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The Relaxation Limit of a Quasi-Linear Hyperbolic-Parabolic Chemotaxis System Modeling Vasculogenesis

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Abstract. This paper is concerned with the relaxation limit of a three-dimensional quasi-linear hyperbolic-parabolic chemotaxis system modeling vasculogenesis when the initial data are prescribed around a constant ground state. When the relaxation time tends to zero (i.e. the damping is strong), we show that the strong-weak limit of the cell density and chemoattractant concentration satisfies a parabolic-elliptic Keller-Segel type chemotaxis system in the sense of distribution.

AMS subject classifications: 35L60, 35L04, 35B40, 35Q92

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1 Introduction

In this paper, we are concerned with the following quasi-linear hyperbolic-parabolic chemotaxis system modeling vasculogenesis – the vitro formation of new blood vessels, proposed by Gamba *et al.* [11]:

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$$\begin{cases} \partial_t\rho+\nabla\cdot(\rho u)=0, & \text{(1.1a)} \\ \partial_t(\rho u)+\nabla\cdot(\rho u\otimes u)+\nabla P(\rho)=-\dfrac{1}{\tau}\rho u+\mu\rho\nabla\phi, & \text{(1.1b)} \\ \partial_t\phi=D\Delta\phi+a\rho-b\phi. & \text{(1.1c)} \end{cases}$$

Here the unknowns $\rho=\rho(x,t)$ and $u=u(x,t)\in\mathbb{R}^3$ denote the density and velocity of the endothelial cell, respectively, and $\phi=\phi(x,t)$ the chemoattractant concentration, at $t>0$ and $x\in\mathbb{R}^3$. The density-dependent quantity P is the pressure function which is smooth and satisfies $P'(\rho)>0$ for $\rho>0$. D,a and b are positive constants representing the diffusion coefficient, production rate and degradation rate of the chemoattractant, $|\mu|$ with $\mu\in\mathbb{R}\backslash\{0\}$ is the cell response intensity to the chemoattractant. $0<\tau\ll 1$ is a relaxation time. The initial data are given by

$$[\rho,u,\phi]|_{t=0}=[\rho_0,u_0,\phi_0](x) \ \rightarrow \ (\bar{\rho},0,\bar{\phi}) \quad \text{as} \quad |x| \ \rightarrow \ \infty \tag{1.2}$$

with constants $\bar{\rho}>0$ and $\bar{\phi}>0$. When the initial value $[\rho_0,u_0,\phi_0]\in H^s(\mathbb{R}^d)$, $s>d/2+1$, is a small perturbation of the constant ground state (i.e. equilibrium) $[\bar{\rho},0,\bar{\phi}]$ with $\bar{\rho}>0$ sufficiently small, the global existence and stability of solutions to (1.1) without vacuum converging to $[\bar{\rho},0,\bar{\phi}]$ was established in [7,8]. By adding a viscous term Δu to the Eq. (1.1b), the linear stability of the constant ground state $[\bar{\rho},0,\bar{\phi}]$ was obtained in [17] under the condition

$$bP'(\bar{\rho})-a\mu\bar{\rho}>0. \tag{1.3}$$

The stationary solutions of (1.1) with vacuum (bump solutions) in a bounded interval with zero-flux boundary condition were constructed in [1,2]. Recently the stability of transition layer solutions of (1.1) on $\mathbb{R}_+=[0,\infty)$ was established in [13] and the convergence to diffusion waves for solutions of (1.1) was obtained in [20] for $x\in\mathbb{R}^3$.

As $\tau\to 0$ (strong damping), it was formally derived in [4] by the asymptotic analysis that the solution of (1.1) converges to the well-known Keller-Segel model. In [6], the authors considered different dissipation relaxation limits of model (1.1), and proved in L^p ($p\geq 1$) space that the convergence limit is the parabolic-elliptic Keller-Segel model by the energy methods and compensated compactness tools. An interesting question is whether the relaxation limit problem of (1.1) can be proved in a stronger sense, such as in H^s ($s\geq 3$) space. For the isothermal compressible Euler equations, namely $P(\rho)=k\rho$ for some constant $k>0$, by using a stream function, Junca and Rascle [15] showed that the solutions to the damped isothermal Euler equations converge to those of the heat equation for large BV initial data. Later, Coulombel and Goudon [5] studied the global existence of smooth solutions and the convergence to the heat equation as the relaxation time tends to zero. For the multidimensional damped isentropic Euler equations, Lin and Coulombel [19] showed that the asymptotic behavior of the global smooth solutions is governed by the porous media equation as the relaxation time tends to zero. We further refer to [3, 25, 26] for other interesting research in this topic. For some other types of hyperbolic or hyperbolic-parabolic chemotaxis models, we refer to [10, 12, 18, 22, 23] and references therein.

The aim of this paper is to construct global smooth solutions to (1.1), and to show that, in an appropriate time scaling, as τ tends to 0, the solution component (ρ,ϕ) of (1.1) converges to the solution of the following Keller-Segel type chemotaxis model:

$$\begin{cases} \partial_t\rho^0-\nabla\cdot\left(\nabla P(\rho^0)-\mu\rho^0\nabla\phi^0\right)=0,\\ \Delta\phi^0+a\rho^0-b\phi^0=0, \end{cases}$$

which is well-known as, if $P(\rho)=\rho$, the parabolic-elliptic minimal Keller-Segel system (cf. [16,21]) extensively studied in the literature [14,23], where the chemotaxis is called "attractive" if $\mu>0$ and repulsive if $\mu<0$.

Throughout this paper, we assume that

$$bP'(\bar\rho)-a\mu\bar\rho>0. \tag{1.4}$$

As will be seen later, the strict positivity of $bP'(\bar\rho)-a\mu\bar\rho$ plays an important role in our analysis in ensuring the dissipation of the system. However, the condition (1.4) is free when $\mu<0$.

Our first result on the existence of global solutions of (1.1)-(1.2) is given below.

Theorem 1.1. *For any $\bar\rho>0$ and $\bar\phi=(a/b)\bar\rho$, if $P(\cdot)$ is smooth on $(0,\infty)$ and satisfies the condition* (1.4) *for given $\bar\rho>0$, there exists a constant $\epsilon>0$ such that for any $\tau\in(0,1]$, if*

$$\left\|[\rho_0-\bar\rho,u_0,\phi_0-\bar\phi]\right\|_{H^N(\mathbb{R}^3)}+\|\nabla\phi_0\|_{H^N(\mathbb{R}^3)}<\epsilon \tag{1.5}$$

with $N\geq 3$, then the Cauchy problem (1.1)-(1.2) *admits a unique global solution with*

$$[\rho(x,t)-\bar\rho,u(x,t),\phi(x,t)-\bar\phi,\nabla\phi(x,t)]\in C\left([0,\infty);H^N(\mathbb{R}^3)\right)$$

satisfying

$$\begin{aligned}&\left\|[\rho(\cdot,t)-\bar\rho,u(\cdot,t),\phi(\cdot,t)-\bar\phi,\nabla\phi(\cdot,t)]\right\|^2_{H^N(\mathbb{R}^3)}+\frac{1}{\tau}\int_0^t\|u(\cdot,s)\|^2_{H^N(\mathbb{R}^3)}ds\\ &\quad+\tau\int_0^t\left(\|\nabla\rho(\cdot,s)\|^2_{H^{N-1}(\mathbb{R}^3)}+\|\nabla\phi(\cdot,s)\|^2_{H^N(\mathbb{R}^3)}+\|\phi_t(\cdot,s)\|^2_{H^N(\mathbb{R}^3)}\right)ds\\ &\leq C_0\left\|[\rho_0-\bar\rho,u_0,\phi_0-\bar\phi,\nabla\phi_0]\right\|^2_{H^N(\mathbb{R}^3)},\end{aligned}$$

where $C_0>0$ is a constant independent of τ and t.

The next theorem asserts the convergence of solutions as $\tau \to 0$. For the convenience of statement, we denote the solution obtained in Theorem 1.1 by $(\rho^\tau, u^\tau, \phi^\tau)$. Then our second main result is given in the following theorem.

Theorem 1.2. *Let $(\rho^\tau, u^\tau, \phi^\tau)$ be the unique global solution obtained in Theorem* 1.1*. Set $s=\tau t$ and define*

$$(\varrho^\tau, m^\tau, \varphi^\tau)(x,s) := \left(\rho^\tau(x,t), \frac{\rho^\tau(x,t) u^\tau(x,t)}{\tau}, \phi^\tau(x,t)\right).$$

Then under the assumptions in Theorem 1.1*, the solutions $(\varrho^\tau, m^\tau, \varphi^\tau)$ satisfy that*

$$\begin{aligned} &\|[\varrho^\tau(s)-\bar{\rho}, \varphi^\tau(s)-\bar{\phi}, \nabla\varphi^\tau(s)]\|^2_{H^N(\mathbb{R}^3)} + \tau^2 \|m^\tau(s)\|^2_{H^N(\mathbb{R}^3)} + \int_0^\infty \|m^\tau(s)\|^2_{H^N(\mathbb{R}^3)} ds \\ &\quad + \int_0^\infty \left(\|\nabla \varrho^\tau(s)\|^2_{H^{N-1}(\mathbb{R}^3)} + \|\nabla\varphi^\tau(s)\|^2_{H^N(\mathbb{R}^3)} + \tau \|\partial_s \varphi^\tau(s)\|^2_{H^N(\mathbb{R}^3)} \right) ds \le C, \end{aligned}$$

where C is a constant independent of τ. Thus, for any $0<T,R<+\infty$, extracting a subsequence if necessary, it follows that as $\tau \to 0$

$$\begin{aligned} \varrho^\tau &\to \varrho^0 \quad \text{strongly in } C\big([0,T); H^m(B(R))\big), \\ \varphi^\tau &\rightharpoonup \varphi^0 \quad \text{weakly in } L^2\big([0,T); H^N(B(R))\big), \end{aligned}$$

where $N-1<m<N$ and $B(R)=:\{x \in \mathbb{R}^3 : |x| \le R\}$. Moreover, (ϱ^0, φ^0) satisfies, in the sense of distributions, the following Keller-Segel type chemotaxis system:

$$\begin{cases} \partial_s \varrho^0 - \nabla \cdot \big(\nabla P(\varrho^0) - \mu \varrho^0 \nabla \varphi^0\big) = 0, \\ \Delta \varphi^0 + a\varrho^0 - b\varphi^0 = 0, \\ \varrho^0|_{s=0} = \rho_0. \end{cases}$$

2 Uniform estimates

We proceed by mentioning some notations frequently used in the paper.

Notation. Throughout this paper, C denotes a generic positive constant (generally large) and λ denotes some positive constant (generally small), where both C and λ may take different values in different places. For two quantities a and $b, a \sim b$ means $ca \le b \le a/c$ for a generic constant $0<c<1$. For any integer $m \ge 0$, we use H^m to denote the usual Sobolev space $H^m(\mathbb{R}^3)$. For simplicity, the norm of H^m is denoted by $\|\cdot\|_m$ with $\|\cdot\|=\|\cdot\|_0$. We use $\langle\cdot,\cdot\rangle$ to denote the inner product of the Hilbert space $L^2(\mathbb{R}^3)$, i.e.

$$\langle f,g\rangle=\int_{\mathbb{R}^3} f(x)g(x)dx, \quad \forall f,g\in L^2(\mathbb{R}^3).$$

For a multi-index $l=(l_1,l_2,l_3)$, we denote $\partial^l=\partial_{x_1}^{l_1}\partial_{x_2}^{l_2}\partial_{x_3}^{l_3}$ and the length of l is $|l|=l_1+l_2+l_3$. For simplicity, we denote

$$\|[A,B]\|_X=\|A\|_X+\|B\|_X$$

for some Sobolev space X.

In this section, we will construct global smooth solutions to (1.1) with initial data (1.2) satisfying (1.5). First, we present the following Sobolev inequality for the L^p estimate on products of derivatives of two functions (cf. [9]).

Lemma 2.1. *Let $\theta=(\theta_1,\cdots,\theta_n)$ and $\eta=(\eta_1,\cdots,\eta_n)$ be two multi-indices with $|\theta|=k_1$, $|\eta|=k_2$ and set $k=k_1+k_2$. Then, for $1\le p,q,r\le\infty$ with $1/p=1/q+1/r$, we have*

$$\left\|\partial^\theta u_1\partial^\eta u_2\right\|_{L^p(\mathbb{R}^n)}\le C\left(\|u_1\|_{L^q(\mathbb{R}^n)}\left\|\nabla^k u_2\right\|_{L^r(\mathbb{R}^n)}+\|u_2\|_{L^q(\mathbb{R}^n)}\left\|\nabla^k u_1\right\|_{L^r(\mathbb{R}^n)}\right), \tag{2.1}$$

where C is a positive constant.

The well-known Aubin-Lions-Simon lemma [24] is cited below for later use.

Lemma 2.2 (Aubin-Lions-Simon Lemma)**.** *Let X_0,X,X_1 be three Banach spaces with $X_0\subseteq X\subseteq X_1$. Suppose that X_0 is compactly embedded in X and that X is continuously embedded in X_1. For $1\le p,q\le\infty$, let*

$$W=\left\{f\in L^p([0,T];X_0)\mid \partial_t f\in L^q([0,T];X_1)\right\}.$$

(i) If $p<\infty$, then the embedding of W into $L^p([0,T];X)$ is compact (that is W is relatively compact in $L^p([0,T];X)$).

(ii) If $p=\infty$ and $q>1$, then the embedding of W into $C([0,T];X)$ is compact.

2.1 Reformulation of the problem

Before constructing the global smooth solutions to (1.1), we first rewrite (1.1) around the constant state $[\bar\rho,0,\bar\phi]$ with $\bar\phi=(a/b)\bar\rho$. By writing $\tilde\rho=\rho-\bar\rho,\tilde u=u-0,$

$\tilde{\phi}=\phi-\bar{\phi}$, we reformulate the Cauchy problem (1.1)-(1.2) as

$$
\begin{cases}
\partial_t\tilde{\rho}+\bar{\rho}\nabla\cdot\tilde{u}=g_1, & (2.2\text{a})\\
\partial_t\tilde{u}+\dfrac{P'(\bar{\rho})}{\bar{\rho}}\nabla\tilde{\rho}+\dfrac{1}{\tau}\tilde{u}-\mu\nabla\tilde{\phi}=g_2, & (2.2\text{b})\\
\partial_t\tilde{\phi}-D\Delta\tilde{\phi}-a\tilde{\rho}+b\tilde{\phi}=0 & (2.2\text{c})
\end{cases}
$$

with initial data given by

$$
[\tilde{\rho},\tilde{u},\tilde{\phi}]|_{t=0}=[\tilde{\rho}_0,\tilde{u}_0,\tilde{\phi}_0]=[\rho_0-\bar{\rho},u_0,\phi_0-\bar{\phi}]. \tag{2.3}
$$

Here g_1,g_2 are defined as follows:

$$
\begin{cases}
g_1=-\nabla\cdot(\tilde{\rho}\tilde{u}),\\
g_2=-\tilde{u}\cdot\nabla\tilde{u}-\left(\dfrac{P'(\tilde{\rho}+\bar{\rho})}{\tilde{\rho}+\bar{\rho}}-\dfrac{P'(\bar{\rho})}{\bar{\rho}}\right)\nabla\tilde{\rho}.
\end{cases}
$$

Next we shall focus on the reformulated problem (2.2)-(2.3) and explore the global existence of solutions. Without confusion, in the rest of this section, we still use $[\rho,u,\phi]$ to denote $[\tilde{\rho},\tilde{u},\tilde{\phi}]$, correspondingly, $[\rho_0,u_0,\phi_0]$ to denote $[\tilde{\rho}_0,\tilde{u}_0,\tilde{\phi}_0]$ for simplicity unless otherwise stated. The main result of this section about the global existence of solutions to the reformulated Cauchy problem (2.2)-(2.3) with small smooth initial data are stated as follows.

Proposition 2.1. *Given $\bar{\rho}>0$ and $N\geq 3$. Let $\bar{\phi}=(a/b)\bar{\rho}$ and $P(\cdot)$ be smooth on $(0,\infty)$ with $bP'(\bar{\rho})-a\mu\bar{\rho}>0$. If $\|[\rho_0,u_0,\phi_0]\|_N^2+\|\nabla\phi_0\|_N^2$ is small enough, then the Cauchy problem* (2.2)-(2.3) *admits a unique global solution $U=[\rho,u,\phi]$ satisfying*

$$
U\in C\big([0,\infty);H^N(\mathbb{R}^3)\big),\quad \nabla\phi\in C\big([0,\infty);H^N(\mathbb{R}^3)\big),
$$

and

$$
\begin{aligned}
&\|[\rho,u,\phi]\|_N^2+\|\nabla\phi\|_N^2+\int_0^t\left(\|\phi_t\|_N^2+\frac{1}{\tau}\|u\|_N^2+\tau\|\nabla\rho\|_{N-1}^2+\tau\|\nabla\phi\|_N^2\right)ds\\
\leq{}& C\left(\|[\rho_0,u_0,\phi_0]\|_N^2+\|\nabla\phi_0]\|_N^2\right),
\end{aligned}
$$

where $C>0$ is independent of τ and t.

The existence of local-in-time solutions of (2.2)-(2.3) can be readily established by the standard iteration method and details are omitted for brevity. Then to prove Proposition 2.1, it suffices to derive the a priori estimates in the following lemma.

Lemma 2.3 (A Priori Estimates). *Assume that the conditions in Proposition* 2.1 *hold. Let*

$$U=[\rho,u,\phi]\in C\big([0,T);H^N(\mathbb{R}^3)\big)$$

be a solution to the system (2.2)-(2.3) *with*

$$\|[\rho,u,\phi]\|_N^2+\|\nabla\phi\|_N^2\ll 1$$

for any $0\le t<T$. *Then*

$$\begin{aligned}
&\frac{d}{dt}\sum_{|l|\le N}\bigg\{\int_{\mathbb{R}^3}\frac{P'(\rho+\bar\rho)}{\rho+\bar\rho}|\partial^l\rho|^2dx+\int_{\mathbb{R}^3}(\rho+\bar\rho)|\partial^l u|^2dx\\
&\qquad\qquad -2\mu\langle\partial^l\phi,\partial^l\rho\rangle+\frac{b\mu}{a}\|\partial^l\phi\|^2+\frac{\mu D}{a}\|\partial^l\nabla\phi\|^2\bigg\}\\
&\quad+\kappa\tau\frac{d}{dt}\sum_{|l|\le N-1}\Big\{2\langle\partial^l u,\partial^l\nabla\rho\rangle+\frac{\mu}{a}\|\partial^l\nabla\phi\|^2\Big\}+\kappa\lambda\tau\|\nabla\rho\|_{N-1}^2\\
&\quad+\kappa\lambda\tau\|\nabla\phi\|_N^2+\frac{\bar\rho}{2\tau}\|u\|_N^2+\frac{2\mu}{a}\|\phi_t\|_N^2\le 0,
\end{aligned}$$

where $\lambda>0$ *and* $0<\kappa\ll 1$ *are constants.*

Proof. Our proof is motivated by the work [20] and consists of three steps.

Step 1. We first claim that

$$\begin{aligned}
&\frac12\frac{d}{dt}\sum_{|l|\le N}\bigg\{\int_{\mathbb{R}^3}\frac{P'(\rho+\bar\rho)}{\rho+\bar\rho}|\partial^l\rho|^2dx+\int_{\mathbb{R}^3}(\rho+\bar\rho)|\partial^l u|^2dx\\
&\qquad\qquad -2\mu\langle\partial^l\phi,\partial^l\rho\rangle+\frac{b\mu}{a}\|\partial^l\phi\|^2+\frac{\mu D}{a}\|\partial^l\nabla\phi\|^2\bigg\}\\
&\quad+\frac{\bar\rho}{2\tau}\|u\|_N^2+\frac{\mu}{a}\|\phi_t\|_N^2\\
&\le C\tau\|[\rho,u,\phi]\|_N^2\big(\|\nabla[\rho,u,\phi]\|_{N-1}^2+\|\nabla\phi\|_N^2\big).
\end{aligned}\tag{2.4}$$

In fact, it is convenient to start from the following reformulated form of (2.2):

$$\begin{cases}
\partial_t\rho+(\rho+\bar\rho)\nabla\cdot u=-u\cdot\nabla\rho, & \text{(2.5a)}\\
\partial_t u+\dfrac{P'(\rho+\bar\rho)}{\rho+\bar\rho}\nabla\rho+\dfrac{1}{\tau}u-\mu\nabla\phi=-u\cdot\nabla u, & \text{(2.5b)}\\
\partial_t\phi-D\Delta\phi-a\rho+b\phi=0. & \text{(2.5c)}
\end{cases}$$

Applying ∂^l to the Eq. (2.5a) for $0\le|l|\le N$, multiplying the result by $P'(\rho+\bar\rho)\times(\rho+\bar\rho)^{-1}\partial^l\rho$ and taking integration in x give

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}\frac{P'(\rho+\bar\rho)}{\rho+\bar\rho}|\partial^l\rho|^2dx+\langle P'(\rho+\bar\rho)\partial^l\nabla\cdot u,\partial^l\rho\rangle\\
=&\frac{1}{2}\left\langle\left(\frac{P'(\rho+\bar\rho)}{\rho+\bar\rho}\right)_t\partial^l\rho,\partial^l\rho\right\rangle-\sum_{k<l}C_l^k\left\langle\partial^{l-k}(\rho+\bar\rho)\partial^k\nabla\cdot u,\frac{P'(\rho+\bar\rho)}{\rho+\bar\rho}\partial^l\rho\right\rangle\\
&-\left\langle u\cdot\partial^l\nabla\rho,\frac{P'(\rho+\bar\rho)}{\rho+\bar\rho}\partial^l\rho\right\rangle-\sum_{k<l}C_l^k\left\langle\partial^{l-k}u\cdot\partial^k\nabla\rho,\frac{P'(\rho+\bar\rho)}{\rho+\bar\rho}\partial^l\rho\right\rangle. \qquad (2.6)
\end{aligned}
$$

Applying ∂^l to the Eq. (2.5b) for $0\le|l|\le N$, multiplying it by $(\rho+\bar\rho)\partial^l u$, and integrating the resulting equation with respect to x, we get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}(\rho+\bar\rho)|\partial^l u|^2dx+\left\langle P'(\rho+\bar\rho)\partial^l\nabla\rho,\partial^l u\right\rangle+\frac{1}{\tau}\int_{\mathbb{R}^3}(\rho+\bar\rho)|\partial^l u|^2dx\\
=&\frac{1}{2}\langle(\rho+\bar\rho)_t\partial^l u,\partial^l u\rangle-\sum_{k<l}C_l^k\left\langle\partial^{l-k}\left(\frac{P'(\rho+\bar\rho)}{\rho+\bar\rho}\right)\partial^k\nabla\rho,(\rho+\bar\rho)\partial^l u\right\rangle\\
&-\langle u\cdot\partial^l\nabla u,(\rho+\bar\rho)\partial^l u\rangle-\sum_{k<l}C_l^k\langle\partial^{l-k}u\cdot\partial^k\nabla u,(\rho+\bar\rho)\partial^l u\rangle\\
&+\mu\langle\partial^l\nabla\phi,(\rho+\bar\rho)\partial^l u\rangle. \qquad (2.7)
\end{aligned}
$$

With integration by parts and (2.5a), we can update the last term in (2.7) as follows:

$$
\begin{aligned}
&\mu\langle\partial^l\nabla\phi,(\rho+\bar\rho)\partial^l u\rangle\\
=&-\mu\langle\partial^l\phi,(\rho+\bar\rho)\partial^l\nabla\cdot u\rangle-\mu\langle\partial^l\phi,\nabla\rho\partial^l u\rangle\\
=&\mu\langle\partial^l\phi,\partial^l\rho_t\rangle+\mu\langle\partial^l\phi,\partial^l(u\cdot\nabla\rho)\rangle+\mu\left\langle\partial^l\phi,\sum_{k<l}C_l^k\partial^{l-k}(\rho+\bar\rho)\partial^k\nabla\cdot u\right\rangle-\mu\langle\partial^l\phi,\nabla\rho\partial^l u\rangle\\
=&\frac{d}{dt}\mu\langle\partial^l\phi,\partial^l\rho\rangle-\mu\langle\partial^l\phi_t,\partial^l\rho\rangle+\mu\langle\partial^l\phi,u\cdot\partial^l\nabla\rho\rangle-\mu\langle\partial^l\phi,\nabla\rho\partial^l u\rangle\\
&+\mu\left\langle\partial^l\phi,\sum_{k<l}C_l^k\partial^{l-k}(\rho+\bar\rho)\partial^k\nabla\cdot u\right\rangle+\mu\left\langle\partial^l\phi,\sum_{k<l}C_l^k\partial^{l-k}u\cdot\partial^k\nabla\rho\right\rangle.
\end{aligned}
$$

Applying ∂^l to the Eq. (2.5c) for $0\le|l|\le N$, multiplying it by $(\mu/a)\partial^l\phi_t$, and integrating the resultant equation with respect to x, we have

$$
\frac{d}{dt}\left\{\frac{b\mu}{2a}\|\partial^l\phi\|^2+\frac{\mu D}{2a}\|\partial^l\nabla\phi\|^2\right\}+\frac{\mu}{a}\|\partial^l\phi_t\|^2=\mu\langle\partial^l\phi_t,\partial^l\rho\rangle. \qquad (2.8)
$$

It follows from (2.6)-(2.8) and the integration by parts that

$$\begin{aligned}
&\frac{d}{dt}\bigg(\frac{1}{2}\int_{\mathbb{R}^3}\frac{P'(\rho+\bar{\rho})}{\rho+\bar{\rho}}|\partial^l\rho|^2dx+\frac{1}{2}\int_{\mathbb{R}^3}(\rho+\bar{\rho})\partial^l u|^2dx\\
&\qquad -\mu\langle\partial^l\phi,\partial^l\rho\rangle+\frac{b\mu}{2a}\|\partial^l\phi\|^2+\frac{\mu D}{2a}\|\partial^l\nabla\phi\|^2\bigg)\\
&\quad+\frac{1}{\tau}\int_{\mathbb{R}^3}(\rho+\bar{\rho})|\partial^l u|^2dx+\frac{\mu}{a}\|\partial^l\phi_t\|^2\\
&=I_1^l(t)+\sum_{k<l}C_l^k I_{k,l}^l(t),
\end{aligned}\tag{2.9}$$

where

$$\begin{aligned}
I_1^l(t)=&\big\langle P''(\rho+\bar{\rho})\nabla\rho\partial^l\rho,\partial^l u\big\rangle+\frac{1}{2}\left\langle\left(\frac{P'(\rho+\bar{\rho})}{\rho+\bar{\rho}}\right)_t,|\partial^l\rho|^2\right\rangle\\
&+\frac{1}{2}\big\langle(\rho+\bar{\rho})_t,|\partial^l u|^2\big\rangle+\frac{1}{2}\left\langle\nabla\cdot\left(u\frac{P'(\rho+\bar{\rho})}{\rho+\bar{\rho}}\right),|\partial^l\rho|^2\right\rangle\\
&+\frac{1}{2}\big\langle\nabla\cdot\big(u(\rho+\bar{\rho})\big),|\partial^l u|^2\big\rangle-\mu\big\langle\partial^l\nabla\phi,u\partial^l\rho\big\rangle\\
&-\mu\big\langle\partial^l\phi,\nabla\cdot u\partial^l\rho\big\rangle-\mu\big\langle\partial^l\phi,\nabla\rho\partial^l u\big\rangle,\\
I_{k,l}^l(t)=&-\left\langle\partial^{l-k}(\rho+\bar{\rho})\partial^k\nabla\cdot u,\frac{P'(\rho+\bar{\rho})}{\rho+\bar{\rho}}\partial^l\rho\right\rangle-\left\langle\partial^{l-k}u\cdot\partial^k\nabla\rho,\frac{P'(\rho+\bar{\rho})}{\rho+\bar{\rho}}\partial^l\rho\right\rangle\\
&-\left\langle\partial^{l-k}\left(\frac{P'(\rho+\bar{\rho})}{\rho+\bar{\rho}}\right)\partial^k\nabla\rho,(\rho+\bar{\rho})\partial^l u\right\rangle-\big\langle\partial^{l-k}u\cdot\partial^k\nabla u,(\rho+\bar{\rho})\partial^l u\big\rangle\\
&+\mu\big\langle\partial^l\phi,\partial^{l-k}(\rho+\bar{\rho})\partial^k\nabla\cdot u\big\rangle+\mu\big\langle\partial^l\phi,\partial^{l-k}u\cdot\partial^k\nabla\rho\big\rangle.
\end{aligned}$$

Next, we estimate $I_1^l(t)$ and $I_{k,l}^l(t)$ term by term. The key part is to get a uniform bound independent of τ. When $|l|=0$, by the Cauchy-Schwarz and Gagliardo-Nirenberg inequalities, $I_1^l(t)$ can be estimated as

$$\begin{aligned}
|I_1^l(t)|\le& C\|\nabla\rho\|\|\rho\|_{L^6}\|u\|_{L^3}+C\|\nabla\cdot u\|\big(\|\rho\|_{L^6}\|\rho\|_{L^3}+\|u\|_{L^6}\|u\|_{L^3}\big)\\
&+C\|\nabla\rho\|\big(\|u\|_{L^6}\|\rho\|_{L^3}+\|u\|_{L^6}\|u\|_{L^3}\big)+C\|\nabla\phi\|_{L^2}\|\rho\|_{L^6}\|u\|_{L^3}\\
&+C\|\phi\|_{L^3}\|\nabla\cdot u\|_{L^2}\|\rho\|_{L^6}+C\|\nabla\rho\|_{L^2}\|\phi\|_{L^6}\|u\|_{L^3}\\
\le&\frac{\bar{\rho}}{8\tau}\|u\|_1^2+C\tau\|[\rho,u,\phi]\|_1^2\|\nabla[\rho,u,\phi]\|^2,
\end{aligned}\tag{2.10}$$

where we have used the facts $\|[\rho,u,\phi]\|_N\ll 1$ and $\partial_t\rho+(\rho+\bar{\rho})\nabla\cdot u=-u\cdot\nabla\rho$. When $|l|\ge 1$, it follows from the Cauchy-Schwarz inequality that

$$|I_1^l(t)|\le C\|\nabla\rho\|_{L^\infty}\|\partial^l\rho\|\|\partial^l u\|+C\|\nabla\cdot u\|_{L^\infty}\big(\|\partial^l\rho\|^2+\|\partial^l u\|^2\big)$$

$$
\begin{aligned}
&+C\|u\|_{L^\infty}\|\nabla\rho\|_{L^\infty}\left(\|\partial^l\rho\|^2+\|\partial^l u\|^2\right)+C\|u\|_{L^\infty}\|\partial^l\rho\|\|\partial^l\nabla\phi\|\\
&+C\|\nabla\cdot u\|_{L^\infty}\|\partial^l\rho\|\|\partial^l\phi\|+C\|\nabla\rho\|_{L^\infty}\|\partial^l u\|\|\partial^l\phi\|\\
\le&\frac{\bar\rho}{8\tau}\|u\|_N^2+C\tau\|[\rho,u]\|_N^2\left(\|\nabla[\rho,u,\phi]\|_{N-1}^2+\|\nabla\phi\|_N^2\right),
\end{aligned}
$$

which along with (2.10) yields for any l,

$$
\left|I_1^l(t)\right|\le\frac{\bar\rho}{8\tau}\|u\|_N^2+C\tau\|[\rho,u,\phi]\|_N^2\left(\|\nabla[\rho,u,\phi]\|_{N-1}^2+\|\nabla\phi\|_N^2\right). \tag{2.11}
$$

For $\sum_{k<l}C_l^kI_{k,l}^l(t)$, we have

$$
\begin{aligned}
\left|\sum_{k<l}C_l^kI_{k,l}^l(t)\right|\le&\sum_{k<l}C_l^k\|\partial^l\rho\|\|\partial^{l-k}(\rho+\bar\rho)\partial^k\nabla\cdot u\|+\sum_{k<l}C_l^k\|\partial^l\rho\|\|\partial^{l-k}u\cdot\partial^k\nabla\rho\|\\
&+\sum_{k<l}C_l^k\|\partial^l u\|\left\|\partial^{l-k}\left(\frac{P'(\rho+\bar\rho)}{\rho+\bar\rho}\right)\partial^k\nabla\rho\right\|+\sum_{k<l}C_l^k\|\partial^l u\|\|\partial^{l-k}u\cdot\partial^k\nabla u\|\\
&+\sum_{k<l}C_l^k\|\partial^l\phi\|\|\partial^{l-k}(\rho+\bar\rho)\partial^k\nabla\cdot u\|+\sum_{k<l}C_l^k\|\partial^l\phi\|\|\partial^{l-k}u\cdot\partial^k\nabla\rho\|\\
=&\sum_{i=1}^6 J_i.
\end{aligned}\tag{2.12}
$$

For J_1, noticing that $|l-k|\ge1$, there exists some multiple index s with $|s|=1$. We have from (2.1) that

$$
\begin{aligned}
|J_1|\le&\sum_{k<l}C_l^k\|\partial^l\rho\|\|\partial^{l-k-s}\partial^s\rho\partial^k\nabla\cdot u\|\\
\le&C\|\partial^l\rho\|\left(\|\partial^s\rho\|_{L^\infty}\|\nabla^{|l|-1}\nabla\cdot u\|+\|\nabla\cdot u\|_{L^\infty}\|\nabla^{|l|-1}\partial^s\rho\|\right)\\
\le&\frac{\bar\rho}{16\tau}\|u\|_N^2+C\tau\|\rho\|_N^2\|\nabla\rho\|_{N-1}^2.
\end{aligned}\tag{2.13}
$$

Similarly, we can estimate J_2-J_6 as

$$
\sum_{i=2}^6|J_i|\le\frac{\bar\rho}{16\tau}\|u\|_N^2+C\tau\|[\rho,u,\phi]\|_N^2\|\nabla[\rho,u,\phi]\|_{N-1}^2. \tag{2.14}
$$

Plugging (2.13) and (2.14) into (2.12), we see

$$
\left|\sum_{k<l}C_l^kI_{k,l}^l(t)\right|\le\frac{\bar\rho}{8\tau}\|u\|_N^2+C\tau\|[\rho,u,\phi]\|_N^2\|\nabla[\rho,u,\phi]\|_{N-1}^2. \tag{2.15}
$$

Substituting (2.11) and (2.15) into (2.9), one has

$$\begin{aligned}
&\frac{d}{dt}\Big(\frac{1}{2}\int_{\mathbb{R}^3}\frac{P'(\rho+\bar\rho)}{\rho+\bar\rho}|\partial^l\rho|^2dx+\frac{1}{2}\int_{\mathbb{R}^3}(\rho+\bar\rho)|\partial^l u|^2dx \\
&\qquad -\mu\langle\partial^l\phi,\partial^l\rho\rangle+\frac{b\mu}{2a}\|\partial^l\phi\|^2+\frac{\mu D}{2a}\|\partial^l\nabla\phi\|^2\Big) \\
&\quad+\frac{1}{\tau}\int_{\mathbb{R}^3}(\rho+\bar\rho)|\partial^l u|^2dx+\frac{\mu}{a}\|\partial^l\phi_t\|^2 \\
&\le\frac{\bar\rho}{4\tau}\|u\|_N^2+C\tau\|[\rho,u,\phi]\|_N^2\big(\|\nabla[\rho,u,\phi]\|_{N-1}^2+\|\nabla\phi\|_N^2\big). \qquad (2.16)
\end{aligned}$$

Taking the summation of (2.16) over $|l|\le N$ and using the fact $\rho+\bar\rho\ge 3\bar\rho/4$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\sum_{|l|\le N}\Big\{\int_{\mathbb{R}^3}\frac{P'(\rho+\bar\rho)}{\rho+\bar\rho}|\partial^l\rho|^2dx+\int_{\mathbb{R}^3}(\rho+\bar\rho)|\partial^l u|^2dx \\
&\qquad -2\mu\langle\partial^l\phi,\partial^l\rho\rangle+\frac{b\mu}{a}\|\partial^l\phi\|^2+\frac{\mu D}{a}\|\partial^l\nabla\phi\|^2\Big\}+\frac{\bar\rho}{2\tau}\|u\|_N^2+\frac{\mu}{a}\|\phi_t\|_N^2 \\
&\le C\tau\|[\rho,u,\phi]\|_N^2\big(\|\nabla[\rho,u,\phi]\|_{N-1}^2+\|\nabla\phi\|_N^2\big),
\end{aligned}$$

which leads to (2.4).

Step 2. We claim that

$$\begin{aligned}
&\tau\frac{d}{dt}\sum_{|l|\le N-1}\Big\{\langle\partial^l u,\partial^l\nabla\rho\rangle+\frac{\mu}{2a}\|\partial^l\nabla\phi\|^2\Big\}+\lambda\tau\|\nabla\rho\|_{N-1}^2+\lambda\tau\|\nabla\phi\|_N^2 \\
&\le\frac{C}{\tau}\|u\|_N^2+C\tau\|[\rho,u]\|_N^2\|\nabla[\rho,u]\|_{N-1}^2. \qquad (2.17)
\end{aligned}$$

Let $0\le|l|\le N-1$. Applying ∂^l to the (2.2b), multiplying it by $\tau\partial^l\nabla\rho$, taking integration in x, using integration by parts, and replacing $\partial_t\rho$ from (2.2a), one has

$$\begin{aligned}
&\tau\frac{d}{dt}\langle\partial^l u,\partial^l\nabla\rho\rangle+\frac{\tau P'(\bar\rho)}{\bar\rho}\|\partial^l\nabla\rho\|^2-\mu\tau\langle\partial^l\nabla\phi,\partial^l\nabla\rho\rangle \\
&=\bar\rho\tau\|\nabla\cdot\partial^l u\|^2-\tau\big\langle\nabla\cdot\partial^l u,\partial^l g_1\big\rangle-\langle\partial^l u,\nabla\partial^l\rho\rangle+\tau\big\langle\partial^l g_2,\nabla\partial^l\rho\big\rangle. \qquad (2.18)
\end{aligned}$$

Applying $\partial^l\nabla$ to (2.2c) we multiply the arising equation by $(\tau\mu/a)\partial^l\nabla\phi$ and integrate the result in x, thus obtaining

$$\frac{\tau\mu}{2a}\frac{d}{dt}\|\partial^l\nabla\phi\|^2+\frac{\tau\mu D}{a}\|\partial^l\nabla^2\phi\|^2-\mu\tau\langle\partial^l\nabla\phi,\partial^l\nabla\rho\rangle+\frac{\mu b\tau}{a}\|\partial^l\nabla\phi\|^2=0. \qquad (2.19)$$

Adding (2.18) to (2.19) yields

$$\begin{aligned}
&\tau\frac{d}{dt}\left\{\langle\partial^l u,\partial^l\nabla\rho\rangle+\frac{\mu}{2a}\|\partial^l\nabla\phi\|^2\right\}+\frac{\tau P'(\bar\rho)}{\bar\rho}\|\partial^l\nabla\rho\|^2\\
&\quad-2\mu\tau\langle\partial^l\nabla\phi,\partial^l\nabla\rho\rangle+\frac{\mu b\tau}{a}\|\partial^l\nabla\phi\|^2+\frac{\mu D\tau}{a}\|\partial^l\nabla^2\phi\|^2\\
&=\bar\rho\tau\|\nabla\cdot\partial^l u\|^2-\tau\langle\nabla\cdot\partial^l u,\partial^l g_1\rangle-\langle\partial^l u,\nabla\partial^l\rho\rangle+\tau\langle\partial^l g_2,\nabla\partial^l\rho\rangle.
\end{aligned}\tag{2.20}$$

Since $bP'(\bar\rho)-a\mu\bar\rho>0$, the following matrix:

$$\begin{pmatrix}\dfrac{P'(\bar\rho)}{\bar\rho} & -\mu\\ -\mu & \dfrac{b\mu}{a}\end{pmatrix}\tag{2.21}$$

is positive definite, which yields a positive constant $C_1>0$ such that

$$\frac{P'(\bar\rho)}{\bar\rho}\|\partial^l\nabla\rho\|^2-2\mu\langle\partial^l\nabla\phi,\partial^l\nabla\rho\rangle+\frac{\mu b}{a}\|\partial^l\nabla\phi\|^2\ge C_1\big(\|\partial^l\nabla\rho\|^2+\|\partial^l\nabla\phi\|^2\big).$$

This together with (2.20) gives

$$\begin{aligned}
&\tau\frac{d}{dt}\left\{\langle\partial^l u,\partial^l\nabla\rho\rangle+\frac{\mu}{2a}\|\partial^l\nabla\phi\|^2\right\}+C_1\tau\|\partial^l\nabla\rho\|^2\\
&\quad+C_1\tau\|\partial^l\nabla\phi\|^2+\frac{\mu D\tau}{a}\|\partial^l\nabla^2\phi\|^2\\
&=\bar\rho\tau\|\nabla\cdot\partial^l u\|^2-\tau\langle\nabla\cdot\partial^l u,\partial^l g_1\rangle-\langle\partial^l u,\nabla\partial^l\rho\rangle+\tau\langle\partial^l g_2,\nabla\partial^l\rho\rangle.
\end{aligned}\tag{2.22}$$

Then, it follows from the Cauchy-Schwarz inequality that

$$\begin{aligned}
&\tau\frac{d}{dt}\left\{\langle\partial^l u,\partial^l\nabla\rho\rangle+\frac{\mu}{2a}\|\partial^l\nabla\phi\|^2\right\}+\frac{\tau C_1}{2}\|\partial^l\nabla\rho\|^2\\
&\quad+\tau C_1\|\partial^l\nabla\phi\|^2+\frac{\mu D\tau}{a}\|\partial^l\nabla^2\phi\|^2\\
&\le\frac{C}{\tau}\big(\|\nabla\cdot\partial^l u\|^2+\|\partial^l u\|^2\big)+C\tau\big(\|\partial^l g_1\|^2+\|\partial^l g_2\|^2\big),
\end{aligned}\tag{2.23}$$

where we have used the fact $0<\tau<1<1/\tau$. Noticing that g_1,g_2 are quadratically nonlinear, one has from (2.1) that

$$\|\partial^l g_1\|^2+\|\partial^l g_2\|^2\le C\|[\rho,u]\|_N^2\|\nabla[\rho,u]\|_{N-1}^2.$$

Substituting this into (2.23) and taking the summation over $|l|\le N-1$, we have

$$\begin{aligned}&\tau\frac{d}{dt}\sum_{|l|\le N-1}\Big\{\langle\partial^l u,\partial^l\nabla\rho\rangle+\frac{\mu}{2a}\|\partial^l\nabla\phi\|^2\Big\}+\lambda\tau\big(\|\nabla\rho\|_{N-1}^2+\|\nabla\phi\|_N^2\big)\\ \le&\frac{C}{\tau}\|u\|_N^2+C\tau\|[\rho,u]\|_N^2\|\nabla[\rho,u]\|_{N-1}^2,\end{aligned}$$

where $\lambda=\min\{C_1/2,\mu D/a\}$. This completes the proof of (2.17).

Step 3. Multiplying (2.17) by κ and adding the resulting inequality to (2.4), we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\sum_{|l|\le N}\Big\{\int_{\mathbb{R}^3}\frac{P'(\rho+\bar\rho)}{\rho+\bar\rho}|\partial^l\rho|^2dx+\int_{\mathbb{R}^3}(\rho+\bar\rho)|\partial^l u|^2dx\\ &\qquad-2\mu\langle\partial^l\phi,\partial^l\rho\rangle+\frac{b\mu}{a}\|\partial^l\phi\|^2+\frac{\mu D}{a}\|\partial^l\nabla\phi\|^2\Big\}\\ &\quad+\kappa\tau\frac{d}{dt}\sum_{|l|\le N-1}\Big\{\langle\partial^l u,\partial^l\nabla\rho\rangle+\frac{\mu}{2a}\|\partial^l\nabla\phi\|^2\Big\}\\ &\quad+\kappa\lambda\tau\|\nabla\rho\|_{N-1}^2+\kappa\lambda\tau\|\nabla\phi\|_N^2+\frac{\bar\rho}{2\tau}\|u\|_N^2+\frac{\mu}{a}\|\phi_t\|_N^2\\ \le&\frac{C\kappa}{\tau}\|u\|_N^2+C\tau\|[\rho,u,\phi]\|_N^2\big(\|\nabla[\rho,u,\phi]\|_{N-1}^2+\|\nabla\phi\|_N^2\big).\end{aligned}$$

By choosing κ and $\|[\rho,u,\phi]\|_N$ small enough, we can obtain that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\sum_{|l|\le N}\Big\{\int_{\mathbb{R}^3}\frac{P'(\rho+\bar\rho)}{\rho+\bar\rho}|\partial^l\rho|^2dx+\int_{\mathbb{R}^3}(\rho+\bar\rho)|\partial^l u|^2dx\\ &\qquad-2\mu\langle\partial^l\phi,\partial^l\rho\rangle+\frac{b\mu}{a}\|\partial^l\phi\|^2+\frac{\mu D}{a}\|\partial^l\nabla\phi\|^2\Big\}\\ &+\kappa\tau\frac{d}{dt}\sum_{|l|\le N-1}\Big\{\langle\partial^l u,\partial^l\nabla\rho\rangle+\frac{\mu}{2a}\|\partial^l\nabla\phi\|^2\Big\}+\frac{\kappa\lambda\tau}{2}\|\nabla\rho\|_{N-1}^2\\ &+\frac{\kappa\lambda\tau}{2}\|\nabla\phi\|_N^2+\frac{\bar\rho}{4\tau}\|u\|_N^2+\frac{\mu}{a}\|\phi_t\|_N^2\le 0.\end{aligned}\tag{2.24}$$

This completes the proof of Lemma 2.3. □

Proof of Proposition 2.1. We denote $\mathcal{E}_N(U(t))$ by

$$\begin{aligned}\mathcal{E}_N(U(t))=&\sum_{|l|\le N}\left\{\int_{\mathbb{R}^3}\frac{P'(\rho+\bar\rho)}{\rho+\bar\rho}|\partial^l\rho|^2dx-2\mu\langle\partial^l\phi,\partial^l\rho\rangle\right.\\&\left.+\frac{b\mu}{a}\|\partial^l\phi\|^2+\int_{\mathbb{R}^3}(\rho+\bar\rho)|\partial^l u|^2dx\right\}\\&+\frac{\mu D}{a}\|\nabla\phi\|_N^2+\kappa\tau\sum_{|l|\le N-1}\left\{2\langle\partial^l u,\partial^l\nabla\rho\rangle+\frac{\mu}{a}\|\partial^l\nabla\phi\|^2\right\},\end{aligned}\tag{2.25}$$

and rewrite $\int_{\mathbb{R}^3}P'(\rho+\bar\rho)/(\rho+\bar\rho)|\partial^l\rho|^2dx$ as

$$\int_{\mathbb{R}^3}\frac{P'(\rho+\bar\rho)}{\rho+\bar\rho}|\partial^l\rho|^2dx=\frac{P'(\bar\rho)}{\bar\rho}\|\partial^l\rho\|^2+\left\langle\left(\frac{P'(\rho+\bar\rho)}{\rho+\bar\rho}-\frac{P'(\bar\rho)}{\bar\rho}\right)\partial^l\rho,\partial^l\rho\right\rangle,\tag{2.26}$$

which updates (2.25) to

$$\begin{aligned}\mathcal{E}_N(U(t))=&\sum_{|l|\le N}\left\{\frac{P'(\bar\rho)}{\bar\rho}\|\partial^l\rho\|^2-2\mu\langle\partial^l\phi,\partial^l\rho\rangle+\frac{b\mu}{a}\|\partial^l\phi\|^2+\int_{\mathbb{R}^3}(\rho+\bar\rho)|\partial^l u|^2dx\right\}\\&+\sum_{|l|\le N}\left\langle\left(\frac{P'(\rho+\bar\rho)}{\rho+\bar\rho}-\frac{P'(\bar\rho)}{\bar\rho}\right)\partial^l\rho,\partial^l\rho\right\rangle+\frac{\mu D}{a}\|\nabla\phi\|_N^2\\&+\kappa\tau\sum_{|l|\le N-1}\left\{2\langle\partial^l u,\partial^l\nabla\rho\rangle+\frac{\mu}{a}\|\partial^l\nabla\phi\|^2\right\}\end{aligned}$$

with constant $0<\kappa\ll 1$. By the fact that the matrix (2.21) is positive definite, along with the smallness of κ and $\|\rho\|_N$, we have that

$$\mathcal{E}_N(U(t))\sim\|[\rho,u,\phi]\|_N^2+\|\nabla\phi\|_N^2.$$

This together with (2.24) leads to

$$\begin{aligned}&\|[\rho,u,\phi]\|_N^2+\|\nabla\phi\|_N^2+\int_0^t\left(\frac{1}{\tau}\|u\|_N^2+\tau\|\nabla\rho\|_{N-1}^2+\tau\|\nabla\phi\|_N^2+\|\phi_t\|_N^2\right)ds\\&\le C\left(\|[\rho_0,u_0,\phi_0]\|_N^2+\|\nabla\phi_0]\|_N^2\right).\end{aligned}\tag{2.27}$$

This a priori estimate combined with the local existence theorem completes the proof of Proposition 2.1, and thus Theorem 1.1. □

3 Convergence to the Keller-Segel model

In this section, we are going to prove Theorem 1.2.

Proof of Theorem 1.2. We have shown that the Cauchy problem (1.1) admits a unique global solution $(\rho^\tau,u^\tau,\phi^\tau)$. Next, we will adapt the compactness argument developed in [5] and justify the convergence of $(\rho^\tau,u^\tau,\phi^\tau)$ to the solution of the Keller-Segel model as $\tau\to 0$. To this end, we introduce the rescaled time variable $s=\tau t$ and define

$$(\varrho^\tau,m^\tau,\varphi^\tau)(x,s):=\left(\rho^\tau(x,t),\frac{\rho^\tau(x,t)u^\tau(x,t)}{\tau},\phi^\tau(x,t)\right).$$

The system (1.1) turns into

$$\begin{cases}\partial_s\varrho^\tau+\nabla\cdot m^\tau=0, & \text{(3.1a)}\\ \tau^2\partial_s m^\tau+\tau^2\nabla\cdot\left(\dfrac{m^\tau\otimes m^\tau}{\varrho^\tau}\right)+\nabla P(\varrho^\tau)=-m^\tau+\mu\varrho^\tau\nabla\varphi^\tau, & \text{(3.1b)}\\ \tau\partial_s\varphi^\tau=D\Delta\varphi^\tau+a\varrho^\tau-b\varphi^\tau. & \text{(3.1c)}\end{cases}$$

Thus, we have from (2.27) that

$$\begin{aligned}&\left\|[\varrho^\tau(s)-\bar\rho,\varphi^\tau(s)-\bar\phi,\nabla\varphi^\tau(s)]\right\|_N^2+\tau^2\|m^\tau(s)\|_N^2+\int_0^\infty\|m^\tau(s)\|_N^2ds\\&+\int_0^\infty\left(\|\nabla\varrho^\tau(s)\|_{N-1}^2+\|\nabla\varphi^\tau(s)\|_N^2+\tau\left\|\partial_s\varphi^\tau(s)\right\|_N^2\right)ds\le C,\end{aligned}\tag{3.2}$$

where C is a constant independent of τ.

From (3.2), we have that ϱ^τ is bounded from below and above by positive constants, τm^τ and $\varphi^\tau-\bar\phi$ are bounded in $L^\infty([0,\infty),L^2(\mathbb{R}^3))$, and $m^\tau\otimes m^\tau/\varrho^\tau$ is bounded in $L^1([0,\infty),L^1(\mathbb{R}^3))$. Hence, as τ goes to 0, we can pass the limit in the Eq. (3.1b) in the sense of distributions as following:

$$\begin{cases}-m^\tau-\nabla P(\varrho^\tau)+\mu\varrho^\tau\nabla\varphi^\tau\rightharpoonup 0,\\ D\Delta\varphi^\tau+a\varrho^\tau-b\varphi^\tau\rightharpoonup 0\quad\text{in }\mathcal{D}'(\mathbb{R}^+\times\mathbb{R}^3),\end{cases}$$

which together with the Eq. (3.1a) yields

$$\begin{cases}\partial_s\varrho^\tau+\nabla\cdot\left(\mu\varrho^\tau\nabla\varphi^\tau-\nabla P(\varrho^\tau)\right)\rightharpoonup 0\quad\text{in }\mathcal{D}'(\mathbb{R}^+\times\mathbb{R}^3),\\ D\Delta\varphi^\tau+a\varrho^\tau-b\varphi^\tau\rightharpoonup 0\quad\text{in }\mathcal{D}'(\mathbb{R}^+\times\mathbb{R}^3).\end{cases}$$

It follows from (3.1), (3.2) that $(\varrho^\tau - \bar{\rho})$ is bounded in $L^\infty([0,\infty);H^N(\mathbb{R}^3))$ and $\nabla\varrho^\tau$ is bounded in $L^2([0,\infty);H^{N-1}(\mathbb{R}^3))$. Furthermore, we obtain that $\partial_s\varrho^\tau$ is bounded in $L^2([0,\infty);H^{N-1}(\mathbb{R}^3))$. This shows that $(\varrho^\tau - \bar{\rho})$ is bounded in the space $H^1([0,T);H^{N-1}(\mathbb{R}^3))$ for any $T>0$. Thus, there exist a subsequence τ_n and a function ϱ^0 such that

$$\varrho^{\tau_n} - \bar{\rho} \rightharpoonup \varrho^0 - \bar{\rho} \quad \text{weakly in } H^1\big([0,T);H^{N-1}(\mathbb{R}^3)\big).$$

It means that $\varrho^0 - \bar{\rho} \in C([0,T);H^{N-1}(\mathbb{R}^3))$, which along with $\varrho^{\tau_n}|_{s=0} = \rho_0$ for all n leads that $\varrho^0|_{s=0} = \rho_0$.

Now, we have that $\partial_s\varrho^{\tau_n}$ is bounded in $L^2([0,\infty);H^{N-1}(\mathbb{R}^3))$, and moreover ϱ^{τ_n} is bounded in $L^\infty([0,\infty);H^N(\mathbb{R}^3))$. By the Aubin-Lions-Simon lemma, we can extract a new subsequence, still denoted by τ_n, such that the following convergence holds:

$$\varrho^{\tau_n} \to \varrho^0 \quad \text{strongly in } C\big([0,T);H^m(B(R))\big),$$

where $N-1<m<N, R>0$ and $B(R)$ denotes the ball $\{x \in \mathbb{R}^3 : |x| \le R\}$. Lastly, we can get the convergence properties to φ^{τ_n} from (3.2)

$$\varphi^{\tau_n} \rightharpoonup \varphi^0 \quad \text{weakly in } L^2\big([0,T);H^N(B(R))\big).$$

The proof of Theorem 1.2 is complete. □

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