tag{4.34}$$

Combining (4.33) and (4.34) gives (4.11) and hence completes the proof. ∎

**4.2. Global stability for $b_3>0$, $c_3>0$.** In this subsection, we investigate the large time behavior of solutions for the system (1.6) with $b_3>0$ and $c_3>0$. For simplicity, we assume $c_3=1$. We underline from Remark 3.6 that $\|v\|_{L^\infty}$ is bounded by a constant independent of $b_1$ as $b_1$ is small. Hence, if $b_1>0$ and $b_3>0$ are suitably small, $1-b_3-b_1\|v\|_{L^\infty}>0$ is warranted. To derive the stability in the case of $b_3,c_3>0$, we first derive a lower bound estimate for $u$.

LEMMA 4.4. *Let $(u,v,w)$ be the unique global bounded classical solution of* (1.6). *Let $b_1>0$ and $b_3>0$ be sufficiently small such that $1-b_3-b_1\|v\|_{L^\infty}>0$. Then there exists a constant $t_0\in(0,\infty)$ such that*

$$u(x,t)\geq \min\{\bar{u},1-b_3-b_1\|v\|_{L^\infty}\} \quad \textit{for all } (x,t)\in\bar{\Omega}\times(t_0,\infty) \tag{4.35}$$

*and*

$$\liminf_{t\to\infty} u(x,t)\geq 1-b_3-b_1\|v\|_{L^\infty} \quad \textit{for all } x\in\bar{\Omega}, \tag{4.36}$$

*where $\bar{u}=\min_{x\in\bar{\Omega}}u(x,t_0)$.*

*Proof.* By the strong maximum principle applied to the first equation in (1.6), there exists a constant $t_0>0$ such that $\min_{x\in\bar{\Omega}}u(x,t_0)=\bar{u}>0$. For convenience, we denote $\mathcal{K}=1-b_1\|v\|_{L^\infty}-b_3$. Then we consider the following problem:

$$\begin{cases} u_t-d_1\Delta u=u\left(1-u-b_1v-\frac{b_3w}{u+w}\right)\geq u(\mathcal{K}-u), & x\in\Omega, t>t_0,\\ \frac{\partial u}{\partial\nu}=0, & x\in\partial\Omega, t>t_0,\\ u|_{t=t_0}=u(x,t_0), & x\in\Omega. \end{cases} \tag{4.37}$$

Let $\tilde{u}(t)$ be the solution of the following ODE problem:

$$\begin{cases} \frac{d\tilde{u}(t)}{dt}=\tilde{u}(\mathcal{K}-\tilde{u}), \quad t>t_0,\\ \tilde{u}(t_0)=\bar{u}>0, \end{cases}$$

which has the explicit solution $\tilde{u}(t)=\frac{\mathcal{K}}{1+(\frac{\mathcal{K}}{\bar{u}}-1)e^{-\mathcal{K}(t-t_0)}}$ such that

$$\tilde{u}(t)\geq \min\{\bar{u},\mathcal{K}\}, \ t>t_0.$$

Recall that $u(x,t_0)\geq \bar{u}$. Then $\tilde{u}$ is a lower solution of the following PDE problem:

$$\begin{cases} U^0_t - d_1\Delta U^0 = U^0(\mathcal{K}-U^0), & x\in\Omega, t>t_0,\\ \frac{\partial U^0}{\partial \nu}(x,t)=0, & x\in\partial\Omega, t>t_0,\\ U^0(x,t_0)=u(x,t_0), & x\in\Omega, \end{cases} \tag{4.38}$$

and consequently,

$$\tilde{u}(t)\leq U^0(x,t) \quad \text{for all } (x,t)\in\bar{\Omega}\times(t_0,\infty). \tag{4.39}$$

Applying the comparison principle to (4.37) and (4.38), and using (4.39), one has

$$\min\{\bar{u},\mathcal{K}\}\leq \tilde{u}(t)\leq U^0(x,t)\leq u(x,t) \quad \text{for all } (x,t)\in\bar{\Omega}\times(t_0,\infty),$$

which indicates (4.35) and (4.36). ◻

Next, we shall show that the coexistence steady state $(u_*,v_*,w_*)$ is globally asymptotically stable under some conditions. Similarly, we introduce the following energy functional:

$$\mathcal{E}_2(t)=\mathcal{J}_u(t)+\mathcal{J}_v(t)+\mathcal{J}_w(t), \tag{4.40}$$

where

$$\mathcal{J}_s(t)=\int_\Omega\left(s-s_*-s_*\ln\frac{s}{s_*}\right), \quad s\in\{u,v,w\},$$

and $(u_*,v_*,w_*)=(u_1^*,v_1^*,w_1^*)$ is defined in (A.1). Then we have the following results.

Lemma 4.5. *Let $(u,v,w)$ be the solution of* (1.6), *and let $(u_*,v_*,w_*)=(u_1^*,v_1^*,w_1^*)$ be the coexistence steady state defined in* (A.1). *If* (1.11) *holds, then there exist $\xi_2>0,\chi_2>0$ such that the solution $(u,v,w)$ will exponentially converge to $(u_*,v_*,w_*)$ in $L^\infty$-norm as $t\to\infty$ whenever $0<\xi<\xi_2$ and $0<\chi<\chi_2$.*

*Proof.* Using the first equation of (1.6) and the fact that $1-u_*-b_1v_*-b_3\frac{w_*}{u_*+w_*}=0$ in (1.10), we derive

$$\begin{aligned}\frac{d}{dt}\mathcal{J}_u(t)&=\int_\Omega\left(1-\frac{u_*}{u}\right)u_t\\ &=-d_1u_*\int_\Omega\frac{|\nabla u|^2}{u^2}+\int_\Omega(u-u_*)\left(1-u-b_1v-\frac{b_3w}{u+w}\right)\\ &=-d_1u_*\int_\Omega\frac{|\nabla u|^2}{u^2}-\int_\Omega(u-u_*)^2-b_1\int_\Omega(u-u_*)(v-v_*)\\ &\quad -b_3\int_\Omega\frac{u_*}{(u_*+w_*)(u+w)}(u-u_*)(w-w_*)\\ &\quad +b_3\int_\Omega\frac{w_*}{(u_*+w_*)(u+w)}(u-u_*)^2.\end{aligned} \tag{4.41}$$

With $1 - v_* + u_* - b_2 w_* = 0$ in (1.10), we can use the second equation of (1.6) to derive that

(4.42)
$$\begin{aligned}\frac{d}{dt}\mathcal{J}_v(t) &= \int_\Omega \left(1 - \frac{v_*}{v}\right) v_t \\ &= -d_2 v_* \int_\Omega \frac{|\nabla v|^2}{v^2} + \xi v_* \int_\Omega \frac{\nabla u \cdot \nabla v}{v} + \int_\Omega (v - v_*)(1 - v + u - b_2 w) \\ &= -d_2 v_* \int_\Omega \frac{|\nabla v|^2}{v^2} + \xi v_* \int_\Omega \frac{\nabla u \cdot \nabla v}{v} - \int_\Omega (v - v_*)^2 + \int_\Omega (u - u_*)(v - v_*) \\ &\quad - b_2 \int_\Omega (v - v_*)(w - w_*).\end{aligned}$$

Similarly, using the third equation of (1.6) with $c_3 = 1$ and the fact $1 - w_* + v_* + \frac{u_*}{u_* + w_*} = 0$ in (1.10), we obtain

(4.43)
$$\begin{aligned}&\frac{d}{dt}\mathcal{J}_w(t) = \int_\Omega \left(1 - \frac{w_*}{w}\right) w_t \\ &= -w_* \int_\Omega \frac{|\nabla w|^2}{w^2} + \chi w_* \int_\Omega \frac{v \nabla u \cdot \nabla w + u \nabla v \cdot \nabla w}{w} + \int_\Omega (w - w_*) \left(1 - w + v + \frac{u}{u + w}\right) \\ &= -w_* \int_\Omega \frac{|\nabla w|^2}{w^2} + \chi w_* \int_\Omega \frac{v \nabla u \cdot \nabla w + u \nabla v \cdot \nabla w}{w} - \int_\Omega (w - w_*)^2 + \int_\Omega (v - v_*)(w - w_*) \\ &\quad + \int_\Omega \frac{w_*}{(u_* + w_*)(u + w)} (u - u_*)(w - w_*) - \int_\Omega \frac{u_*}{(u_* + w_*)(u + w)} (w - w_*)^2.\end{aligned}$$

Using the definition of $\mathcal{E}_2(t)$ in (4.40) and the identities (4.41)–(4.43), we have

$$\frac{d}{dt}\mathcal{E}_2(t) = -\int_\Omega X_1 A_2 X_1^T - \int_\Omega Y_1 B_2 Y_1^T, \tag{4.44}$$

where $X_1 = (u - u_*, v - v_*, w - w_*)$, $Y_1 = \left(\frac{\nabla u}{u}, \frac{\nabla v}{v}, \frac{\nabla w}{w}\right)$, and $A_2, B_2$ are symmetric matrices denoted by

$$A_2 := \begin{pmatrix} 1 - \frac{b_3 w_*}{(u_* + w_*)(u + w)} & \frac{b_1 - 1}{2} & \frac{b_3 u_* - w_*}{2(u_* + w_*)(u + w)} \\ \frac{b_1 - 1}{2} & 1 & \frac{b_2 - 1}{2} \\ \frac{b_3 u_* - w_*}{2(u_* + w_*)(u + w)} & \frac{b_2 - 1}{2} & 1 + \frac{u_*}{(u_* + w_*)(u + w)} \end{pmatrix},$$
$$B_2 := \begin{pmatrix} d_1 u_* & -\frac{\xi v_* u}{2} & -\frac{\chi w_* u v}{2} \\ -\frac{\xi v_* u}{2} & d_2 v_* & -\frac{\chi w_* u v}{2} \\ -\frac{\chi w_* u v}{2} & -\frac{\chi w_* u v}{2} & w_* \end{pmatrix}.$$

Recalling that $\|u\|_{L^\infty}$ and $\|v\|_{L^\infty}$ are independent of parameters $\xi$ and $\chi$ (see Theorem 1.1), one can find appropriate numbers $\xi_2 > 0$ and $\chi_2 > 0$ such that if $\xi \in (0, \xi_2)$ and $\chi \in (0, \chi_2)$, then

$$4 d_1 d_2 u_* v_* > \xi \chi^2 v_* w_* \|u\|_{L^\infty}^3 \|v\|_{L^\infty}^2 + \chi^2 w_* (d_2 v_* + d_1 u_*) \|u\|_{L^\infty}^2 \|v\|_{L^\infty}^2 + \xi^2 (v_*)^2 \|u\|_{L^\infty}^2,$$

which gives rises to

$$\begin{vmatrix} d_1 u_* & -\frac{\xi v_* u}{2} \\ -\frac{\xi v_* u}{2} & d_2 v_* \end{vmatrix} = \frac{v_* (4 d_1 d_2 u_* - \xi^2 v_* u^2)}{4} \geq \frac{v_* (4 d_1 d_2 u_* - \xi^2 v_* \|u\|_{L^\infty}^2)}{4} > 0$$

and

$$\begin{aligned}|B_2| &= -\frac{w_*}{4}\left[\xi\chi^2 v_* w_* u^3 v^2 + \chi^2 w_*(d_1 u_* + d_2 v_*)u^2 v^2 + \xi^2 v_*^2 u^2 - 4d_1 d_2 u_* v_*\right]\\ &\geq -\frac{w_*}{4}\left[\xi\chi^2 v^* w_* \|u\|_{L^\infty}^3 \|v\|_{L^\infty}^2 + \chi^2 w_*(d_1 u_* + d_2 v_*)\|u\|_{L^\infty}^2 \|v\|_{L^\infty}^2\right.\\ &\quad \left. + \xi^2 (v_*)^2 \|u\|_{L^\infty}^2 - 4d_1 d_2 u_* v_*\right]\\ &> 0.\end{aligned}$$

These imply that the matrix $B_2$ is positive definite and hence

$$Y_1 B_2 Y_1^T > 0. \tag{4.45}$$

Next, we claim the following:

- If $\frac{1}{10} \leq b_2 < \sqrt{2}$ and $b_1 = b_3 = 0$, then the matrix $A_2$ is positive definite.

In fact, if $b_1 = b_3 = 0$ and $0 < b_2 < \sqrt{2}$, one can check that the system (1.6) with $c_3 = 1$ has a unique positive coexistence steady state $(\bar{u}_*, \bar{v}_*, \bar{w}_*)$ satisfying (see also (A.3) in appendix)

$$\begin{cases}\bar{u}_* = 1,\\ \bar{v}_* = \frac{b_2^2 + 2b_2 + 4 - b_2\sqrt{(b_2+2)(b_2+10)}}{2(b_2+1)},\\ \bar{w}_* = \frac{2 - b_2 + \sqrt{(b_2+2)(b_2+10)}}{2(b_2+1)}.\end{cases} \tag{4.46}$$

Moreover, when $b_1 = b_3 = 0$, the corresponding matrix $A_2$ becomes

$$\tilde{A}_2 := \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{\bar{w}_*}{2(\bar{u}_* + \bar{w}_*)(u+w)}\\ -\frac{1}{2} & 1 & \frac{b_2 - 1}{2}\\ -\frac{\bar{w}_*}{2(\bar{u}_* + \bar{w}_*)(u+w)} & \frac{b_2-1}{2} & 1 + \frac{\bar{u}_*}{(\bar{u}_* + \bar{w}_*)(u+w)}\end{pmatrix},$$

which is positive definite if $|\tilde{A}_2| > 0$. After some calculations, one can check that

$$4|\tilde{A}_2| = 3 - (b_2 - 1)^2 - \frac{(\bar{w}_*)^2}{(\bar{u}_* + \bar{w}_*)^2 (u+w)^2} + \frac{3\bar{u}_* + (b_2 - 1)\bar{w}_*}{(\bar{u}_* + \bar{w}_*)(u+w)}. \tag{4.47}$$

From Lemma 4.4, we know that if $b_1 = b_3 = 0$, it holds that

$$\liminf_{t\to\infty} u(x,t) \geq 1,$$

which implies there exists $T_1 > 0$ such that $u(x,t) \geq \frac{\sqrt{2}}{2}$ for all $t \geq T_1$ and hence

$$u(x,t) + w(x,t) \geq \frac{\sqrt{2}}{2} \quad \text{for all } \ x \in \Omega \text{ and } t \geq T_1. \tag{4.48}$$

With (4.48) in hand and using the facts $\bar{u}_* = 1$ and $0 < b_2 < \sqrt{2}$, we can directly calculate that

(4.49)

$$3 - (b_2 - 1)^2 - \frac{(\bar{w}_*)^2}{(\bar{u}_* + \bar{w}_*)^2 (u+w)^2} \geq 3 - (b_2 - 1)^2 - 2\frac{(\bar{w}_*)^2}{(1 + \bar{w}_*)^2} \geq 1 - (b_2 - 1)^2 > 0.$$

On the other hand, if $b_2 \geq \frac{1}{10}$, one can derive that

$$3\bar{u}_* + (b_2 - 1)\bar{w}_* = 3 + (b_2 - 1)\bar{w}_* = \frac{9b_2 + 4 - b_2^2 + (b_2 - 1)\sqrt{(b_2+2)(b_2+10)}}{2(b_2+1)} > 0,$$

and hence

$$\frac{3\bar{u}_* + (b_2 - 1)\bar{w}_*}{(\bar{u}_* + \bar{w}_*)(u + w)} > 0. \tag{4.50}$$

Combining (4.47), (4.49), and (4.50), one finds that the matrix $\tilde{A}_2$ is positive definite if $\frac{1}{10} \le b_2 < \sqrt{2}$. By the continuity of $(u_*, v_*, w_*)$ with respect to $b_1$ and $b_3$ (see Remark A.1), if $b_1$ and $b_3$ are small enough and $\frac{1}{10} \le b_2 < \sqrt{2}$, $A_2$ is positive definite for all $t \ge T_1$. Then there exists a constant $\beta > 0$ such that

$$X_1 A_2 X_1^T \ge \beta X_1^2 \quad \text{for all } t \ge T_1. \tag{4.51}$$

Combining (4.44), (4.45), and (4.51), one has

$$\frac{d}{dt}\mathcal{E}_2(t) + \beta \mathcal{F}_2(t) \le 0 \quad \text{for all} \quad t \ge T_1,$$

where

$$\mathcal{F}_2(t) := \int_\Omega (u - u_*)^2 + (v - v_*)^2 + (w - w_*)^2.$$

Using the similar argument as in the proof of Lemma 4.4, we can show that

$$\|u(\cdot,t) - u_*\|_{L^\infty} + \|v(\cdot,t) - v_*\|_{L^\infty} + \|u(\cdot,t) - u_*\|_{L^\infty} \le Ce^{-\sigma t} \quad \text{for all } t > T_2$$

hold for some positive constants $C, \sigma$, and $T_2$. ◻

*Proof of Theorems* 1.2 *and* 1.4. Theorem 1.2 is a consequence of Lemma 4.3 and Theorem 1.4 results from Lemma 4.5. ◻

**Appendix.** The homogeneous coexistence steady state $(u_*, v_*, w_*)$ in (1.6) satisfies the following equations:

$$\begin{cases} 1 - u_* - b_1 v_* - b_3 \frac{w_*}{u_* + w_*} = 0, \\ 1 - v_* + u_* - b_2 w_* = 0, \\ 1 - w_* + v_* + \frac{u_*}{u_* + w_*} = 0, \end{cases}$$

which has two explicit solutions $(u_1^*, v_1^*, w_1^*)$ and $(u_2^*, v_2^*, w_2^*)$ as follows:

$$\begin{cases} u_1^* = \frac{2b_1^2(b_2^2-1)+b_1(b_2+2)(b_2-3b_3+2b_2b_3)+(1+b_2)(2+(b_2-2)b_3-b_3^2)}{2(1+b_1+b_2)(1+b_1(1+b_2)+(2+b_2)b_3)} \\ \qquad + \frac{(b_1b_2+b_2b_3+b_3)\sqrt{20+12b_2+b_2^2+4b_1^2(b_2^2-1)+4b_1(3+b_2)(b_2-b_3)-20b_3-14b_2b_3+b_3^2}}{2(1+b_1+b_2)(1+b_1(1+b_2)+(2+b_2)b_3)}, \\ v_1^* = \frac{4+2b_2+b_2^2+2b_3-4b_2b_3-4b_2^2b_3-b_3^2-2b_1(1+b_2)(b_2+b_3-2)}{2(1+b_1+b_2)(1+b_1(1+b_2)+(2+b_2)b_3)} \\ \qquad + \frac{(b_3-b_2)\sqrt{20+12b_2+b_2^2+4b_1^2(b_2^2-1)+4b_1(3+b_2)(b_2-b_3)-20b_3-14b_2b_3+b_3^2}}{2(1+b_1+b_2)(1+b_1(1+b_2)+(2+b_2)b_3)}, \\ w_1^* = \frac{2+4b_1+2b_1^2-b_2+5b_1b_2+2b_1^2b_2+9b_3+5b_1b_3+7b_2b_3+2b_1b_2b_3-b_3^2}{2(1+b_1+b_2)(1+b_1(1+b_2)+(2+b_2)b_3)} \\ \qquad + \frac{(1+b_1+b_3)\sqrt{20+b_2^2+4b_1^2(b_2^2-1)+4b_1(b_2+3)(b_2-b_3)-20b_3+b_3^2-2b_2(7b_3-6)}}{2(1+b_1+b_2)(1+b_1(1+b_2)+(2+b_2)b_3)} \end{cases} \tag{A.1}$$

and

$$\text{(A.2)}\quad \begin{cases} u_2^* = \frac{2b_1^2(b_2^2-1)+b_1(b_2+2)(b_2-3b_3+2b_2b_3)+(1+b_2)(2+(b_2-2)b_3-b_3^2)}{2(1+b_1+b_2)(1+b_1(1+b_2)+(2+b_2)b_3)} \\ \qquad -\frac{(b_1b_2+b_2b_3+b_3)\sqrt{20+12b_2+b_2^2+4b_1^2(b_2^2-1)+4b_1(3+b_2)(b_2-b_3)-20b_3-14b_2b_3+b_3^2}}{2(1+b_1+b_2)(1+b_1(1+b_2)+(2+b_2)b_3)}, \\ v_2^* = \frac{4+2b_2+b_2^2+2b_3-4b_2b_3-4b_2^2b_3-b_3^2-2b_1(1+b_2)(b_2+b_3-2)}{2(1+b_1+b_2)(1+b_1(1+b_2)+(2+b_2)b_3)} \\ \qquad -\frac{(b_3-b_2)\sqrt{20+12b_2+b_2^2+4b_1^2(b_2^2-1)+4b_1(3+b_2)(b_2-b_3)-20b_3-14b_2b_3+b_3^2}}{2(1+b_1+b_2)(1+b_1(1+b_2)+(2+b_2)b_3)}, \\ w_2^* = \frac{2+4b_1+2b_1^2-b_2+5b_1b_2+2b_1^2b_2+9b_3+5b_1b_3+7b_2b_3+2b_1b_2b_3-b_3^2}{2(1+b_1+b_2)(1+b_1(1+b_2)+(2+b_2)b_3)} \\ \qquad -\frac{(1+b_1+b_3)\sqrt{20+b_2^2+4b_1^2(b_2^2-1)+4b_1(b_2+3)(b_2-b_3)-20b_3+b_3^2-2b_2(7b_3-6)}}{2(1+b_1+b_2)(1+b_1(1+b_2)+(2+b_2)b_3)}. \end{cases}$$

*Remark* A.1. It can be seen from (A.1) and (A.2) that $(u_1^*, v_1^*, w_1^*)$ and $(u_2^*, v_2^*, w_2^*)$ are continuous with respect to $b_1, b_2$, and $b_3$ for any $b_1, b_2, b_3 \geq 0$. If $b_1 = b_3 = 0$, we have

$$\text{(A.3)}\quad \begin{cases} u_1^* = 1, \\ v_1^* = \frac{b_2^2+2b_2+4-b_2\sqrt{(b_2+2)(b_2+10)}}{2(b_2+1)}, \\ w_1^* = \frac{2-b_2+\sqrt{(b_2+2)(b_2+10)}}{2(b_2+1)} \end{cases}$$

and

$$\begin{cases} u_2^* = 1, \\ v_2^* = \frac{b_2^2+2b_2+4+b_2\sqrt{(b_2+2)(b_2+10)}}{2(b_2+1)}, \\ w_2^* = \frac{2-b_2-\sqrt{(b_2+2)(b_2+10)}}{2(b_2+1)}. \end{cases}$$

Clearly, if $0 < b_2 < \sqrt{2}$, one has

$$\text{(A.4)}\quad u_1^*, v_1^*, w_1^* > 0 \quad \text{and} \quad u_2^* > 0, v_2^* > 0, w_2^* < 0.$$

Therefore, by the continuity, we still have (A.4) if $b_1$ and $b_3$ are small enough and $0 < b_2 < \sqrt{2}$. In this case, we only have one positive coexistence steady state $(u_1^*, v_1^*, w_1^*)$.

**Acknowledgments.** We are very thankful to the referees for carefully reading our paper and giving us various corrections and very insightful suggestions/comments, which greatly improve the precision of our results and exposition of our manuscript.

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