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Strong solutions to a nonlinear stochastic aggregation-diffusion equation

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It is well-known that solutions to deterministic nonlocal aggregation-diffusion models may blow up in two or higher dimensions. Various mechanisms hence have been proposed to "regularize" the deterministic aggregation-diffusion equations in a manner that allows pattern formation without blow-up. However, stochastic effect has not been ever considered among other things. In this work, we consider a nonlocal aggregation-diffusion model with multiplicative noise and establish the local existence and uniqueness of strong solutions on $\mathbb{R}^d (d \geq 2)$. If the noise is non-autonomous and linear, we establish the global existence and large-time behavior of strong solutions with decay properties by combining the Moser-Alikakos iteration technique and some decay estimates of Girsanov type processes. If the noise is nonlinear and strong enough, we show that blow-up can be prevented. As such, our results assert that certain multiplicative noise can also regularize the aggregation-diffusion model.

Keywords: Stochastic aggregation-diffusion equations; Regularization effect; Global existence; Large-time behavior.

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1. Introduction

Aggregation-diffusion equations via nonlocal interactions are ubiquitous in the modeling of various biological processes/phenomena from microscopic to macroscopic

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levels. Among a large class of equations, the following nonlocal aggregation-diffusion equation has recently received extensive attention:

$$\frac{\partial u}{\partial t} - \Delta u^m + \chi \mathrm{div}(u\nabla G * u) = 0, \quad x \in \mathbb{R}^d, \;\; t > 0, \tag{1.1}$$

where $m \geq 1$ is the diffusion parameter, $\chi \in \mathbb{R}$ is the aggregation coefficient, $u(t,x)$ represents the density of species (cells) at position $x \in \mathbb{R}^d$ ($d \geq 2$) at time t, and $G : \mathbb{R}^d \to \mathbb{R}$ is an interaction kernel. The equation (1.1) can be derived as the continuum limit of many particle system [7, 47] and has a range of applications arising in physics and biology depending on the choice of interaction kernel and diffusion parameter $m \geq 1$, such as self-organization of chemotactic movement [5, 36, 49], biological swarm [12, 57], cancer invasion [21, 26], and so on (see a survey article [13]). While the linear random motion is indicated by $m = 1$, the nonlinear degenerate diffusion with $m > 1$ describes the repulsion between species to account for the over-crowding effect. When the interaction kernel G is a Newtonian or Bessel potential, equation (1.1) is well-known as the Keller–Segel chemotaxis model, for which many interesting results have been available. Among other things, the most prominent feature of the Keller–Segel model is that there is a critical mass in the critical regime $m = 2 - 2/d$ such that the solution to (1.1) may blow up in finite time for super-critical mass and exist globally for sub-critical mass. This was established first for the case $m = 1$ in [6, 20, 45, 46], and later extended to any $m > 0$ (see [53, 4, 3, 38] for subcritical case $m > 2 - 2/d$, [4, 53] for critical case $m = 2 - 2/d$, and [53, 3, 4] for super-critical case $m < 2 - 2/d$). Moreover, various modifications/mechanisms have been proposed to "regularize" the equation (1.1) with $m = 1$ in a manner that allows pattern formation but without blow-up (see a survey article [31]).

As is well-known, an additional logistic term, as one of the mechanisms shown in [31], has been shown to being able to regularize the (1.1) in the literature (cf. [48, 58, 59]). However, in a fluctuating or noisy environment, an additional stochastic process may be more appropriate to capture the reality (cf. [43]). The purpose of this paper is to consider the aggregation-diffusion model (1.1) with a multiplicative noise and investigate whether this randomness can affect the global dynamics of the system such as global well-posededness/blow-up and asymptotic behavior of solutions. For simplicity we consider $m = 1$ in this paper and for definiteness we assume G is the Bessel kernel, i.e., G is the Green function of the Helmholtz operator $I - \Delta$ (namely $(I - \Delta)^{-1}u = G * u$). Then, we consider the following stochastic aggregation-diffusion model

$$\mathrm{d}u - \Delta u \,\mathrm{d}t + \chi \mathrm{div}(u\nabla G * u)\,\mathrm{d}t = \sigma(t,u)\,\mathrm{d}\mathcal{W}, \quad x \in \mathbb{R}^d \;(d \geq 2), \;\; t > 0, \tag{1.2}$$

where $\mathcal{W}$ is a cylindrical Wiener process which will be specified in next section and $\sigma(t,u)\,\mathrm{d}\mathcal{W}$ accounts for the noise arising from the fluctuating or noisy environment. To simplify notations, we define a linear differential operator $Q(\cdot)$ with order -1

and the nonlocal nonlinear term $F(u)$ as follows:

$$\begin{cases} Q(u) = \nabla G * u = \nabla (I - \Delta)^{-1} u, \\ F(u) = \mathrm{div}(uQ(u)) = (Q(u) \cdot \nabla) u + u \mathrm{div} Q(u). \end{cases} \tag{1.3}$$

Then (1.2) can be reformulated as

$$\mathrm{d}u - \Delta u \, \mathrm{d}t + \chi F(u) \, \mathrm{d}t = \sigma(t, u) \mathrm{d}\mathcal{W}, \quad x \in \mathbb{R}^d \ (d \geq 2), \ \ t > 0.$$

In contrast to abundant results available to its deterministic counterpart, the stochastic aggregation-diffusion model (1.2) has not been studied and basic questions like well-posedness (even local well-posedness) and large-time behavior of solutions are still unknown. Hence it would be of interest to establish some analytical results for the stochastic aggregation-diffusion models. Therefore, the first goal of this paper is to

- Establish local existence and uniqueness of strong solutions to the following stochastic aggregation-diffusion model:

$$\begin{cases} \mathrm{d}u - \Delta u \, \mathrm{d}t + \chi F(u) \, \mathrm{d}t = \sigma(t, u) \, \mathrm{d}\mathcal{W}, \quad x \in \mathbb{R}^d, \ \ t > 0, \\ u(\omega, 0, x) = u_0(\omega, x) \in H^s, \end{cases} \tag{1.4}$$

  where ω belongs to some sample space Ω. The relevant results are stated in Theorem 2.1.

On the other hand, what kind of effects that the noise may bring is a question worthwhile to study. For example, it is known that the well-posedness of linear stochastic transport equation with noise can be established under weaker hypotheses than its deterministic counterpart (cf. [22, 23]). For stochastic Euler equations, certain noise may prevent coalescence of vortices (singularity) in two-dimensional space [24]. With a focus on (1.2), it is natural to study how the noise affects its global dynamics. As shown in [39, 28], the linear noise $\sigma(t, u) \, \mathrm{d}\mathcal{W} = \beta u \, \mathrm{d}W$, where $\beta \in \mathbb{R} \backslash \{0\}$ and W is a standard 1-D Brownian motion, is a dissipative factor for many SPDEs. Motivated by these works, we consider the global dynamics of (1.2) with non-autonomous linear multiplicative noise, namely $\sigma(t, u) \, \mathrm{d}\mathcal{W} = \beta(t) u \, \mathrm{d}W$. Therefore, our second goal is set to

- Establish the global boundedness and large-time behavior of strong solutions to the following stochastic aggregation-diffusion model with non-autonomous linear noise:

$$\begin{cases} \mathrm{d}u - \Delta u \, \mathrm{d}t + \chi F(u) \, \mathrm{d}t = \beta(t) u \, \mathrm{d}W, \quad x \in \mathbb{R}^d, \ \ t > 0 \\ u(\omega, 0, x) = u_0(\omega, x) \in H^s, \end{cases} \tag{1.5}$$

  where W is a standard 1-D Brownian motion. The detailed results for (1.5) are stated in Theorem 2.2 ($d = 2, s > 4$) and Theorem 2.3 ($d \geq 2, s > \frac{d}{2} + 3$).

We outline here that linear noise can bring some "regularization" effects on the aggregation-diffusion model as stated in Theorems 2.2 and 2.3. Without noise, it is well-known that the solution to (1.5) may blow up in two dimensions with a critical mass and three (or higher) dimensions for small mass (cf. [19, 42, 45]). In Theorem 2.2, if the initial data is small in L^1 sense, then L^∞ norm of the solution decays exponentially almost surely. In Theorem 2.3, a linear large noise can guarantee exponential decay of H^s norm with high probability.

However, the above results hold true either with some smallness conditions on initial data (Theorem 2.2) or with probability (Theorem 2.3). It is therefore very natural to ask *when global solvability holds without smallness condition on initial data or with probability one.* Theorems 2.2 and 2.3 indicate that linear noise is not enough. Our final goal in this paper is to find out such noise structure. The mathematical interest of finding such noise is important because it is helpful to understand the mechanisms which stabilize the equation, and this is the first step as searching for the real correct and physical noise which provides regularization effect.

As we will see in (b) in Theorem 2.1 below, for the solution to (1.4), its H^s-norm blows up if and only if its H^γ-norm blows up, where $\gamma \in (\frac{d}{2}+1, s]$. This suggests choosing a noise coefficient involving the H^γ-norm of u. For convenience, to be consistent with the proof of Theorem 2.1, in this work we consider the case that $\sigma(t,u)\,\mathrm{d}\mathcal{W} = \alpha(1+\|u\|_{H^r})^\varrho u\,\mathrm{d}W$, where $\alpha \in \mathbb{R}\backslash\{0\}$, W is a standard 1-D Brownian motion, $r \in (\frac{d}{2}+1, s-2)$ and $\varrho > 0$ is a parameter. More precisely, the third target of this paper is to

- Determine the range of α and ϱ such that the solution to the following problem exists globally in time:
$$\begin{cases} \mathrm{d}u - \Delta u\,\mathrm{d}t + \chi F(u)\,\mathrm{d}t = \alpha(1+\|u\|_{H^r})^\varrho u\,\mathrm{d}W, & x \in \mathbb{R}^d, \ t > 0 \\ u(\omega,0,x) = u_0(\omega,x), \end{cases} \tag{1.6}$$
where $r \in (\frac{d}{2}+1, s-2)$ and W is a standard 1-D Brownian motion. The result is stated in Theorem 2.4.

The rest of this paper is organized as follows. In Sec. 2, we introduce some notations, state our main results and then briefly sketch the proof strategies. In Sec. 3, we present some basic results that will be frequently used. In Sec. 4, we prove Theorem 2.1. In Sec. 5, we consider the problems (1.5) and (1.6). We prove Theorems 2.2 and 2.3 in Sec. 5.1 and prove Theorem 2.4 in Sec. 5.2.

2. Main Results

In this section, we shall introduce some notions regarding strong solutions to (1.2), recall some results from abstract probability theory and functional analysis, and then state our main results.

2.1. *Notations and background*

Let $L^p(\mathbb{R}^d)$ with $d \geq 1$ and $1 \leq p < \infty$ be the standard Lebesgue space of measurable p-integrable defined on $\mathbb{R}^d$ and let $L^\infty(\mathbb{R}^d)$ be the space of essentially bounded functions. Particularly, $L^2(\mathbb{R}^d)$ has an inner product $(f,g)_{L^2} = \int_{\mathbb{R}^d} f \cdot \overline{g}\,\mathrm{d}x$, where $\overline{g}$ denotes the complex conjugation of g. The Fourier transform and inverse Fourier transform of $f(x) \in L^2(\mathbb{R}^d)$ are defined by $\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x)\mathrm{e}^{-\mathrm{i}x\cdot\xi}\,\mathrm{d}x$, and $f(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \widehat{f}(\xi)\mathrm{e}^{\mathrm{i}x\cdot\xi}\,\mathrm{d}\xi$, respectively. For any $s \in \mathbb{R}$, the operator $\mathcal{D}^s = (I-\Delta)^{s/2}$ is defined by

$$\widehat{\mathcal{D}^s f}(\xi) := (1+|\xi|^2)^{s/2}\widehat{f}(\xi).$$

Then, for $s \geq 0$, the Sobolev spaces H^s on $\mathbb{R}^d$ with its inner product $(\cdot,\cdot)_{H^s}$ can be defined as:

$$H^s := \{f \in L^2 : \|f\|_{H^s}^2 := (f,f)_{H^s} < +\infty\}, \quad (f,g)_{H^s} := (\mathcal{D}^s f, \mathcal{D}^s g)_{L^2}.$$

Here, and in the sequel, all the function spaces are defined on $\mathbb{R}^d$ and we drop $\mathbb{R}^d$ for brevity if there is no ambiguity. It is clear that $(I-\Delta)^{-1}$ is a bounded operator from H^s to H^{s+2}.

We denote the commutator between linear operators A and B by $[A,B]$, i.e., $[A,B] = AB - BA$. For a set E, $\mathbf{1}_E(x)$ is the indicator function on E, i.e., it is equal to 1 when $x \in E$, and zero otherwise. We will use $\lesssim$ to denote an inequality that holds up to some constants, which may be different from line to line.

We next briefly recall some background on stochastic analysis which we use below (see [25, 33, 18] for more details). Let $\mathcal{W}$ be a cylindrical Brownian motion on a separable Hilbert space $\mathbb{U}$:

$$\mathcal{W}(t) := \sum_{k\geq 1} W_k(t)e_k, \quad t \geq 0,$$

where $\{e_k\}_{k\geq 1}$ is a complete orthonormal basis of $\mathbb{U}$, and $\{W_k\}_{k\geq 1}$ is a sequence of independent 1-D Brownian motions on a right-continuous complete filtration probability space $(\Omega, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. However, the above formal summation is not convergent on $\mathbb{U}$. Therefore, we consider a larger separable Hilbert space $\mathbb{U}_0$ such that the canonical embedding $\mathbb{U} \hookrightarrow \mathbb{U}_0$ is Hilbert–Schmidt. Then we have that for any $T > 0$, cf. [18, 25, 34], $\mathcal{W} \in C([0,T];\mathbb{U}_0)$ $\mathbb{P}$-a.s. From now on, we call $\mathcal{S} = (\Omega, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{W})$ a stochastic basis and let $\mathcal{L}_2(\mathbb{U};\mathcal{X})$ be the collection of Hilbert–Schmidt operators from $\mathbb{U}$ to some separable Hilbert space $\mathcal{X}$. As in [18, 50], for $\mathcal{X}$-valued predictable process $G \in L^2\left(\Omega; L^2_{\mathrm{loc}}([0,\infty);\mathcal{L}_2(\mathbb{U};\mathcal{X}))\right)$, one can define the Itô stochastic integral

$$\int_0^t G\,\mathrm{d}\mathcal{W} := \sum_{k=1}^\infty \int_0^t Ge_k\,\mathrm{d}W_k.$$

Here we remark that the stochastic integral $\int_0^t G\,\mathrm{d}\mathcal{W}$ is independent of the choice of the space $\mathbb{U}_0$, cf. [18, 50]. For example, $\mathbb{U}_0$ can be defined as:

$$\mathbb{U}_0 = \left\{ v = \sum_{k=1}^{\infty} a_k e_k : \sum_{k=1}^{\infty} \frac{a_k^2}{k^2} < \infty \right\}, \quad \|v\|_{\mathbb{U}_0} = \sum_{k=1}^{\infty} \frac{a_k^2}{k^2}.$$

Moreover, we have that for all almost surely bounded stopping times τ,

$$\left(\int_0^\tau G\,\mathrm{d}\mathcal{W}, v \right)_{\mathcal{X}} = \sum_{k=1}^{\infty} \int_0^\tau (Ge_k, v)_{\mathcal{X}}\,\mathrm{d}W_k \ \mathbb{P}\text{-a.s.}$$

In particular the Burkholder-Davis-Gundy (BDG) inequality in the present context reads as

$$\mathbb{E}\left(\sup_{t\in[0,T]} \left\| \int_0^t G\,\mathrm{d}\mathcal{W} \right\|_{\mathcal{X}}^p \right) \le C\mathbb{E}\left(\int_0^T \|G\|_{\mathcal{L}_2(\mathbb{U};\mathcal{X})}^2\,\mathrm{d}t \right)^{\frac{p}{2}}, \quad p \ge 1,$$

or in terms of the coefficients,

$$\mathbb{E}\left(\sup_{t\in[0,T]} \left\| \sum_{k=1}^{\infty} \int_0^t Ge_k\,\mathrm{d}W_k \right\|_{\mathcal{X}}^p \right) \le C\mathbb{E}\left(\int_0^T \sum_{k=1}^{\infty} \|Ge_k\|_{\mathcal{X}}^2\,\mathrm{d}t \right)^{\frac{p}{2}}, \quad p \ge 1.$$

2.2. *Assumptions and definitions*

We first prescribe some conditions on the noise coefficient σ.

Assumption $\mathbf{A}_1$. *Assume that $\sigma : [0,\infty) \times H^s \ni (t,u) \mapsto \sigma(t,u) \in \mathcal{L}_2(\mathbb{U}; H^s)$ for $u \in H^s$ with $s > \frac{d}{2}+1$ such that σ is continuous in (t,u). Furthermore, we assume the following*:

(1) *There is an increasing function $f(\cdot) : [0,+\infty) \to [0,+\infty)$ such that for any $t > 0$ and $s > \frac{d}{2}+1$,*

$$\|\sigma(t,u)\|_{\mathcal{L}_2(\mathbb{U};H^s)} \le f(\|u\|_{W^{1,\infty}})(1 + \|u\|_{H^s}).$$

(2) *There is an increasing function $g(\cdot) : [0,+\infty) \to [0,+\infty)$ such that for any $K > 0$ and $s > \frac{d}{2}+1$,*

$$\sup_{t\ge 0,\, \|u\|_{H^s}\vee\|v\|_{H^s}\le K} \|\sigma(t,u) - \sigma(t,v)\|_{\mathcal{L}_2(\mathbb{U};H^s)} \le g(K)\|u-v\|_{H^s}.$$

Assumption $\mathbf{A}_2$. *When the non-negativity of solutions is considered, we assume that there is a $C > 0$ such that for any $t > 0$,*

$$\|\sigma(t,v)\|_{\mathcal{L}_2(\mathbb{U};L^2)}^2 - 2\|\nabla v\|_{L^2}^2 \le C\|v\|_{L^2}^2, \quad v \in H^1.$$

Assumption $\mathbf{A}_3$. *When* (1.5) *with non-autonomous linear noise $\beta(t)u\,\mathrm{d}W$ is considered, we assume that*: *$\beta(t) \in C([0,\infty))$ and there are β^* and β_* such that $0 < \beta_* \le \beta^2(t) \le \beta^*$ for all $t \ge 0$.*

We remark here that if Assumption A_3 is satisfied, then Assumptions A_1 and A_2 are also verified for $\sigma(t,u) = \beta(t)u$. This fact will be used in Sec. 5.1.

Before we formulate our main results, we give the definitions for the strong solutions to the problem (1.4).

Definition 2.1 (Strong solutions). Let $s > \frac{d}{2} + 3$. Fix a stochastic basis $\mathcal{S}$ and assume $\sigma(\cdot,\cdot) : [0,\infty) \times H^s \ni u \mapsto \sigma(t,u) \in \mathcal{L}_2(\mathbb{U}; H^s)$. Let u_0 be an H^s-valued $\mathcal{F}_0$ measurable random variable. A local strong solution to (1.4) is a pair (u,τ), where τ is a stopping time satisfying $\mathbb{P}\{\tau > 0\} = 1$ and $(u(t))_{t\in[0,\tau]}$ is an $\mathcal{F}_t$ predictable process satisfying

$$\mathbb{P}\{u \in C([0,\tau]; H^s)\} = 1,$$

and the following equation holds true almost surely:

$$u(t\wedge\tau) - u(0) + \int_0^{t\wedge\tau} (-\Delta u + \chi F(u))\mathrm{d}t' = \int_0^{t\wedge\tau} \sigma(t',u)\,\mathrm{d}\mathcal{W}, \quad t \geq 0, \ \ x \in \mathbb{R}^d.$$

Definition 2.2 (Uniqueness). Let $\mathcal{S}$ be a fixed stochastic basis. The local solutions to (1.4) are said to be (pathwise) unique, if any two local solutions (u_1,τ_1) and (u_2,τ_2) satisfy that $\mathbb{P}\left\{u_1(0,x) = u_2(0,x),\ x \in \mathbb{R}^d\right\} = 1$ can imply

$$\mathbb{P}\{u_1(t,x) = u_2(t,x),\ (t,x) \in [0,\tau_1\wedge\tau_2] \times \mathbb{R}^d\} = 1.$$

Definition 2.3 (Maximal solution). Let the conditions be exactly as in Definition 2.1 above. A maximal strong solution to (1.4) is a triple $(u, \{\tau_n\}_{n\geq 1}, \tau^*)$ such that

(1) For any $n \in \mathbb{N}$, (u,τ_n) is a strong solution;
(2) $\tau_n \to \tau^*$ increasingly and

$$\sup_{t\in[0,\tau_n]} \|u\|_{H^s} \geq n, \quad \text{on } \{\tau^* < \infty\}.$$

If $\tau^* = \infty$ $\mathbb{P}$-a.s., then such a solution is called global.

When it is clear from the context, we just write (u,τ^*) instead of $(u, \{\tau_n\}_{n\geq 1}, \tau^*)$ for simplicity.

2.3. *Main results and remarks*

Theorem 2.1. *Let $d \geq 2$, $\chi \in \mathbb{R}\backslash\{0\}$ and $s > \frac{d}{2} + 3$. Given a stochastic basis $\mathcal{S} = (\Omega, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{W})$, if u_0 is an H^s-valued $\mathcal{F}_0$ measurable random variable such that $\mathbb{E}\|u_0\|_{H^s}^2 < \infty$ and $\sigma(t,u)$ satisfies Assumption A_1, then*

(a) *There is a unique strong solution (u,τ) to* (1.4) *in the sense of Definitions* 2.1 *and* 2.2. *Moreover, (u,τ) satisfies*

$$\mathbb{E}\left[\sup_{t\in[0,\tau]} \|u(t)\|_{H^s}^2\right] < \infty, \tag{2.1}$$

and it can be extended to a maximal solution (u, τ^*) *in the sense of Definition* 2.3.

(b) *For all* $\gamma \in (\frac{d}{2}+1, s]$, (u, τ^*) *satisfies*

$$\mathbf{1}_{\{\limsup_{t\to\tau^*}\|u(t)\|_{H^s}=\infty\}} = \mathbf{1}_{\{\limsup_{t\to\tau^*}\|u(t)\|_{H^\gamma}=\infty\}} \quad \mathbb{P}\text{-}a.s. \tag{2.2}$$

(c) *If Assumption* A_2 *is also satisfied and* $u_0 \geq 0$ $\mathbb{P}$*-a.s., then*

$$\mathbb{P}\{u \geq 0,\ t \in [0, \tau^*)\} = 1.$$

Remark 2.1. Now we give a remark to discuss Theorem 2.1, the main difficulties encountered in the proof and the main strategies we used. We first notice that the nonlocal term $F(\cdot)$ in 1.4 is not monotone (see (3.6), H^{s+1}-norm appears) in the sense of [50] so that we will not use the Galerkin approximation under a Gelfand triple developed in [50].

- (Mollifying and cut-off) The starting point of our analysis is to consider (1.4) as an SDE in H^s, which can be achieved by mollifying the equation. Then we have a sequence of approximation solution $\{u_\varepsilon\} \in C([0, T_\varepsilon); H^s)$ for some $T_\varepsilon > 0$. In the *a priori* estimate, since the estimate on $\mathbb{E}\|u_\varepsilon\|^2_{H^s}$ involves $\mathbb{E}\|u_\varepsilon\|_{H^r}\|u_\varepsilon\|^2_{H^s}$ for some $r > \frac{d}{2}+1$ (see the estimate for the nonlinear term in Lemma 3.4 and Remark 3.1), which can not be split. To close the estimates, we will add a *cut-off* function $\theta_R(\|\cdot\|_{H^r})$ to cut the H^r-norm (see (4.1)). Though the cut-off technique is insufficient to make the equation global Lipschitz in H^s, it still provides linear growth in H^s, and hence $T_\varepsilon = \infty$ almost surely. Otherwise we have to show $\inf_\varepsilon T_\varepsilon > 0$ $\mathbb{P}$-a.s. However, how to find such a quantitative lower bound is generally not clear.
- (Convergence of the approximation solutions) To obtain a strong solution, one needs to take limit in the mollified problem. Our method is different from the martingale approach (by first establishing martingale solutions and then obtaining strong solution via (pathwise) uniqueness) used in many previous works. For example, we refer to [10, 11] for different examples in unbounded domains with linear growing noise. However, the method used in [10, 11] is not applicable in our case. This is because, firstly, in our case we need a *cut-off* function $\theta_R(\|\cdot\|_{H^r})$ as mentioned above, and secondly, even though one can actually establish the probabilistic compactness $L^2(\Omega; H^s_{\text{loc}})$"$\hookrightarrow\hookrightarrow$"$L^2(\Omega; H^{s-2}_{\text{loc}})$ (cf. Prokhorov's Theorem and Skorokhod's Theorem) and obtain the converging in H^{s-2}_{loc} (cf. Skorokhod's Theorem), we can *not* pass the limit because $\|\cdot\|_{H^r}$ is a global object involving all $x \in \mathbb{R}^d$ and it can *not* be controlled by the H^{s-2}_{loc}-topology even in the case $s-2 > r$. In this paper, following [41] (see also [2, 54]), we will show that there is a subsequence of the approximation solutions converging in $C([0,T]; H^r)$ $\mathbb{P}$-a.s. directly (see Lemma 4.3 below). We outline that the convergence holds true on $[0,T]$, which *a priori* may look surprising because a sequence of stopping times is usually needed to estimate nonlinear terms, and as is mentioned above, the lower bound of such stopping times is difficult to obtain in stochastic setting. To

take limit in $\theta_R(\|\cdot\|_{H^r})$, we choose $r < s-2$. This and the previous condition $r > \frac{d}{2}+1$ imply that $s > \frac{d}{2}+3$.
- (Removal of cut-off) For almost surely bounded initial variable u_0, one can introduce a stopping time $\tau = \tau(u_0, R)$ (as in (4.15)) to remove the cut-off. Inspired by [30, 28, 44], we use a cutting-combining argument to remove all the additional conditions on u_0 and guarantee that τ is positive almost surely. Roughly speaking, the cutting-combining argument is as follows: If $\{\Omega_k\}$ is mutually disjoint, $\sum_k \mathbf{1}_{\Omega_k} = \mathbf{1}$ and $u_0 = \sum_k \mathbf{1}_{\Omega_k} u_0$ almost surely, and u_k is the solution to (1.4) with initial data $\mathbf{1}_{\Omega_k} u_0$, then $\sum_k \mathbf{1}_{\Omega_k} u_k$ is a solution to u_0.
- (Generalized blow-up criterion) By Definition 2.3 and (2.2), one has
$$\mathbf{1}_{\{\tau^*<\infty\}} = \mathbf{1}_{\{\limsup_{t\to\tau^*} \|u(t)\|_{H^\gamma}=\infty\}}, \quad \gamma \in (d/2+1, s] \ \mathbb{P}\text{-a.s.}$$
However, (2.2) is a stronger statement, because the H^s-norm of solution may also tend to ∞ at $\tau^* = \infty$, and even in this case (2.2) still holds (due to (4.16)). Hence we call it a generalized blow-up criterion to include the case of blow-up at ∞, which can be also viewed as a kind of global existence.
- (Non-negativity) The idea of showing the almost surely non-negativity is to show that $u^- = 0$ $\mathbb{P}$-a.s., where u^- is the negative part of u. In the deterministic PDEs, this can be achieved by showing that $\|u^-(t)\|_{L^2} = 0$ for all $t > 0$. Now, we prove it by showing that $\mathbb{E}\|u^-\|_{L^2} = 0$ for all $t > 0$. To this end, we need to consider Itô's formula for $\|u^-\|_{L^2}^2$, where the main difficulty is that the nonlinear functional $\|\cdot^-\|_{L^2}^2 : H^s \ni u \mapsto \|u^-\|_{L^2}^2 \in \mathbb{R}$ is not C^2. To overcome this, motivated by [14, 16, 15], we consider smooth C^2-approximations of $\|u^-\|_{L^2}$.

Then we consider (1.5), where $\sigma(t,u)\,\mathrm{d}\mathcal{W} = \beta(t)u\,\mathrm{d}W$. This particular structure only requires a single standard 1-D Brownian motion W rather than a cylindrical Wiener process $\mathcal{W}$. Hence for (1.5), the stochastic basis becomes $\mathcal{S} = (\Omega, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0}, W)$, where W is a standard 1-D Brownian motion. Define

$$\varphi(t) = \varphi(\omega,t) := \mathrm{e}^{\int_0^t \beta(t')\,\mathrm{d}W(t') - \int_0^t \frac{\beta^2(t')}{2}\,\mathrm{d}t'}, \quad \mathcal{A} = \mathcal{A}(\omega) := \sup_{t>0} \varphi(\omega,t). \tag{2.3}$$

When Assumption A_3 holds true, we outline that (see (1) in Lemma 3.6 below)

$$\mathcal{A} < \infty \ \mathbb{P}\text{-a.s.}$$

According to Theorem 2.1, for the maximal solution (u, τ^*) to (1.5), even though the H^s-norm of u may blow up at τ^*, the L^∞-norm of u may survive for all the time. Indeed, in next theorem, we see that noise has decay effect on $\|u\|_{L^\infty}$.

Theorem 2.2 (Decay of L^∞-norm in $\mathbb{R}^2$ almost surely). *Let $d = 2$, $\chi > 0$, $s > 4$ and $(\varphi, \mathcal{A})$ be given in* (2.3). *Let Assumption A_3 hold true. Assume u_0 is an $H^s \cap L^1$-valued $\mathcal{F}_0$ measurable random variable satisfying $u_0 \geq 0$ $\mathbb{P}$-a.s. and*

$\mathbb{E}\|u_0\|^2_{H^s} < \infty$. *If for some large constant* $C > 0$,

$$\mathbb{P}\left\{\|u_0\|_{L^1} \leq \frac{1}{4C\chi\mathcal{A}}\right\} = 1, \tag{2.4}$$

then there is a random variable $0 < K = K(\omega) < \infty$ $\mathbb{P}$-*a.s. such that the solution* u *to* (1.5) *satisfies*

$$\mathbb{P}\{\|u(t)\|_{L^\infty} \leq CK \max\{\|u_0\|_{L^1}, \|u_0\|_{L^\infty}\}\varphi(t), \quad t > 0\} = 1. \tag{2.5}$$

That is to say, $\mathbb{P}\{\|u(t)\|_{L^\infty}$ decays with (least) rate $\mathrm{e}^{\int_0^t \beta(t')\,\mathrm{d}W_{t'} - \int_0^t \frac{\beta^2(t')}{2}\,\mathrm{d}t'}\} = 1$.

For the H^s-norm of u, the next theorem shows that noise can also bring decay effect with high probability.

Theorem 2.3 (Decay of H^s-norm in $\mathbb{R}^d$ with high probability). *Let* $\chi \in \mathbb{R}\backslash\{0\}$, $d \geq 2$ *and* $s > \frac{d}{2} + 3$. *Let Assumption* A_3 *hold. If* u_0 *is an* H^s*-valued* $\mathcal{F}_0$ *measurable random variable such that for some large enough* $C = C(s) > 0$,

$$\|u_0\|_{H^s} \leq \frac{\beta_*}{C\lambda_1|\chi|R}, \quad R > 1, \quad \lambda_1 > 2 \ \mathbb{P}\text{-a.s.}, \tag{2.6}$$

then the maximal solution (u, τ^*) *to* (1.5) *satisfies that for any* $\lambda_2 > \frac{2\lambda_1}{\lambda_1 - 2}$,

$$\mathbb{P}\left\{\|u(t)\|_{H^s} \leq \frac{\beta_*}{C\lambda_1|\chi|}\mathrm{e}^{-\frac{((\lambda_1-2)\lambda_2-2\lambda_1)}{2\lambda_1\lambda_2}\int_0^t \beta^2(t')\,\mathrm{d}t'}, \quad t > 0\right\} \geq 1 - \left(\frac{1}{R}\right)^{2/\lambda_2},$$

which means, $\mathbb{P}\{\|u(t)\|_{H^s}$ decays with (least) rate $\mathrm{e}^{-\frac{((\lambda_1-2)\lambda_2-2\lambda_1)}{2\lambda_1\lambda_2}\int_0^t \beta^2(t')\,\mathrm{d}t'}\} \geq 1 - (\frac{1}{R})^{2/\lambda_2}$.

Theorem 2.4 (Strong noise prevents blow-up almost surely). *Let* $d \geq 2$, $s > \frac{d}{2} + 3$, $\chi \in \mathbb{R}\backslash\{0\}$ *and* $u_0 \in H^s$ *be an* H^s*-valued* $\mathcal{F}_0$*-measurable random variable with* $\mathbb{E}\|u_0\|^2_{H^s} < \infty$. *If* q *and* a *satisfy*

$$\alpha \in \mathbb{R}\backslash\{0\} \quad \text{if } \varrho > \frac{1}{2} \text{ or } \alpha^2 > 2D|\chi| \quad \text{if } \varrho = \frac{1}{2}, \tag{2.7}$$

where D *is the constant given in Lemma* 3.4, *then* (1.6) *has a unique global solution starting from* u_0.

Remark 2.2. We give the following remarks concerning Theorems 2.2, 2.3 and 2.4.

- In the proof of Theorem 2.2, the Moser-Alikakos iteration technique and decay estimate of Girsanov type process are used (for the Moser iteration involving expectation, we refer to [29]). Theorem 2.2 entails that if the initial mass is small, then the solution to (1.5) is bounded globally and decays to zero. In contrast to the deterministic counterpart of (1.5), where the decay rate of $\|u\|_{L^\infty}$ is only algebraic (cf. [32]), Theorem 2.2 shows that the multiplicative noise $\beta(t)u\,\mathrm{d}W$ brings more dissipation in the sense that $\|u\|_{L^\infty}$ decays exponentially with (least) rate $\mathrm{e}^{\int_0^t \beta(t')\,\mathrm{d}W_{t'} - \int_0^t \frac{\beta^2(t')}{2}\,\mathrm{d}t'}$.

- The proof of Theorem 2.3 involves extracting a damping part from the transformation (5.1) (see (5.14)) and using some estimates for the exit times of Girsanov type process. In Theorem 2.3, for fixed $\lambda_1 > 2$ and $\lambda_2 > \frac{2\lambda_1}{\lambda_1 - 2}$, if we let $R \gg 1, \beta_* \gg 1$ such that $1/R^{2/\lambda_2}$ is small enough but $\beta_* \gg C\lambda_1|\chi|R$ is large, then the H^s norm of initial data can be large. Moreover, the H^s norm of the corresponding solution decays exponentially with high probability.
- Theorem 2.4 is proved by using a Lyapunov function $\log(1 + x^2)$ and the result means that if the nonlinear noise is strong enough, i.e., (2.7) is satisfied, then the global existence holds almost surely without any smallness assumption on initial data. Our approach is motivated by [9, 51].
- Without noise, it is well-known that the solution to (1.5) may blow up in two dimensions with a critical mass and three (or higher) dimensions for small mass (cf. [19, 42, 45]). The results of Theorems 2.3 and 2.4 indicate that large multiplicative noise can provide some "regularization" effects on the global boundedness or decay properties of the solutions (In Theorem 2.2, decay of L^∞ norm becomes faster; In Theorem 2.3, H^s norm decays exponentially with high probability, and in Theorem 2.4, solution exists globally without any kind smallness conditions on the initial data). For deterministic aggregation-diffusion equations, different mechanisms have been proposed to ensure the pattern formation without blow-up. Theorems 2.2, 2.3 and 2.4 show that certain noise can also induce some dissipation/regularization effects to aggregation-diffusion model.

3. Some Preliminary Results

Now we gather some necessary results from analysis. For any $\varepsilon \in (0, 1)$, J_ε is the Friedrichs mollifier defined by $J_\varepsilon f(x) = j_\varepsilon * f(x)$, where $*$ stands for the convolution, $j_\varepsilon(x) = \frac{1}{\varepsilon^d} j(\frac{x}{\varepsilon})$ and $j(x)$ is a Schwartz function satisfying $0 \le \widehat{j}(\xi) \le 1$ for $\xi \in \mathbb{R}^d$ and $\widehat{j}(\xi) = 1$ for all $\xi \in \mathbb{R}^d$ with $|\xi| \le 1$. It is easy to find that (cf. [41])

$$\|I - J_\varepsilon\|_{\mathcal{L}(H^s;H^r)} \lesssim \varepsilon^{s-r}, \quad r < s, \tag{3.1}$$

$$\|J_\varepsilon\|_{\mathcal{L}(H^s;H^r)} \lesssim O(\varepsilon^{s-r}), \quad r > s. \tag{3.2}$$

In addition, we have the following properties which will be used in the sequel without further notice:

$$\mathcal{D}^s J_\varepsilon = J_\varepsilon \mathcal{D}^s, \quad (J_\varepsilon f, g)_{L^2} = (f, J_\varepsilon g)_{L^2}, \quad \|J_\varepsilon u\|_{H^s} \le \|u\|_{H^s},$$

where $\mathcal{D}^s = (I - \Delta)^{s/2}$ is defined in Sec. 2.1.

We also notice the following estimates:

Lemma 3.1 ([55]). *Let $d \ge 1$. Let $f, g : \mathbb{R}^d \to \mathbb{R}^d$ such that $g \in W^{1,\infty}$ and $f \in L^2$. Then for some $C > 0$,*

$$\|[J_\varepsilon, (g \cdot \nabla)]f\|_{L^2} \le C\|\nabla g\|_{L^\infty}\|f\|_{L^2}.$$

Lemma 3.2 ([35]). *If $f, g \in H^s \cap W^{1,\infty}$ for $s > 0$, then*

$$\|[\mathcal{D}^s, (f \cdot \nabla)]g\|_{L^2} \le C_s(\|\mathcal{D}^s f\|_{L^2}\|\nabla g\|_{L^\infty} + \|\nabla f\|_{L^\infty}\|\mathcal{D}^{s-1}\nabla g\|_{L^2}).$$

If $f, g \in H^s \cap L^\infty$, *then*

$$\|fg\|_{H^s} \leq C_s(\|f\|_{H^s}\|g\|_{L^\infty} + \|f\|_{L^\infty}\|g\|_{H^s}).$$

Lemma 3.3. *Let* $F(\cdot)$ *be defined in* (1.3). *For any* $u, v \in H^{s+1}$ *with* $s > \frac{d}{2}+1$ *and* $\delta > \frac{d}{2}$, *we have*

$$\|F(v)\|_{H^s} \lesssim \|v\|_{H^\delta}(\|v\|_{H^s} + \|v\|_{H^{s+1}}), \tag{3.3}$$

$$\|F(u) - F(v)\|_{H^s} \lesssim \|u\|_{H^{s+1}}\|u-v\|_{H^s} + \|v\|_{H^s}\|u-v\|_{H^{s+1}}, \tag{3.4}$$

$$|(F(u) - F(v), u-v)_{L^2}| \lesssim (\|u\|_{W^{1,\infty}} + \|v\|_{H^\delta})\|u-v\|_{L^2}^2, \tag{3.5}$$

$$|(F(u) - F(v), u-v)_{H^s}| \lesssim (\|u\|_{H^{s+1}} + \|v\|_{H^s})\|u-v\|_{H^s}^2. \tag{3.6}$$

Proof. Using Lemma 3.2, $H^\delta \hookrightarrow L^\infty$ with noticing that $(I-\Delta)^{-1}$ is bounded from H^s to H^{s+2}, we have

$$\begin{aligned}
\|F(v)\|_{H^s} &\lesssim \|vQ(v)\|_{H^{s+1}} \lesssim \|v\|_{L^\infty}\|Q(v)\|_{H^{s+1}} + \|v\|_{H^{s+1}}\|Q(v)\|_{L^\infty} \\
&\lesssim \|v\|_{H^\delta}(\|v\|_{H^s} + \|v\|_{H^{s+1}}),
\end{aligned}$$

which is (3.3). Set $w = u - v$. Then we have

$$\begin{aligned}
&\|F(u) - F(v)\|_{H^s} \\
&\lesssim \|\mathrm{div}(uQ(w))\|_{H^s} + \|\mathrm{div}(wQ(v))\|_{H^s} \\
&\lesssim \|u\|_{H^{s+1}}\|Q(w)\|_{H^{s+1}} + \|w\|_{H^{s+1}}\|Q(v)\|_{H^{s+1}},
\end{aligned}$$

which implies (3.4). Similarly, it follows that

$$\begin{aligned}
&|(F(u) - F(v), w)_{L^2}| \\
&\lesssim |(\nabla u \cdot Q(w), w)_{L^2}| + |(u\mathrm{div}Q(w), w)_{L^2}| \\
&\quad + |(\nabla w \cdot Q(v), w)_{L^2}| + |(w\mathrm{div}Q(v), w)_{L^2}| \\
&\lesssim \|\nabla u\|_{L^\infty}\|Q(w)\|_{L^2}\|w\|_{L^2} + \|u\|_{L^\infty}\|\mathrm{div}Q(w)\|_{L^2}\|w\|_{L^2} \\
&\quad + \|\mathrm{div}Q(v)\|_{L^\infty}\|w\|_{L^2}^2 \\
&\lesssim \|u\|_{W^{1,\infty}}\|w\|_{L^2}^2 + \|v\|_{H^\delta}\|w\|_{L^2}^2,
\end{aligned}$$

which is (3.5). As for (3.6), we first notice that $H^s \hookrightarrow W^{1,\infty}$ and hence

$$\|\mathrm{div}Q(v)\|_{L^\infty} \lesssim \|\nabla Q(v)\|_{L^\infty} \lesssim \|Q(v)\|_{H^s} \lesssim \|v\|_{H^{s-1}}.$$

Then we use Lemma 3.2 and integration by parts to deduce that

$$\begin{aligned}
&|(F(u) - F(v), w)_{H^s}| \\
&\lesssim |(\mathcal{D}^s\mathrm{div}(uQ(w)), \mathcal{D}^s w)_{L^2}| + |([\mathcal{D}^s, (Q(v)\cdot\nabla)]w, \mathcal{D}^s w)_{L^2}| \\
&\quad + |(Q(v)\cdot\nabla\mathcal{D}^s w, \mathcal{D}^s w)_{L^2}| + |(\mathcal{D}^s(w\mathrm{div}Q(v)), \mathcal{D}^s w)_{L^2}| \\
&\lesssim \|uQ(w)\|_{H^{s+1}}\|w\|_{H^s} + \|v\|_{H^s}\|w\|_{H^s}^2 + \|w\mathrm{div}Q(v)\|_{H^s}\|w\|_{H^s} \\
&\lesssim \|u\|_{H^{s+1}}\|w\|_{H^s}^2 + \|v\|_{H^s}\|w\|_{H^s}^2,
\end{aligned}$$

which yields (3.6). □

Recall that $F(\cdot)$ is defined in (1.3) and J_ε is the Friedrichs mollifier in Sec. 3. Then we have

Lemma 3.4. *Let $s \geq \delta > \frac{d}{2}+1$. For all $u \in H^s$, there is a constant $D = D(s) > 0$ such that for all $\varepsilon > 0$, we have*

$$|(J_\varepsilon F(u), J_\varepsilon u)_{H^s}| \leq D\|u\|_{H^\delta}\|u\|_{H^s}^2.$$

Proof. Using Lemmas 3.1 and 3.2, integration by parts, the fact that $\nabla(I-\Delta)^{-1}$ is bounded from H^s to H^{s+1} and the embedding $H^\delta \hookrightarrow W^{1,\infty}$, we obtain that for some $D > 0$,

$$\begin{aligned}
&|(\mathcal{D}^s J_\varepsilon F(u), \mathcal{D}^s J_\varepsilon u)_{L^2}| \\
&= |(\mathcal{D}^s J_\varepsilon[(Q(u)\cdot\nabla)u], \mathcal{D}^s J_\varepsilon u)_{L^2} + (\mathcal{D}^s J_\varepsilon[u\mathrm{div}Q(u)], \mathcal{D}^s J_\varepsilon u)_{L^2}| \\
&\leq |([\mathcal{D}^s, (Q(u)\cdot\nabla)]u, \mathcal{D}^s J_\varepsilon^2 u)_{L^2}| + |([J_\varepsilon, (Q(u)\cdot\nabla)]\mathcal{D}^s u, \mathcal{D}^s J_\varepsilon u)_{L^2}| \\
&\quad + \left|\left((Q(u)\cdot\nabla)\mathcal{D}^s J_\varepsilon u, \mathcal{D}^s J_\varepsilon u\right)_{L^2}\right| + \left|\left(\mathcal{D}^s J_\varepsilon[u\mathrm{div}Q(u)], \mathcal{D}^s J_\varepsilon u\right)_{L^2}\right| \\
&\lesssim \|Q(u)\|_{H^s}\|\nabla u\|_{L^\infty}\|u\|_{H^s} + \|\nabla Q(u)\|_{L^\infty}\|u\|_{H^s}^2 + \|u\mathrm{div}Q(u)\|_{H^s}\|u\|_{H^s} \\
&\leq D\|u\|_{H^\delta}\|u\|_{H^s}^2,
\end{aligned}$$

which gives the desired result. $\square$

Remark 3.1. We remark that if $u \in H^{s+1}$, we can omit the mollifier J_ε in the proof of Lemma 3.4 to deduce that

$$|(F(u), u)_{H^s}| \leq D\|u\|_{H^\delta}\|u\|_{H^s}^2.$$

However, in applications, sometimes we can only know $u \in H^s$, and hence $(F(u), u)_{H^s}$ is *not* well-defined because $F(u) \in H^{s-1}$ (cf. Lemma 3.3). This means that we can not apply Itô's formula to $\|u(t)\|_{H^s}^2$ directly (cf. (4.13) below). In this case, we need Lemma 3.4, where the constant D does not depend on ε.

Now we recall the following lemma (cf. [15, Lemma 3.1]) which will be used in the study of the non-negativity of the solutions.

Lemma 3.5 ([15]). *Define the following functions*:

$$\rho(x) = -\mathbf{1}_{\{x<0\}}x, \quad \kappa(\cdot) = \rho^2(\cdot),$$

$$a(x) = \begin{cases} 0, & x \geq 0, \\ 1, & x < 0, \end{cases} \quad \text{and} \quad \kappa_\varepsilon(x) = \begin{cases} x^2 - \dfrac{\varepsilon^2}{6}, & x < -\varepsilon, \\ \dfrac{-x^3}{\varepsilon}\left(\dfrac{x}{2\varepsilon} + \dfrac{4}{3}\right), & -\varepsilon \leq x < 0, \\ 0, & x \geq 0. \end{cases}$$

Then the following properties hold:

- $\kappa(x) = 0$ *if* $x \geq 0$ *and* $\kappa(x) = x^2$ *if* $x < 0$;
- $\kappa_\varepsilon'(x)$ *and* $\kappa_\varepsilon''(x)$ *are continuous*;
- $\kappa_\varepsilon' \leq 0$, $\kappa_\varepsilon'' \geq 0$, $\kappa_\varepsilon'(x) = 0$ *if* $x \geq 0$;
- $\kappa_\varepsilon(x) \to \kappa(x)$, $\kappa_\varepsilon'(x) \to -2\rho(x)$ and $\kappa_\varepsilon''(x) \to 2a(x)$ *uniformly on* $\mathbb{R}$.

Lemma 3.6 ([41]). *Let Assumption* A_3 *hold true and* $\vartheta(t) \in C([0,\infty))$ *be a bounded function. Let*

$$X(t) = \mathrm{e}^{\int_0^t \beta(t')\,\mathrm{d}W_{t'} + \int_0^t \vartheta(t') - \frac{\beta^2(t')}{2}\,\mathrm{d}t'}, \quad t \geq 0.$$

Then we have the following properties:

(1) *Let* $\phi(t) := \int_0^t \beta^2(t')\,\mathrm{d}t'$ *and* $\phi^{-1}(t)$ *be the inverse function of* ϕ. *If*

$$\limsup_{t\to\infty} \frac{1}{\sqrt{2t\log\log t}} \left(\int_0^{\phi^{-1}(t)} \vartheta(t')\,\mathrm{d}t' - \frac{t}{2} \right) < -1,$$

then $\lim_{t\to\infty} X(t) = 0$ $\mathbb{P}$-*a.s. If*

$$\liminf_{t\to\infty} \frac{1}{\sqrt{2t\log\log t}} \left(\int_0^{\phi^{-1}(t)} \vartheta(t')\,\mathrm{d}t' - \frac{t}{2} \right) > 1,$$

then $\lim_{t\to\infty} X(t) = +\infty$ $\mathbb{P}$-*a.s.*

(2) *Let* $\vartheta(t) = p\beta^2(t)$ *with* $p < \frac{1}{2}$ *and* $\tau_R := \inf\{t \geq 0 : X(t) > R\}$ *with* $R > 1$, *then*

$$\mathbb{P}\{\tau_R = +\infty\} \geq 1 - \left(\frac{1}{R}\right)^{1-2p}.$$

4. Proof of Theorem 2.1

For clarity, we complete the proof of Theorem 2.1 in the following several subsections/steps.

4.1. *Approximation scheme and associated estimates*

Now we construct the approximation scheme.

Cut-off. Let $s > \frac{d}{2} + 3$ and $r \in (\frac{d}{2} + 1, s - 2)$. For any $R > 0$, we let $\theta_R(x) : [0,\infty) \to [0,1]$ be a C^∞ function such that $\theta_R(x) = 1$ for $|x| \in [0,R]$ and $\theta_R(x) = 0$ for $|x| > 2R$. Then we consider the problem by cutting the nonlinearities in (1.4) as follows:

$$\begin{cases} \mathrm{d}u + [-\Delta u + \chi\theta_R(\|u\|_{H^r})F(u)]\mathrm{d}t = \theta_R(\|u\|_{H^r})\sigma(t,u)\mathrm{d}\mathcal{W}, \quad t>0, \\ u(\omega,0,x) = u_0(\omega,x) \in H^s. \end{cases} \tag{4.1}$$

Mollifying. Recall that J_ε is the Friedrichs mollifier defined in the previous section. Then we mollify (4.1) and consider the following approximate problem:

$$\begin{cases} \mathrm{d}u + G_{1,\varepsilon}(u)\,\mathrm{d}t = G_2(t,u)\,\mathrm{d}\mathcal{W}, \\ \qquad G_{1,\varepsilon}(u) = -J_\varepsilon^2\Delta u + \chi\theta_R(\|u\|_{H^r})J_\varepsilon F(J_\varepsilon u), \\ \qquad G_2(t,u) = \theta_R(\|u\|_{H^r})\sigma(t,u), \\ \qquad u(\omega,0,x) = u_0(\omega,x). \end{cases} \tag{4.2}$$

Lemma 4.1. *Let $\chi \in \mathbb{R}\backslash\{0\}$, $s > \frac{d}{2}+3$ and $r \in (\frac{d}{2}+1, s-2)$. Fix a stochastic basis $\mathcal{S}$ and let $u_0 \in L^2(\Omega; H^s)$ be an H^s-valued $\mathcal{F}_0$ measurable random variable. Assume σ satisfies Assumption A_1. Then for any $R>1$ and $\varepsilon \in (0,1)$, (4.2) admits a unique solution $u_\varepsilon \in C([0,\infty); H^s)$ $\mathbb{P}$-a.s. Moreover, for any $T>0$, there is a constant $C = C(\chi, R, T, u_0) > 0$ such that*

$$\sup_{\varepsilon>0}\mathbb{E}\left\{\sup_{t\in[0,T]}\|u_\varepsilon(t)\|_{H^s}^2 + 2\int_0^T \|\nabla J_\varepsilon u_\varepsilon(t)\|_{H^s}^2\,\mathrm{d}t\right\} \le C.$$

Proof. Using Assumption A_1, (3.2) and Lemma 3.3, it is easy to obtain that for any $T>0$ and $R>1$, there exist $l_1 = l_1(R,\varepsilon,\chi)$ and $l_2 = l_2(R)$ such that for all $u \in C([0,T]; H^q)$ with $q > \frac{d}{2}+1$,

$$\|G_{1,\varepsilon}(u)\|_{H^q} \le l_1(1+\|u\|_{H^q}), \quad \|G_2(t,u)\|_{\mathcal{L}_2(\mathbb{U};H^q)} \le l_2(1+\|u\|_{H^q}), \quad t\in[0,T].$$

For any $R>1$, $s>\frac{d}{2}+3$ and $\varepsilon \in (0,1)$, the above estimate implies that (4.2) defines an SDE in H^s with linear growth condition. Similarly, we can infer from Assumption A_1 and Lemma 3.3 that for any $t>0$, $G_{1,\varepsilon}(u)$ and $G_2(t,u)$ are locally Lipschitz in $u \in H^s$.

Then the theory of SDE in Hilbert space (see for example [50, Theorem 4.2.4 with Example 4.1.3]) shows that (4.2) admits a unique solution $u_\varepsilon \in C([0,\infty); H^s)$ almost surely.

Now we establish the uniform-in-ε estimate for (4.2). By Itô's formula, we have

$$\mathrm{d}\|u_\varepsilon\|_{H^s}^2 + 2\|\nabla J_\varepsilon u_\varepsilon\|_{H^s}^2\,\mathrm{d}t = A_1 + \sum_{i=2}^4 A_i\,\mathrm{d}t,$$

where

$$A_1 := 2\theta_R(\|u_\varepsilon\|_{H^r})(\sigma(t,u_\varepsilon)\,\mathrm{d}\mathcal{W}, u_\varepsilon)_{H^s}$$

$$A_2 := -2\chi\theta_R(\|u_\varepsilon\|_{H^r})(\mathcal{D}^s J_\varepsilon[(Q(J_\varepsilon u_\varepsilon)\cdot\nabla)J_\varepsilon u_\varepsilon], \mathcal{D}^s u_\varepsilon)_{L^2}$$

$$A_3 := -2\chi\theta_R(\|u_\varepsilon\|_{H^r})(\mathcal{D}^s J_\varepsilon[J_\varepsilon u_\varepsilon \mathrm{div} Q(J_\varepsilon u_\varepsilon)], \mathcal{D}^s u_\varepsilon)_{L^2}$$

$$A_4 := \theta_R^2(\|u_\varepsilon\|_{H^r})\|\sigma(t,u_\varepsilon)\|_{\mathcal{L}_2(\mathbb{U};H^s)}^2.$$

Let $T > 0$. Integrating the above equation, taking a supremum for $t \in [0, T]$ and using the BDG inequality yield

$$\begin{aligned}
&\mathbb{E} \sup_{t\in[0,T]} \|u_\varepsilon(t)\|_{H^s}^2 + 2\mathbb{E}\int_0^T \|\nabla J_\varepsilon u_\varepsilon(t)\|_{H^s}^2 \,\mathrm{d}t \\
&\leq \mathbb{E}\|u_0\|_{H^s}^2 + \sum_{i=2}^{4} \int_0^T \mathbb{E} \sup_{t'\in[0,t]} |A_i(t')| \,\mathrm{d}t \\
&\quad + \frac{1}{2}\mathbb{E} \sup_{t\in[0,T]} \|u_\varepsilon\|_{H^s}^2 + Cf^2(2R)\int_0^T \left(1 + \mathbb{E} \sup_{t'\in[0,t]} \|u_\varepsilon(t')\|_{H^s}^2\right) \mathrm{d}t.
\end{aligned}$$

Since J_ε is self-adjoint, in the same way as in Lemma 3.4, we derive that

$$|A_2| + |A_3| \leq C|\chi|R\|u_\varepsilon\|_{H^s}^2.$$

Therefore, we arrive at

$$\int_0^T \mathbb{E} \sup_{t'\in[0,t]} (|A_2(t')| + |A_3(t')|)\mathrm{d}t \leq C|\chi|R\int_0^T \mathbb{E} \sup_{t'\in[0,t]} \|u_\varepsilon(t')\|_{H^s}^2 \,\mathrm{d}t.$$

Similarly, we use $H^r \hookrightarrow W^{1,\infty}$ and Assumption A_1 to deduce that

$$|A_4| \leq \theta_R^2(\|u_\varepsilon\|_{H^r})f^2(\|u_\varepsilon\|_{H^r})(1 + \|u_\varepsilon\|_{H^s})^2.$$

Consequently, we find a constant $C = C(\chi, R) > 0$ such that

$$\sum_{i=2}^{4} \int_0^T \mathbb{E} \sup_{t'\in[0,t]} |A_i(t')| \,\mathrm{d}t \leq C\int_0^T \left(1 + \mathbb{E} \sup_{t'\in[0,t]} \|u_\varepsilon(t')\|_{H^s}^2\right) \mathrm{d}t'.$$

Combining the above estimates, we see that u_ε satisfies

$$\begin{aligned}
&\mathbb{E} \sup_{t\in[0,T]} \|u_\varepsilon(t)\|_{H^s}^2 + 4\mathbb{E}\int_0^T \|J_\varepsilon \nabla u_\varepsilon(t)\|_{H^s}^2 \,\mathrm{d}t \\
&\qquad \leq 2\mathbb{E}\|u_0\|_{H^s}^2 + C(\chi, R)\int_0^T \left(1 + \mathbb{E} \sup_{t'\in[0,t]} \|u_\varepsilon(t')\|_{H^s}^2\right) \mathrm{d}t.
\end{aligned}$$

Thanks to the Grönwall inequality, we obtain the desired estimate. □

Lemma 4.2. *Assume the conditions in Lemma* 4.1 *hold true. For any* $T > 0$ *and* $K > 0$, *we define*

$$\tau_{\varepsilon,K}^T := \inf\{t \geq 0 : \|u_\varepsilon(t)\|_{H^s} \geq K\} \wedge T, \quad \tau_{\varepsilon,\eta,K}^T := \tau_{\varepsilon,K}^T \wedge \tau_{\eta,K}^T. \tag{4.3}$$

Then we have

$$\lim_{\varepsilon\to 0}\sup_{\eta\le\varepsilon}\mathbb{E}\sup_{t\in[0,\tau^T_{\varepsilon,\eta,K}]}\|u_\varepsilon-u_\eta\|_{H^r}=0,\quad K>1. \tag{4.4}$$

Proof. For the solutions u_ε and u_η to (4.2), we consider the following problem for $v_{\varepsilon,\eta}=u_\varepsilon-u_\eta$,

$$\mathrm{d}v_{\varepsilon,\eta}+[G_{1,\varepsilon}(u_\varepsilon)-G_{1,\eta}(u_\eta)]\mathrm{d}t=[G_2(t,u_\varepsilon)-G_2(t,u_\eta)]\mathrm{d}\mathcal{W},\ \ v_{\varepsilon,\eta}(0)=0.$$

We notice that

$$\begin{aligned}
&G_{1,\varepsilon}(u_\varepsilon)-G_{1,\eta}(u_\eta)\\
=&-J_\varepsilon^2\Delta u_\varepsilon+J_\eta^2\Delta u_\eta+\chi\theta_R(\|u_\varepsilon\|_{H^r})J_\varepsilon F(J_\varepsilon u_\varepsilon)-\chi\theta_R(\|u_\eta\|_{H^r})J_\eta F(J_\eta u_\eta)\\
=&\left(J_\eta^2-J_\varepsilon^2\right)\Delta u_\varepsilon+J_\eta^2\Delta\left(u_\eta-u_\varepsilon\right)\\
&+\chi\left[\theta_R(\|u_\varepsilon\|_{H^r})-\theta_R(\|u_\eta\|_{H^r})\right]J_\varepsilon F(J_\varepsilon u_\varepsilon)\\
&+\chi\theta_R(\|u_\eta\|_{H^r})(J_\varepsilon-J_\eta)\left[F(J_\varepsilon u_\varepsilon)\right]\\
&+\chi\theta_R(\|u_\eta\|_{H^r})J_\eta\left[F(J_\varepsilon u_\varepsilon)-F(J_\eta u_\varepsilon)\right]\\
&+\chi\theta_R(\|u_\eta\|_{H^r})J_\eta\left[F(J_\eta u_\varepsilon)-F(J_\eta u_\eta)\right]\\
:=&\sum_{i=1}^6\mathcal{R}_i,
\end{aligned}\tag{4.5}$$

and

$$\begin{aligned}
&G_2(t,u_\varepsilon)-G_2(t,u_\eta)\\
=&\,\theta_R(\|u_\varepsilon\|_{H^r})\sigma(t,u_\varepsilon)-\theta_R(\|u_\eta\|_{H^r})\sigma(t,u_\eta)\\
=&\left[\theta_R(\|u_\varepsilon\|_{H^r})-\theta_R(\|u_\eta\|_{H^r})\right]\sigma(t,u_\varepsilon)+\theta_R(\|u_\eta\|_{H^r})[\sigma(t,u_\varepsilon)-\sigma(t,u_\eta)]\\
:=&\sum_{i=7}^8\mathcal{R}_i.
\end{aligned}\tag{4.6}$$

Then we use Itô's formula with noticing (4.5) and (4.6) to find that for any $t>0$,

$$\|v_{\varepsilon,\eta}(t)\|^2_{H^r}+2\int_0^t(\mathcal{R}_2,v_{\varepsilon,\eta})_{H^r}\,\mathrm{d}t'=R_1-\int_0^t R_2\,\mathrm{d}t'+\int_0^t R_3\,\mathrm{d}t', \tag{4.7}$$

where

$$\begin{cases}
R_1:=2\displaystyle\int_0^t([G_2(t,u_\varepsilon)-G_2(t,u_\eta)]\mathrm{d}\mathcal{W},v_{\varepsilon,\eta})_{H^r},\\
R_2:=2\displaystyle\sum_{i\in\{1,3,4,5,6\}}(\mathcal{R}_i,v_{\varepsilon,\eta})_{H^r},\\
R_3:=\left\|\displaystyle\sum_{i=7}^8\mathcal{R}_i\right\|^2_{\mathcal{L}_2(\mathbb{U};H^r)}.
\end{cases}$$

Obviously, integration by parts and the fact that J_η is self-adjoint imply

$$(\mathcal{R}_2, v_{\varepsilon,\eta})_{H^r} = -(J_\eta^2 \Delta v_{\varepsilon,\eta}, v_{\varepsilon,\eta})_{H^r} = \|\nabla J_\eta v_{\varepsilon,\eta}\|_{H^r}^2 \geq 0,$$

Then (4.7) yields

$$\|v_{\varepsilon,\eta}(t)\|_{H^r}^2 \leq |R_1| + \int_0^t |R_2|\,\mathrm{d}t' + \int_0^t |R_3|\,\mathrm{d}t' \quad \mathbb{P}\text{-a.s.} \tag{4.8}$$

Applying the BDG inequality to (4.8) with noticing Assumption A_1, we derive

$$\begin{aligned}
&\mathbb{E} \sup_{t\in[0,\tau_{\varepsilon,\eta,K}^T]} \|v_{\varepsilon,\eta}(t)\|_{H^r}^2 \\
&\leq C\mathbb{E}\left(\int_0^{\tau_{\varepsilon,\eta,K}^T} \|v_{\varepsilon,\eta}\|_{H^r}^2 |R_3| \mathrm{d}t\right)^{\frac{1}{2}} + \sum_{i=2}^{3} \mathbb{E}\int_0^{\tau_{\varepsilon,\eta,K}^T} |R_i|\,\mathrm{d}t \\
&\leq \frac{1}{2}\mathbb{E} \sup_{t\in[0,\tau_{\varepsilon,\eta,K}^T]} \|v_{\varepsilon,\eta}\|_{H^r}^2 + C\mathbb{E}\int_0^{\tau_{\varepsilon,\eta,K}^T} |R_3|\,\mathrm{d}t + \mathbb{E}\int_0^{\tau_{\varepsilon,\eta,K}^T} |R_2|\,\mathrm{d}t.
\end{aligned}$$

By (4.3), the mean value theorem for $\theta_R(\cdot)$ and Assumption A_1, we find

$$\|\mathcal{R}_7\|_{\mathcal{L}_2(\mathbb{U};H^r)} \leq C\|v_{\varepsilon,\eta}\|_{H^r} f(\|u_\varepsilon\|_{H^s})(1 + \|u_\varepsilon\|_{H^s}).$$

and

$$\|\mathcal{R}_8\|_{\mathcal{L}_2(\mathbb{U};H^r)} \leq Cg(K)\|v_{\varepsilon,\eta}\|_{H^r}, \quad t \in [0, \tau_{\varepsilon,\eta,K}^T] \quad \mathbb{P}\text{-a.s.},$$

where $\mathcal{R}_7$ and $\mathcal{R}_8$ are given in (4.6) and g is given in Assumption A_1. Consequently, we can find a constant $C(K) > 0$ such that

$$\begin{aligned}
&\mathbb{E} \sup_{t\in[0,\tau_{\varepsilon,\eta,K}^T]} \|v_{\varepsilon,\eta}(t)\|_{H^r}^2 \\
&\leq C(K)\int_0^T \mathbb{E} \sup_{t'\in[0,\tau_{\varepsilon,\eta,K}^t]} \|v_{\varepsilon,\eta}(t')\|_{H^r}^2\,\mathrm{d}t + 2\mathbb{E}\int_0^{\tau_{\varepsilon,\eta,K}^T} |R_2|\,\mathrm{d}t.
\end{aligned} \tag{4.9}$$

Claim: There is an increasing function $\Psi : [0,\infty) \to (0,\infty)$ and a function $\varpi : (0,\infty)\times(0,\infty) \to (0,\infty)$ satisfying $\lim_{(x,y)\to 0} \varpi(x,y) = 0$ such that

$$|R_2| \lesssim \Psi(\|u_\varepsilon\|_{H^s} + \|u_\eta\|_{H^s})(\varpi(\varepsilon,\eta) + \|v_{\varepsilon,\eta}\|_{H^r}^2). \tag{4.10}$$

To show this, we first notice that $J_\varepsilon J_\eta = J_\eta J_\varepsilon$, which means $J_\varepsilon^2 - J_\eta^2 = (J_\varepsilon + J_\eta)(J_\varepsilon - J_\eta)$, and then we have,

$$\begin{aligned}
|(\mathcal{R}_1, v_{\varepsilon,\eta})_{H^r}| &\leq \|(J_\varepsilon^2 - J_\eta^2)u_\varepsilon\|_{H^{r+2}}\|v_{\varepsilon,\eta}\|_{H^r} \\
&\lesssim \max\{\varepsilon^{s-r-2}, \eta^{s-r-2}\}\|u_\varepsilon\|_{H^s}\|v_{\varepsilon,\eta}\|_{H^r} \\
&\lesssim \max\{\varepsilon^{2s-2r-4}, \eta^{2s-2r-4}\}\|u_\varepsilon\|_{H^s}^2 + \|v_{\varepsilon,\eta}\|_{H^r}^2.
\end{aligned}$$

We use the mean value theorem for $\theta_R(\cdot)$, (3.1) and Lemma 3.3 to derive

$$\begin{aligned}|(\mathcal{R}_3, v_{\varepsilon,\eta})_{H^r}| &\le |\chi|[\theta_R(\|u_\varepsilon\|_{H^r}) - \theta_R(\|u_\eta\|_{H^r})]\|J_\varepsilon F(J_\varepsilon u_\varepsilon)\|_{H^r}\|v_{\varepsilon,\eta}\|_{H^r}\\ &\le C|\chi|\|v_{\varepsilon,\eta}\|_{H^r}^2\|F(J_\varepsilon u_\varepsilon)\|_{H^r}\\ &\le C|\chi|\|v_{\varepsilon,\eta}\|_{H^r}^2\|u_\varepsilon\|_{H^r}(\|u_\varepsilon\|_{H^r} + \|u_\varepsilon\|_{H^{s-1}})\\ &\le C|\chi|\|v_{\varepsilon,\eta}\|_{H^r}^2\|u_\varepsilon\|_{H^s}^2.\end{aligned}$$

Similarly,

$$\begin{aligned}|(\mathcal{R}_4, v_{\varepsilon,\eta})_{H^r}| &\le |\chi|\|(J_\varepsilon - J_\eta)[F(J_\varepsilon u_\varepsilon)]\|_{H^r}\|v_{\varepsilon,\eta}\|_{H^r}\\ &\le C|\chi|\max\{\varepsilon,\eta\}\|u_\varepsilon\|_{H^s}^2\|v_{\varepsilon,\eta}\|_{H^r}\\ &\lesssim \max\{\varepsilon^2,\eta^2\}\|u_\varepsilon\|_{H^s}^4 + \|v_{\varepsilon,\eta}\|_{H^r}^2,\end{aligned}$$

$$\begin{aligned}&|(\mathcal{R}_5, v_{\varepsilon,\eta})_{H^r}|\\ &\le |\chi|\|F(J_\varepsilon u_\varepsilon) - F(J_\eta u_\varepsilon)\|_{H^r}\|v_{\varepsilon,\eta}\|_{H^r}\\ &\lesssim (\|u_\varepsilon\|_{H^{s-1}}\|J_\varepsilon u_\varepsilon - J_\eta u_\varepsilon\|_{H^r} + \|u_\varepsilon\|_{H^r}\|J_\varepsilon u_\varepsilon - J_\eta u_\varepsilon\|_{H^{s-1}})\|v_{\varepsilon,\eta}\|_{H^r}\\ &\lesssim \max\{\varepsilon,\eta\}\|u_\varepsilon\|_{H^s}^2\|v_{\varepsilon,\eta}\|_{H^r}\\ &\lesssim \max\{\varepsilon^2,\eta^2\}\|u_\varepsilon\|_{H^s}^4 + \|v_{\varepsilon,\eta}\|_{H^r}^2,\end{aligned}$$

and

$$\begin{aligned}|(\mathcal{R}_6, v_{\varepsilon,\eta})_{H^r}| &\le C|\chi||(F(J_\eta u_\varepsilon) - F(J_\eta u_\eta), J_\eta u_\varepsilon - J_\eta u_\eta)_{H^r}|\\ &\lesssim (\|u_\varepsilon\|_{H^{s-1}} + \|u_\eta\|_{H^r})\|v_{\varepsilon,\eta}\|_{H^r}^2.\end{aligned}$$

Summarizing the above estimates for $(\mathcal{R}_i, v_{\varepsilon,\eta})_{H^r}$ with $i = 1,3,4,5,6$, we obtain (4.10).

Combining (4.9) and (4.10), we find a function $\varpi : (0,\infty) \times (0,\infty) \to (0,\infty)$ such that $\lim_{(x,y)\to 0}\varpi(x,y) = 0$ and

$$\begin{aligned}&\mathbb{E}\sup_{t\in[0,\tau_{\varepsilon,\eta,K}^T]}\|v_{\varepsilon,\eta}(t)\|_{H^r}^2\\ &\le C(K,\chi)\int_0^T \mathbb{E}\sup_{t'\in[0,\tau_{\varepsilon,\eta,K}^t]}\|v_{\varepsilon,\eta}(t)\|_{H^r}^2\,\mathrm{d}t + C(K)T\varpi(\varepsilon,\eta),\end{aligned}$$

which, together with Grönwall's inequality, implies

$$\mathbb{E}\sup_{t\in[0,\tau_{\varepsilon,\eta,K}^T]}\|v_{\varepsilon,\eta}(t)\|_{H^r}^2 \le C(K,T,\chi)\varpi(\varepsilon,\eta),$$

and hence (4.4) holds. $\square$

We first let $\varepsilon = \varepsilon_n$ be countable set in Lemma 4.3 to guarantee that for all n, u_{ε_n} can be defined on the same set $\widetilde{\Omega}$ with $\mathbb{P}\{\widetilde{\Omega}\} = 1$. Then we can find a subsequence converging almost surely. Precisely, we have

Lemma 4.3. *Let $\chi \in \mathbb{R}\backslash\{0\}$, $s > \frac{d}{2}+3$, $r \in (\frac{d}{2}+1, s-2)$, $T > 0$ and $\{u_\varepsilon\}$ be given in Lemma 4.1. There is a countable subsequence of $\{u_\varepsilon\}$ (still denoted as $\{u_\varepsilon\}$), and an $\{\mathcal{F}_t\}_{t\geq 0}$ progressive measurable H^s-valued process u such that*

$$u_\varepsilon \xrightarrow{\varepsilon\to 0} u \quad \text{in } C([0,T];H^r) \quad \mathbb{P}\text{-a.s.}, \tag{4.11}$$

and for some $C = C(T) > 0$,

$$\mathbb{E}\left\{\sup_{t\in[0,T]} \|u(t)\|_{H^s}^2 + 2\int_0^T \|\nabla u(t)\|_{H^s}^2 \,\mathrm{d}t\right\} \leq C. \tag{4.12}$$

Proof. Recall (4.3). For any $\epsilon > 0$, by using Lemmas 4.1 and 4.2 and Chebyshev's inequality, we see that

$$\begin{aligned}
&\mathbb{P}\left\{\sup_{t\in[0,T]} \|u_\varepsilon - u_\eta\|_{H^r} > \epsilon\right\} \\
&= \mathbb{P}\left\{(\{\tau_{\varepsilon,\eta,K}^T < T\} \cup \{\tau_{\varepsilon,\eta,K}^T = T\}) \cap \left\{\sup_{t\in[0,T]} \|u_\varepsilon - u_\eta\|_{H^r} > \epsilon\right\}\right\} \\
&\leq \mathbb{P}\{\tau_{\varepsilon,K}^T < T\} + \mathbb{P}\{\tau_{\eta,K}^T < T\} + \mathbb{P}\left\{\sup_{t\in[0,\tau_{\varepsilon,\eta,K}^T]} \|u_\varepsilon - u_\eta\|_{H^r} > \epsilon\right\} \\
&\leq \frac{2C(R,T,u_0)}{K^2} + \mathbb{P}\left\{\sup_{t\in[0,\tau_{\varepsilon,\eta,K}^T]} \|u_\varepsilon - u_\eta\|_{H^r} > \epsilon\right\}.
\end{aligned}$$

Now (4.4) clearly forces

$$\limsup_{\varepsilon\to 0} \sup_{\eta\leq\varepsilon} \mathbb{P}\left\{\sup_{t\in[0,T]} \|u_\varepsilon - u_\eta\|_{H^r} > \epsilon\right\} \leq \frac{2C(R,T,u_0)}{K^2}, \quad K > 1.$$

Letting $K \to \infty$, we see that u_ε converges in probability in $C([0,T];H^r)$. Therefore, up to a further subsequence, (4.11) holds true. (4.12) comes from Lemma 4.1, (4.11) and Fatou's lemma. □

4.2. *Solving the cut-off problem*

With the help of Lemma 4.3, we can obtain the existence of a solution to (4.1).

Proposition 4.1. *Let $\chi \in \mathbb{R}\backslash\{0\}$, $s > \frac{d}{2}+3$ and $r \in (\frac{d}{2}+1, s-2)$. Fix a stochastic basis $\mathcal{S}$ and let $u_0 \in L^2(\Omega;H^s)$ be $\mathcal{F}_0$ measurable. Assume σ satisfies Assumption A_1. For any $R > 1$ and $T > 0$, (4.1) has a solution $u \in L^2(\Omega;C([0,T];H^s))$.*

Proof. Since for each $\varepsilon \in (0,1)$, u_ε is $\{\mathcal{F}_t\}_{t\ge 0}$ progressive measurable, so is u. By Lemma 4.3, we can send $\varepsilon \to 0$ in (4.2) to conclude that u solves (4.1). Due to (4.12), we only need to prove that $u \in C([0,T];H^s)$ almost surely. We first notice that Lemma 4.3 means $u \in C([0,T];H^r) \cap L^\infty(0,T;H^s)$ almost surely. Since H^s is dense in H^r, we know that (cf. [56, page 263, Lemma 1.4]) $u \in C_w([0,T];H^s)$, where $C_w([0,T];H^s)$ is the space of weakly continuous functions with values in H^s. Therefore, we only need to prove the continuity of $[0,T] \ni t \mapsto \|u(t)\|_{H^s}$.

In order to use Itô's formula in a Hilbert space, we recall the mollifier J_ε defined in Sec. 3, and then we consider Itô's formula for $\|J_\varepsilon u(t)\|^2_{H^s}$ rather than $\|u(t)\|^2_{H^s}$ (cf. Remark 3.1). Then we arrive at

$$\begin{aligned} \mathrm{d}\|J_\varepsilon u(t)\|^2_{H^s} + 2\|\nabla J_\varepsilon u\|_{H^s}\,\mathrm{d}t &= 2\theta_R(\|u\|_{H^r})(J_\varepsilon \sigma(t,u)\,\mathrm{d}\mathcal{W}, J_\varepsilon u)_{H^s} \\ &\quad - 2\chi\theta_R(\|u\|_{H^r})(J_\varepsilon F(u), J_\varepsilon u)_{H^s}\,\mathrm{d}t \\ &\quad + \theta_R^2(\|u\|_{H^r})\|J_\varepsilon \sigma(t,u)\|^2_{\mathcal{L}_2(\mathbb{U};H^s)}\,\mathrm{d}t. \end{aligned} \tag{4.13}$$

By (4.12),

$$\tau_N := \inf\left\{t \ge 0 : \|u(t)\|_{H^s} + 2\int_0^t \|\nabla u(t')\|^2_{H^s}\,\mathrm{d}t' > N\right\} \xrightarrow{N\to\infty} \infty \quad \mathbb{P}\text{-a.s.}$$

Then we only need to prove the continuity up to time $\tau_N^T := \tau_N \wedge T$ for each $N \ge 1$. For any $[t_2, t_1] \subset [0,T]$ with $t_1 - t_2 < 1$, we use Lemma 3.4, Assumption A_1 and the above estimate to find

$$\mathbb{E}\left[\left|H_\varepsilon\left(t_1 \wedge \tau_N^T\right) - H_\varepsilon\left(t_2 \wedge \tau_N^T\right)\right|^4\right] \le C(N,T)|t_1 - t_2|^2,$$

where $H_\varepsilon(t) := \|J_\varepsilon u(t)\|^2_{H^s} + 2\int_0^t \|J_\varepsilon \nabla u(t')\|^2_{H^s}\,\mathrm{d}t'$. Using Fatou's lemma, for $H(t) := \|u(t)\|^2_{H^s} + 2\int_0^t \|\nabla u(t')\|^2_{H^s}\,\mathrm{d}t'$, we arrive at

$$\mathbb{E}\left[\left|H\left(t_1 \wedge \tau_N^T\right) - H\left(t_2 \wedge \tau_N^T\right)\right|^4\right] \le C(N,T)|t_1 - t_2|^2.$$

This and Kolmogorov's continuity theorem ensure the continuity of $t \mapsto H(t)$. By (4.12), $t \mapsto 2\int_0^t \|\nabla u(t')\|^2_{H^s}\,\mathrm{d}t'$ is continuous almost surely, and hence

$$\|u(t)\|^2_{H^s} = H(t) - 2\int_0^t \|\nabla u(t')\|^2_{H^s}\,\mathrm{d}t' \in C([0,\tau_N^T];\mathbb{R}), \quad N \ge 1 \ \mathbb{P}\text{-a.s.}$$

We finish the proof. $\square$

4.3. *Solving the original problem*

Now we are going to prove (a) in Theorem 2.1. We first consider the uniqueness for the original problem (1.4) since we need similar estimate later.

Lemma 4.4. *Let $\chi \in \mathbb{R}\backslash\{0\}$, $s > \frac{d}{2} + 3$ and let Assumption A_1 be verified. Suppose that u_0 is an H^s-valued $\mathcal{F}_0$ measurable random variable satisfying*

$\mathbb{E}\|u_0\|_{H^s}^2 < \infty$. *If* (u_1, τ_1) *and* (u_2, τ_2) *are two local solutions to* (1.4) *such that* $\mathbb{P}\{u_1(0) = u_2(0) = u_0(x)\} = 1$, *then*

$$\mathbb{P}\{u_1(t,x) = u_2(t,x),\ (t,x) \in [0, \tau_1 \wedge \tau_2] \times \mathbb{R}^d\} = 1.$$

Proof. We first define the stopping time

$$\tau_K^T := \tau_K \wedge T, \quad \tau_K := \inf\{t \geq 0 : \|u_1(t)\|_{H^s} + \|u_2(t)\|_{H^s} > K\}, \quad T > 0. \tag{4.14}$$

Let $w = u_1 - u_2$. We have

$$\mathrm{d}w + [-\Delta w + \chi(F(u_1) - F(u_2))]\mathrm{d}t = [\sigma(t,u_1) - \sigma(t,u_2)]\mathrm{d}\mathcal{W}.$$

Then we use Itô's formula and neglect the positive term $\int_0^t \|\nabla w\|_{L^2}^2\,\mathrm{d}t'$ on the left hand side of the resulting equation to find that

$$\begin{aligned}
\|w(t)\|_{L^2}^2 \leq\ & 2\left|\left(\int_0^t (\sigma(t,u_1) - \sigma(t,u_2))\,\mathrm{d}\mathcal{W}, w\right)_{L^2}\right| \\
& + 2|\chi| \int_0^t |(F(u_1) - F(u_2), w)_{L^2}|\mathrm{d}t' \\
& + \int_0^t \|\sigma(t',u_1) - \sigma(t',u_2)\|_{\mathcal{L}_2(\mathbb{U};L^2)}^2\,\mathrm{d}t' \\
:=\ & B_1 + \int_0^t B_2\,\mathrm{d}t' + \int_0^t B_3\,\mathrm{d}t'.
\end{aligned}$$

Taking a supremum over $t \in [0, \tau_K^T]$ and using Assumption A_1, the BDG inequality, (4.14) and the Cauchy–Schwarz inequality yield a constant $C = C(K) > 0$ such that

$$\begin{aligned}
& \mathbb{E} \sup_{t\in[0,\tau_K^T]} \|w(t)\|_{L^2}^2 \\
& \leq C\mathbb{E}\left(\int_0^{\tau_K^T} \|\sigma(t,u_1) - \sigma(t,u_2)\|_{\mathcal{L}_2(\mathbb{U};L^2)}^2 \|w\|_{L^2}^2\,\mathrm{d}t\right)^{\frac{1}{2}} + \sum_{i=2}^3 \mathbb{E}\int_0^{\tau_K^T} |B_i|\mathrm{d}t \\
& \leq Cg^2(K)\mathbb{E}\left(\sup_{t\in[0,\tau_K^T]} \|w\|_{L^2}^2 \cdot \int_0^{\tau_K^T} \|w\|_{L^2}^2\,\mathrm{d}t\right)^{\frac{1}{2}} + \sum_{i=2}^3 \mathbb{E}\int_0^{\tau_K^T} |B_i|\mathrm{d}t \\
& \leq \frac{1}{2}\mathbb{E}\sup_{t\in[0,\tau_K^T]} \|w\|_{L^2}^2 + C\int_0^T \mathbb{E}\sup_{t'\in[0,\tau_K^t]} \|w(t')\|_{L^2}^2\,\mathrm{d}t + \sum_{i=2}^3 \int_0^T \mathbb{E}\sup_{t'\in[0,\tau_K^t]} |B_i(t')|\mathrm{d}t.
\end{aligned}$$

Using Lemma 3.3, Assumption A_1 and (4.14), we get

$$\sum_{i=2}^3 \mathbb{E}\int_0^{\tau_K^T} B_i\,\mathrm{d}t \leq C\int_0^T \mathbb{E}\sup_{t'\in[0,\tau_K^t]} \|w(t')\|_{L^2}^2\,\mathrm{d}t, \quad C = C(K,\chi).$$

As a result, we find that for some $C = C(K,\chi)$,

$$\mathbb{E} \sup_{t\in[0,\tau_k^T]} \|w(t)\|_{L^2}^2 \le C \int_0^T \mathbb{E} \sup_{t'\in[0,\tau_K^t]} \|w(t')\|_{L^2}^2 \,\mathrm{d}t.$$

Hence $\mathbb{E}\sup_{t\in[0,\tau_K^T]} \|w(t)\|_{L^2}^2 = 0$. For $i = 1,2$, since (u_i,τ_i) are local solutions, we have

$$\mathbb{P}\left\{\liminf_{K\to\infty} \tau_K^T > \tau_1 \wedge \tau_2 \wedge T\right\} = 1.$$

Therefore, sending $K, T \to \infty$ and using the monotone convergence theorem yield $\mathbb{E}\sup_{t\in[0,\tau_1\wedge\tau_2]} \|w(t)\|_{L^2}^2 = 0$, which is the desired uniqueness. □

According to Proposition 4.1 and Lemma 4.4, to prove (a) in Theorem 2.1, we only need to remove the cut-off function. The method used here is inspired by the works [30, 41].

Proof of (a) in Theorem 2.1. For $u_0(\omega,x) \in L^2(\Omega; H^s)$ with $s > \frac{d}{2} + 3$, we consider

$$\Omega_k := \{k - 1 \le \|u_0\|_{H^s} < k\}, \quad k \ge 1.$$

Since $\mathbb{E}\|u_0\|_{H^s}^2 < \infty$, we have

$$u_0(\omega,x) = \sum_{k\ge1} u_{0,k}(\omega,x) := \sum_{k\ge1} u_0(\omega,x)\mathbf{1}_{k-1\le\|u_0\|_{H^s}<k} \quad \mathbb{P}\text{-a.s.}$$

On account of Proposition 4.1, we let $u_{k,R}$ be the global strong solution to the cut-off problem (4.1) with initial value $u_{0,k}$ and cut-off function $\theta_R(\cdot)$. Define

$$\tau_{k,R} := \inf\left\{t > 0 : \sup_{t'\in[0,t]} \|u_{k,R}(t')\|_{H^s}^2 > \|u_{0,k}\|_{H^s}^2 + 2\right\}. \tag{4.15}$$

Then for any $R > 0$ and $k \in \mathbb{N}$, we have $\mathbb{P}\{\tau_{k,R} > 0\} = 1$. Assign $R = R_k > \sqrt{k^2+2}$ to be discrete for each $k \ge 1$ and then denote $(u_k,\tau_k) = (u_{k,R_k},\tau_{k,R_k})$. Obviously, $\mathbb{P}\{\tau_k > 0,\ \forall k \ge 1\} = 1$. Then it follows from the inequality $\|\cdot\|_{H^r} \le \|\cdot\|_{H^s}$ that

$$\mathbb{P}\big\{\|u_k\|_{H^r}^2 \le \|u_k\|_{H^s}^2 \le \|u_{0,k}\|_{H^s}^2 + 2 < R_k^2,\ t \in [0,\tau_k],\ k \ge 1\big\} = 1.$$

From this and the definition of $\theta_R(\cdot)$, we see that (u_k,τ_k) is the strong solution to (1.4) with initial value $u_{0,k}$. As a result, we find that

$$\begin{aligned}
&\mathbf{1}_{\Omega_k} u_k(t\wedge\tau_k) - \mathbf{1}_{\Omega_k} u_{0,k} \\
&= \int_0^{t\wedge\mathbf{1}_{\Omega_k}\tau_k} \mathbf{1}_{\Omega_k}[\Delta u_k - \chi F(u_k)]\mathrm{d}t' + \int_0^{t\wedge\mathbf{1}_{\Omega_k}\tau_k} \mathbf{1}_{\Omega_k}\sigma(t,u_k)\mathrm{d}\mathcal{W}.
\end{aligned}$$

Besides, since $\mathbf{1}_{\Omega_k}\sigma(t,u_k) = \sigma(t,\mathbf{1}_{\Omega_k}u_k) - \mathbf{1}_{\Omega_k^C}\sigma(t,0)$, and $\|\sigma(t,0)\|_{\mathcal{L}_2(\mathbb{U};H^s)} < \infty$ (cf. Assumption A_1), we have

$$\int_0^{t\wedge\mathbf{1}_{\Omega_k}\tau_k} \mathbf{1}_{\Omega_k}\sigma(t,u_k)\mathrm{d}\mathcal{W} = \int_0^{t\wedge\mathbf{1}_{\Omega_k}\tau_k} \sigma(t,\mathbf{1}_{\Omega_k}u_k)\mathrm{d}\mathcal{W}.$$

Similarly, $\mathbf{1}_{\Omega_k}[\Delta u_k - \chi F(u_k)] = [\Delta(\mathbf{1}_{\Omega_k} u_k) - \chi F(\mathbf{1}_{\Omega_k} u_k)]$, and hence

$$\begin{aligned}&\mathbf{1}_{\Omega_k} u_k(t\wedge\tau_k) - \mathbf{1}_{\Omega_k} u_{0,k}\\ &= \int_0^{t\wedge\mathbf{1}_{\Omega_k}\tau_k} [\Delta(\mathbf{1}_{\Omega_k} u_k) - \chi F(\mathbf{1}_{\Omega_k} u_k)]\mathrm{d}t' + \int_0^{t\wedge\mathbf{1}_{\Omega_k}\tau_k} \sigma(t, \mathbf{1}_{\Omega_k} u_k)\mathrm{d}\mathcal{W},\end{aligned}$$

which implies that $(\mathbf{1}_{\Omega_k} u_k, \mathbf{1}_{\Omega_k}\tau_k)$ is a solution to (1.4) with initial data $u_{0,k}$. Since $\Omega_k \bigcap \Omega_{k'} = \emptyset$ for $k \neq k'$ and $\bigcup_{k\geq 1}\Omega_k$ is a set of full measure, we see that

$$\left(u = \sum_{k\geq 1} \mathbf{1}_{\Omega_k} u_k,\ \tau = \sum_{k\geq 1} \mathbf{1}_{\Omega_k}\tau_k \right)$$

is a strong solution to (1.4) corresponding to the initial condition u_0. Besides, using (4.15), we have

$$\sup_{t\in[0,\tau]} \|u\|_{H^s}^2 = \sum_{k\geq 1} \mathbf{1}_{\Omega_k} \sup_{t\in[0,\tau_k]} \|u_k\|_{H^s}^2 \leq 2\|u_0\|_{H^s}^2 + 4.$$

Taking expectation gives rise to (2.1). Uniqueness comes from Lemma 4.4. The extension from a local solution to the maximal solution can be obtained as in [28, 8], here we omit the details. □

4.4. *Generalized blow-up criterion*

Now we prove (b) in Theorem 2.1. The key part in the proof is to analyze the explosion time of $\|u(t)\|_{H^s}$ and that of $\|u(t)\|_{H^r}$. It is motivated by the recent work for the stochastic Euler equation [17], see also [2, 54].

Proof of (b) in Theorem 2.1. Let (u, τ) be the strong solution to (1.4) guaranteed by (a) in Theorem 2.1. Recall that $\gamma \in (\frac{d}{2}+1, s]$. Define

$$\tau_1 := \lim_{m\to\infty} \tau_{1,m}, \quad \tau_{1,m} := \inf\{t \geq 0 : \|u(t)\|_{H^s} \geq m\},$$

and

$$\tau_2 := \lim_{n\to\infty} \tau_{2,n}, \quad \tau_{2,n} := \inf\{t \geq 0 : \|u(t)\|_{H^\gamma} \geq n\}.$$

To prove (b) in Theorem 2.1, we only need to show

$$\tau_1 = \tau_2 \ \mathbb{P}\text{-a.s.} \tag{4.16}$$

Since $H^s \hookrightarrow H^\gamma$, it is easy to see $\tau_{1,m} \leq \tau_{2,m} \leq \tau_2$ $\mathbb{P}$-a.s. Therefore, we have $\tau_1 \leq \tau_2$ $\mathbb{P}$-a.s. Now we prove the converse inequality. We first notice that for all $n, k \geq 1$,

$$\left\{ \sup_{t\in[0,\tau_{2,n}\wedge k]} \|u(t)\|_{H^s} < \infty \right\} \subset \bigcup_{m\geq 1} \{\tau_{2,n}\wedge k \leq \tau_{1,m}\} \subset \{\tau_{2,n}\wedge k \leq \tau_1\}.$$

If we can show

$$\mathbb{P}\left\{ \sup_{t\in[0,\tau_{2,n}\wedge k]} \|u(t)\|_{H^s} < \infty \right\} = 1, \quad n,\ \ k \geq 1, \tag{4.17}$$

then for all $n,k\geq 1$, $\mathbb{P}\{\tau_{2,n}\wedge k\leq\tau_1\}=1$ and

$$\mathbb{P}\{\tau_2\leq\tau_1\}=\mathbb{P}\left\{\bigcap_{n\geq 1}\{\tau_{2,n}\leq\tau_1\}\right\}=\mathbb{P}\left\{\bigcap_{n,k\geq 1}\{\tau_{2,n}\wedge k\leq\tau_1\}\right\}=1. \tag{4.18}$$

Since (4.18) requires the assumption (4.17), the proof is completed by proving (4.17). As is mentioned in Proposition 4.1 (see also Remark 3.1), we will first consider Itô's formula for $\|J_\varepsilon u\|^2_{H^s}$ instead of $\|u\|^2_{H^s}$. Then for any $t>0$, we have

$$\|J_\varepsilon u(t)\|^2_{H^s}-\|J_\varepsilon u(0)\|^2_{H^s}+2\int_0^t\|\nabla J_\varepsilon u(t')\|^2_{H^s}\,\mathrm{d}t'=L_1+\sum_{j=2}^3\int_0^t L_j\,\mathrm{d}t',$$

where

$$\begin{aligned}
L_1&:=2\left(\int_0^t J_\varepsilon\sigma(t',u)\,\mathrm{d}\mathcal{W},J_\varepsilon u\right)_{H^s},\\
L_2&:=-2\chi\big(J_\varepsilon F(u),J_\varepsilon u\big)_{H^s},\\
L_3&:=\|J_\varepsilon\sigma(t',u)\|^2_{\mathcal{L}_2(\mathbb{U};H^s)}.
\end{aligned}$$

It follows from the properties of J_ε, the BDG inequality, $H^\gamma\hookrightarrow W^{1,\infty}$ and Assumption A_1 that

$$\mathbb{E}\left(\sup_{t\in[0,\tau_{2,n}\wedge k]}|L_1(t)|\right)\leq\frac{1}{2}\mathbb{E}\sup_{t\in[0,\tau_{2,n}\wedge k]}\|J_\varepsilon u\|^2_{H^s}+Cf^2(n)\int_0^k(1+\mathbb{E}\|u\|^2_{H^s})\mathrm{d}t.$$

We use Lemma 3.4 to find

$$\mathbb{E}\int_0^{\tau_{2,n}\wedge k}|L_2|\mathrm{d}t\leq C|\chi|\mathbb{E}\int_0^{\tau_{2,n}\wedge k}\|u\|_{H^\gamma}\|u\|^2_{H^s}\,\mathrm{d}t\leq C|\chi|n\int_0^k\mathbb{E}\|u\|^2_{H^s}\,\mathrm{d}t$$

Similarly, Assumption A_1 yields

$$\mathbb{E}\int_0^{\tau_{2,n}\wedge k}|L_3|\mathrm{d}t\leq Cf^2(n)\int_0^k(1+\mathbb{E}\|u\|^2_{H^s})\mathrm{d}t.$$

Therefore, we combine the above estimates to find

$$\begin{aligned}
&\mathbb{E}\sup_{t\in[0,\tau_{2,n}\wedge k]}\|J_\varepsilon u(t)\|^2_{H^s}\\
&\leq 2\mathbb{E}\|u_0\|^2_{H^s}+C\int_0^k\left(1+\mathbb{E}\sup_{t'\in[0,t\wedge\tau_{2,n}]}\|u(t')\|^2_{H^s}\right)\mathrm{d}t,
\end{aligned}$$

where C depends on χ and n. Notice that the right hand side of the above estimate does not depend on ε and J_ε tends to u for all $u\in C([0,T];H^s)$ with any $T>0$.

We send $\varepsilon \to 0$ to find

$$\begin{aligned}&\mathbb{E}\sup_{t\in[0,\tau_{2,n}\wedge k]}\|u(t)\|_{H^s}^2\\&\le 2\mathbb{E}\|u_0\|_{H^s}^2 + C\int_0^k\left(1+\mathbb{E}\sup_{t'\in[0,t\wedge\tau_{2,n}]}\|u(t')\|_{H^s}^2\right)\mathrm{d}t.\end{aligned}\tag{4.19}$$

Therefore, (4.17) comes from the above inequality and the Grönwall inequality. The proof is completed. □

4.5. *Non-negativity of solutions*

Proof of (c) in Theorem 2.1. Let

$$\tau_N := \inf\{t\ge 0 : \|u\|_{H^s}\ge N\},\quad N\ge 1.$$

Then (a) in Theorem 2.1 implies $\lim_{N\to\infty}\tau_N = \tau^*$ $\mathbb{P}$-a.s. Hence it suffices to prove

$$\mathbb{P}\{u(t\wedge\tau_N)\ge 0,\ t\ge 0\} = 1.\quad N\ge 1.\tag{4.20}$$

We recall the definitions of $\kappa, \kappa_\varepsilon$ and ρ in Lemma 3.5 and then define the nonlinear functional:

$$\Phi_\varepsilon : L^2 \ni f \mapsto \Phi_\varepsilon(f)\in\mathbb{R},\quad \Phi_\varepsilon(f(x)) := \int_{\mathbb{R}^d}\kappa_\varepsilon(f(x))\mathrm{d}x.$$

Using Itô's formula (see Theorem 2.10 in [25]) for $\Phi_\varepsilon(u)$ yields that for all $t\ge 0$,

$$\begin{aligned}&\Phi_\varepsilon(u(t\wedge\tau_N))\\&=\Phi_\varepsilon(u(0)) + \int_0^t\left(\kappa_\varepsilon'(u),\Delta u\right)_{L^2}\mathrm{d}t' - \chi\int_0^t\left(\kappa_\varepsilon'(u),F(u)\right)_{L^2}\mathrm{d}t'\\&\quad+\left(\int_0^t\sigma(t',u)\,\mathrm{d}\mathcal{W},\kappa_\varepsilon'(u)\right)_{L^2} + \frac{1}{2}\int_0^t\sum_{j=1}^\infty\left(\kappa_\varepsilon''(u)\left(\sigma(t,u)[\sigma(t,u)]^*y_j\right),y_j\right)_{L^2}\mathrm{d}t',\end{aligned}$$

where $\{y_j\}$ is a complete orthonormal basis of L^2. Then we arrive at

$$\begin{aligned}&\mathbb{E}\Phi_\varepsilon(u(t\wedge\tau_N)) - \mathbb{E}\Phi_\varepsilon(u(0))\\&=-\mathbb{E}\int_0^{t\wedge\tau_N}\left(\kappa_\varepsilon''(u),|\nabla u|^2\right)_{L^2}\mathrm{d}t' - \chi\mathbb{E}\int_0^{t\wedge\tau_N}\left(\kappa_\varepsilon'(u),F(u)\right)_{L^2}\mathrm{d}t'\\&\quad+\frac{1}{2}\mathbb{E}\int_0^{t\wedge\tau_N}\sum_{j=1}^\infty\left(\kappa_\varepsilon''(u)\left(\sigma(t,u)[\sigma(t,u)]^*y_j\right),y_j\right)_{L^2}\mathrm{d}t'.\end{aligned}$$

We notice that $\nabla(\mathbf{1}_{\{u<0\}}u) = \nabla u$ when $u<0$ (see Lemma 7.6 in [27]) and Assumption A_2 implies $\sigma(t,0) = 0$ (hence $\sqrt{a(u)}\sigma(t,u) = \sigma(t,-\rho(u))$ with $a(\cdot)$ being given in Lemma 3.5). Thanks to Lemma 3.5, one can send $\varepsilon \to 0$ to find (see also [14, Proposition 3.4])

$$\begin{aligned}
&\mathbb{E}\|\rho(u(t\wedge\tau_N))\|_{L^2}^2 - \mathbb{E}\|\rho(u_0)\|_{L^2}^2 \\
&= -2\mathbb{E}\int_0^{t\wedge\tau_N} \left(a(u), |\nabla u|^2\right)_{L^2} \mathrm{d}t' + 2\chi\mathbb{E}\int_0^{t\wedge\tau_N} (\rho(u), F(u))_{L^2} \mathrm{d}t' \\
&\quad + \mathbb{E}\int_0^{t\wedge\tau_N} \sum_{j=1}^{\infty} \left(a(u)\sigma(t,u)[\sigma(t,u)]^* y_j, y_j\right)_{L^2} \mathrm{d}t' \\
&= -2\mathbb{E}\int_0^{t\wedge\tau_N} \|\nabla(-\rho(u))\|_{L^2}^2 \mathrm{d}t' + 2\chi\mathbb{E}\int_0^{t\wedge\tau_N} (\rho(u), F(u))_{L^2} \mathrm{d}t' \\
&\quad + \mathbb{E}\int_0^{t\wedge\tau_N} \|\sigma(t,-\rho(u))\|_{\mathcal{L}_2(\mathbb{U};L^2)}^2 \mathrm{d}t'.
\end{aligned}$$

It is easily seen from (1.3) that

$$\|F(u)\|_{L^2} \le \|Q(u)\|_{L^2}\|\nabla u\|_{L^\infty} + \|u\|_{L^2}\|\mathrm{div}Q(u)\|_{L^\infty}.$$

Hence we use the fact $\nabla(\mathbf{1}_{\{u<0\}}u) = \nabla u$ for $u<0$ again and the above estimate to obtain

$$|(\rho(u), F(u))_{L^2}| = |(\mathbf{1}_{\{u<0\}}u, F(u))_{L^2}| \le CN\|\rho(u)\|_{L^2}^2, \quad t' \in [0, t\wedge\tau_N], \quad N \ge 1.$$

Combining Assumption A_2 and the above observation, we have

$$\mathbb{E}\|\rho(u(t\wedge\tau_N))\|_{L^2}^2 \le C(\chi, N)\int_0^t \mathbb{E}\|\rho(u(t'\wedge\tau_N))\|_{L^2}^2 \mathrm{d}t', \quad N \ge 1,$$

which implies that for all $N \ge 1$,

$$\mathbb{E}\|\rho(u(t\wedge\tau_N))\|_{L^2}^2 = 0, \quad t \in [0,\infty).$$

Hence (4.20) holds true. The proof is completed. □

5. Regularization Effect of Multiplicative Noise

5.1. *Non-autonomous linear case*

Now we consider (1.5). This case covers the linear noise case $\beta u\,\mathrm{d}W$ in [39, 28]. We introduce the following Girsanov type transform:

$$z(\omega,t) := \frac{1}{\varphi(\omega,t)}u(\omega,t), \quad \varphi(\omega,t) \quad \text{is given in (2.3).} \tag{5.1}$$

Then we have the following results for z:

Proposition 5.1. *Let $s > \frac{d}{2}+3$ and Assumption A_3 be verified. If $u_0(\omega,x)$ is an H^s-valued $\mathcal{F}_0$ measurable random variable with $\mathbb{E}\|u_0\|_{H^s}^2 < \infty$, then* (1.5) *admits* *a unique maximal solution* (u, τ^*). *For* $t \in [0, \tau^*)$, z *defined by* (5.1) *solves the following problem almost surely,*

$$z_t - \Delta z + \chi \varphi F(z) = 0, \quad z(\omega, 0, x) = u_0(\omega, x), \quad x \in \mathbb{R}^d, \ t > 0, \tag{5.2}$$

where $F(\cdot)$ *is given by* (1.3). *Moreover,* $z \in C([0, \tau^*); H^s) \cap C^1([0, \tau^*); H^{s-2})\mathbb{P}$*-a.s. Furthermore, if* $u_0 \geq 0$ $\mathbb{P}$*-a.s., then* $z \geq 0$ $\mathbb{P}$*-a.s. and* $\|z\|_{L^1} = \|u_0\|_{L^1}$ $\mathbb{P}$*-a.s.*

Proof. Since $\beta(t)$ satisfies Assumption A_3, $\sigma(t, u) = \beta(t)u$ satisfies Assumptions A_1 and A_2. Consequently, (a) in Theorem 2.1 implies that (1.5) has a unique maximal solution (u, τ^*). Direct computation with Itô's formula yields $\mathrm{d}\frac{1}{\varphi} = -\beta(t)\frac{1}{\varphi}\,\mathrm{d}W + \beta^2(t)\frac{1}{\varphi}\,\mathrm{d}t$. Then (5.1) and the fact $z(0) = u_0$ imply (5.2). The regularity of u (and hence the regularity of z) comes from (a) in Theorem 2.1. If $u_0 \geq 0$ $\mathbb{P}$-a.s., then (c) in Theorem 2.1 gives that $z \geq 0$ $\mathbb{P}$-a.s. and the conservation of $\|z\|_{L^1}$ comes from (5.2). $\square$

5.1.1. *Decay of* L^∞*-norm in* $\mathbb{R}^2$ *almost surely*

Proof of Theorem 2.2. Keep in mind that $s > 4$, $z(0) = u_0 \in H^s \hookrightarrow C^2$ and $u_0 \geq 0$ $\mathbb{P}$-a.s. Then, for a.e. $\omega \in \Omega$, by following the proof of [52, Theorem 1.1] (see also [40]) with noticing that $\mathcal{A} = \mathcal{A}(\omega) = \sup_{t>0} \varphi(\omega, t) < \infty$ $\mathbb{P}$-a.s., one has that

$$z \in C([0, \widetilde{\tau}^*); C_{\mathrm{B,U}}(\mathbb{R}^2)), \quad z(t) \geq 0, \quad t \in [0, \widetilde{\tau}^*) \ \ \mathbb{P}\text{-a.s.},$$

where

$$\widetilde{\tau}^* := \lim_{n \to \infty} \widetilde{\tau}_n, \quad \widetilde{\tau}_n := \inf\{t \geq 0 : \|z\|_{L^\infty} \geq n\}, \tag{5.3}$$

and $C_{\mathrm{B,U}}$ is the space of bounded uniformly continuous functions (see [52, (1.5)]). Besides, we notice that z and $b = (I - \Delta)^{-1} z$ satisfy

$$\begin{cases} z_t - \Delta z + \chi \varphi \mathrm{div}(z \nabla b) = 0, & x \in \mathbb{R}^2, \quad t > 0, \\ -\Delta b = z - b, & x \in \mathbb{R}^2, \quad t > 0, \\ z(\omega, 0, x) = u_0(\omega, x), \end{cases} \tag{5.4}$$

which means that $b \geq 0$ on $[0, \widetilde{\tau}^*)$ $\mathbb{P}$-a.s. Even though $\tau^* < \infty$ may occur (the H^s-norm of u may blow up), we will show that $\widetilde{\tau}^* = \infty$ $\mathbb{P}$-a.s.

Step 1: The estimate of $\|z\|_{L^p}$ for $1 < p < \infty$. We multiply the first equation of (5.4) by z^{p-1} $(1 < p < \infty)$ with noticing $z, b \geq 0$ to find that

$$\begin{aligned}
&\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} z^p \,\mathrm{d}x \\
&= \int_{\mathbb{R}^d} z^{p-1}\Delta z \,\mathrm{d}x - \chi\varphi(t)\int_{\mathbb{R}^d} z^{p-1}\mathrm{div}(z\nabla b)\,\mathrm{d}x \\
&= -(p-1)\int_{\mathbb{R}^d} z^{p-2}|\nabla z|^2 \,\mathrm{d}x + \chi(p-1)\varphi(t)\int_{\mathbb{R}^d}(z^{p-1}\nabla z\cdot\nabla b)\,\mathrm{d}x \\
&= -\frac{4(p-1)}{p^2}\int_{\mathbb{R}^d}|\nabla z^{\frac{p}{2}}|^2\,\mathrm{d}x + \chi\frac{p-1}{p}\varphi(t)\int_{\mathbb{R}^d} z^p(z-b)\,\mathrm{d}x \\
&\leq -\frac{4(p-1)}{p^2}\int_{\mathbb{R}^d}|\nabla z^{\frac{p}{2}}|^2\,\mathrm{d}x + \chi\frac{p-1}{p}\varphi(t)\int_{\mathbb{R}^d} z^{p+1}\,\mathrm{d}x, \quad t\in[0,\widetilde{\tau}^*)\ \mathbb{P}\text{-a.s.} \qquad (5.5)
\end{aligned}$$

When $d = 2$, we use the Gagliardo-Nirenberg-Sobolev inequality to find that

$$\begin{aligned}
\varphi(t)\int_{\mathbb{R}^2} z^{p+1}\,\mathrm{d}x &= \varphi(t)\|z^{\frac{p+1}{2}}\|^2_{L^2(\mathbb{R}^2)} \\
&\leq C\varphi(t)\|\nabla z^{\frac{p+1}{2}}\|^2_{L^1(\mathbb{R}^2)} \\
&= C\varphi(t)\left(\frac{p+1}{p}\right)^2\left(\int_{\mathbb{R}^2} z^{\frac{1}{2}}|\nabla z^{p/2}|\mathrm{d}x\right)^2 \\
&\leq \|\nabla z^{p/2}\|^2_{L^2(\mathbb{R}^2)}C\left(\frac{p+1}{p}\right)^2\mathcal{A}\|u_0\|_{L^1(\mathbb{R}^2)}\ \mathbb{P}\text{-a.s.}
\end{aligned}$$

Using the above estimate and (5.5), one has, almost surely, for $t \in [0, \widetilde{\tau}^*)$ and $1 < p < \infty$ that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2} z^p\,\mathrm{d}x \leq (p-1)\|\nabla z^{p/2}\|^2_{L^2}\left[\frac{-4}{p} + \chi C\left(\frac{p+1}{p}\right)^2\mathcal{A}\|u_0\|_{L^1}\right]. \qquad (5.6)$$

If (2.4) holds true, then

$$\mathbb{P}\left\{\|u_0\|_{L^1} \leq \frac{4}{\chi C p\mathcal{A}}\left(\frac{p}{p+1}\right)^2,\ 2\leq p\leq 4\right\} = 1,$$

which together with (5.6) gives that

$$\mathbb{P}\left\{\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2} z^p\,\mathrm{d}x \leq 0,\ 2\leq p\leq 4,\ t\in[0,\widetilde{\tau}^*)\right\} = 1.$$

That is to say,

$$\mathbb{P}\{\|z\|_{L^p} \leq \|u_0\|_{L^p},\ 2\leq p\leq 4,\ t\in[0,\widetilde{\tau}^*)\} = 1. \qquad (5.7)$$

For a.e. $\omega \in \Omega$ and for $1 < p < \infty$, using (5.7), the Gagliardo-Nirenberg interpolation inequality and Young's inequality, we have

$$\begin{aligned}\int_{\mathbb{R}^2} z^{p+1}\,\mathrm{d}x &\leq \|z\|_{L^2}\|z^{\frac{p}{2}}\|_{L^4}^2 \\ &\leq C\|u_0\|_{L^2}\|z^{\frac{p}{2}}\|_{L^2}\|\nabla z^{\frac{p}{2}}\|_{L^2} \\ &\leq \frac{C^2\|u_0\|_{L^2}^2 p\chi\mathcal{A}}{8}\|z\|_{L^p}^p + \frac{2}{p\chi\mathcal{A}}\|\nabla z^{\frac{p}{2}}\|_{L^2}^2.\end{aligned}$$

Combining the above estimate and (5.5), we have almost surely that for $t \in [0, \widetilde{\tau}^*)$ and $1 < p < \infty$,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2} z^p\,\mathrm{d}x \\ &\leq -\frac{4(p-1)}{p}\int_{\mathbb{R}^d}|\nabla z^{\frac{p}{2}}|^2\,\mathrm{d}x + \chi(p-1)\varphi(t)\int_{\mathbb{R}^2} z^{p+1}\,\mathrm{d}x \\ &\leq -\frac{2(p-1)}{p}\int_{\mathbb{R}^2}|\nabla z^{\frac{p}{2}}|^2\,\mathrm{d}x + C^2\|u_0\|_{L^2}^2\chi^2\mathcal{A}^2 p(p-1)\|z\|_{L^p}^p.\end{aligned}$$

Let $M_1(\omega) = C^2\|u_0\|_{L^2}^2\chi^2\mathcal{A}^2$. As $\mathcal{A}(\omega) < \infty$ almost surely and $\mathbb{E}\|u_0\|_{H^s}^2 < \infty$, we have $0 < M_1(\omega) < \infty$ almost surely. Then for a.e. $\omega \in \Omega$, $t \in [0, \widetilde{\tau}^*)$ and $1 < p < \infty$, one derives

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2} z^p\,\mathrm{d}x + \frac{2(p-1)}{p}\int_{\mathbb{R}^2}|\nabla z^{\frac{p}{2}}|^2\,\mathrm{d}x \leq M_1 p(p-1)\int_{\mathbb{R}^2} z^p\,\mathrm{d}x. \tag{5.8}$$

On account of the Grönwall inequality and the above inequality, we see that for any $1 < p < \infty$,

$$\int_{\mathbb{R}^2} z^p\,\mathrm{d}x \leq \|u_0\|_{L^p}^p \mathrm{e}^{M_1 p(p-1)t}, \quad t \in [0, \widetilde{\tau}^*)\ \mathbb{P}\text{-a.s.}$$

However, this L^p bound depends on t. Now we need to establish a uniform-in-time L^∞ estimate for z.

Step 2: Estimating $\|z\|_{L^\infty}$: the Moser-Alikakos iteration. We will extend the well-known Moser-Alikakos iteration technique (see [1]) to this non-autonomous random system to obtain the (pathwise) uniform-in-time L^∞ estimate for z.

To this end, we let

$$a_k := 2^k, \quad k \geq 1.$$

Then, almost surely, we can infer from (5.8) that for all $t \in [0, \widetilde{\tau}^*)$ and $k \geq 1$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2} z^{a_k}\,\mathrm{d}x + 2\frac{a_k - 1}{a_k}\int_{\mathbb{R}^2}|\nabla z^{a_{k-1}}|^2\,\mathrm{d}x \leq M_1 a_k^2\int_{\mathbb{R}^2} z^{a_k}\,\mathrm{d}x. \tag{5.9}$$

Using the Gagliardo-Nirenberg interpolation inequality and Young's inequality, we arrive at

$$\int_{\mathbb{R}^2} z^{a_k}\,\mathrm{d}x = \|z^{a_{k-1}}\|_{L^2}^2 \le C\|z^{a_{k-1}}\|_{L^1}\|\nabla z^{a_{k-1}}\|_{L^2}$$
$$\le C\left(\frac{1}{4\varepsilon}\|z^{a_{k-1}}\|_{L^1}^2 + \varepsilon\|\nabla z^{a_{k-1}}\|_{L^2}^2\right), \quad \varepsilon > 0.$$

Therefore, we add both sides of (5.9) by $\varepsilon \int_{\mathbb{R}^2} z^{a_k}\,\mathrm{d}x$ and then use the above estimate to find that for $k \ge 1$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2} z^{a_k}\,\mathrm{d}x + 2\frac{a_k-1}{a_k}\int_{\mathbb{R}^2}|\nabla z^{a_{k-1}}|^2\,\mathrm{d}x + \varepsilon\int_{\mathbb{R}^2} z^{a_k}\,\mathrm{d}x$$
$$\le (M_1a_k^2+\varepsilon)(C\varepsilon\int_{\mathbb{R}^2}|\nabla z^{a_{k-1}}|^2\,\mathrm{d}x) + (M_1a_k^2+\varepsilon)\left(C\frac{1}{4\varepsilon}\|z^{a_{k-1}}\|_{L^1}^2\right) \quad \mathbb{P}\text{-a.s.}$$

Remember that $M_1 < \infty$ almost surely. For a.e. $\omega \in \Omega$ and for each $k \ge 1$, we pick up $\varepsilon = \varepsilon_k(\omega) > 0$ sufficiently small such that

$$(M_1a_k^2+\varepsilon_k)C\varepsilon_k \le 2\frac{a_k-1}{a_k}. \tag{5.10}$$

Let

$$c_k := (M_1a_k^2+\varepsilon_k)\frac{C}{4\varepsilon_k},$$

and then we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2} z^{a_k}\,\mathrm{d}x + \varepsilon_k\int_{\mathbb{R}^2} z^{a_k}\,\mathrm{d}x \le c_k\left(\sup_{t\in[0,\widetilde{\tau}^*)}\int_{\mathbb{R}^2} z^{a_{k-1}}\,\mathrm{d}x\right)^2, \quad k\ge 1\ \mathbb{P}\text{-a.s.},$$

which gives us

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{\varepsilon_k t}\int_{\mathbb{R}^2} z^{a_k}\,\mathrm{d}x\right) \le \mathrm{e}^{\varepsilon_k t}c_k\left(\sup_{t\in[0,\widetilde{\tau}^*)}\int_{\mathbb{R}^2} z^{a_{k-1}}\,\mathrm{d}x\right)^2 \quad k\ge 1\ \mathbb{P}\text{-a.s.}$$

Then, almost surely we have that for all $k \ge 1$,

$$\int_{\mathbb{R}^2} z^{a_k}(t)\,\mathrm{d}x \le \max\left\{\int_{\mathbb{R}^2} u_0^{a_k}\,\mathrm{d}x,\ \delta_k\left(\sup_{t\in[0,\widetilde{\tau}^*)}\int_{\mathbb{R}^2} z^{a_{k-1}}\,\mathrm{d}x\right)^2\right\}, \tag{5.11}$$

where

$$\delta_k := \frac{c_k}{\varepsilon_k} = (M_1a_k^2+\varepsilon_k)\frac{C}{4\varepsilon_k^2}. \tag{5.12}$$

Moreover, we let

$$I := \max\left\{\|u_0\|_{L^1}, \|u_0\|_{L^\infty}\right\}, \quad A_k := \sup_{t\in[0,\widetilde{\tau}^*)}\int_{\mathbb{R}^2} z^{a_k}\,\mathrm{d}x.$$

It follows from (5.10) that for a.e. $\omega \in \Omega$ and for each $k \ge 1$, ε_k is of order $\frac{1}{a_k^2}$. Hence $\delta_k \ge 1$. Then (5.11) now reads as $A_k \le \max\{I^{a_k}, \delta_k A_{k-1}^2\}$, $k \ge 1$ $\mathbb{P}$-a.s., from which we obtain recursively

$$
\begin{aligned}
A_k &\le \max\{I^{a_k}, \delta_k \max\{I^{a_{k-1}}, \delta_{k-1}A_{k-2}^2\}^2\} \\
&= \max\{I^{a_k}, \max\{\delta_k I^{a_k}, \delta_k \delta_{k-1}^2 A_{k-2}^4\}\} \\
&= \max\{\delta_k I^{a_k}, \delta_k \delta_{k-1}^{a_1} A_{k-2}^{a_2}\} \\
&\le \max\{\delta_k \delta_{k-1}^{a_1} I^{a_k}, \delta_k \delta_{k-1}^{a_1}\delta_{k-2}^{a_2} A_{k-3}^{a_3}\} \\
&\le \cdots \\
&\le \max\{\delta_k \delta_{k-1}^{a_1}\cdots\delta_2^{a_{k-2}} I^{a_k}, \delta_k \delta_{k-1}^{a_1}\cdots\delta_2^{a_{k-2}}\delta_1^{a_{k-1}} A_0^{a_k}\} \\
&\le \delta_k \delta_{k-1}^{a_1}\cdots\delta_2^{a_{k-2}}\delta_1^{a_{k-1}} I^{a_k} \ \mathbb{P}\text{-a.s.}
\end{aligned}
$$

By (5.12) and the fact that ε_k is of order $\frac{1}{a_k^2}$ almost surely, we can find a random variable $1 < M_2 = M_2(\omega) < \infty$ such that

$$
\mathbb{P}\left\{\frac{\delta_k}{a_k^6} \le M_2 \text{ for all } k \ge 1\right\} = 1,
$$

and

$$
\begin{aligned}
A_k^{\frac{1}{a_k}} &\le \delta_k^{\frac{1}{a_k}} \delta_{k-1}^{\frac{a_1}{a_k}} \cdots \delta_2^{\frac{a_{k-2}}{a_k}} \delta_1^{\frac{a_{k-1}}{a_k}} I \\
&\le \prod_{i=1}^{k}((M_2 a_i^6)^{\frac{1}{a_i}})I = M_2^{(1-\frac{1}{2^k})} 2^{3\sum_{i=1}^k \frac{i}{2^{i-1}}} I \le 2^{15} M_2 I \ \mathbb{P}\text{-a.s.}
\end{aligned}
$$

Sending $k \to \infty$, we finally conclude that

$$
\sup_{t\in[0,\widetilde{\tau}^*)} \|z(t)\|_{L^\infty} \le CM_2 \max\{\|u_0\|_{L^1}, \|u_0\|_{L^\infty}\} \quad \mathbb{P}\text{-a.s.} \tag{5.13}
$$

According to the definition of $\widetilde{\tau}^*$ in (5.3),

$$
\limsup_{t\to\widetilde{\tau}^*} \|z(t)\|_{L^\infty} = \infty \text{ on } \{\widetilde{\tau}^* < \infty\}.
$$

However, we notice that the right hand side of (5.13) does not depend on $\widetilde{\tau}^*$. Therefore, the L^∞-norm of z will survive for all the time. That is,

$$
\sup_{t\ge 0} \|z(t)\|_{L^\infty} \le CM_2 \max\{\|u_0\|_{L^1}, \|u_0\|_{L^\infty}\} \quad \mathbb{P}\text{-a.s.}
$$

Step 3: Completing the proof. By (5.1), we have

$$
\|u(t)\|_{L^\infty} \le CM_2 \max\{\|u_0\|_{L^1}, \|u_0\|_{L^\infty}\} e^{\int_0^t \beta(t')\,dW_{t'} - \int_0^t \frac{\beta^2(t')}{2}\,dt'}, \quad t > 0 \quad \mathbb{P}\text{-a.s.},
$$

which gives (2.5). Then the decay of $\|u\|_{L^\infty}$ comes from (1) in Lemma 3.6. The proof is completed. □

5.1.2. *Decay of H^s norm in $\mathbb{R}^d$ with high probability*

Proof of Theorem 2.3. Using similar estimates as we obtain (4.19), we can conclude that there is a constant $C = C(s)$ such that for a.e. $\omega \in \Omega$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|z(t)\|_{H^s}^2 \leq C|\chi|\varphi(t)\|z(t)\|_{H^{s-2}}\|z(t)\|_{H^s}^2, \quad \varphi(t) \quad \text{is given in (2.3).}$$

For $w(t) := \mathrm{e}^{-\int_0^t \beta(t')\,\mathrm{d}W_{t'}} u(t) = \mathrm{e}^{-\int_0^t \frac{\beta^2(t')}{2}\,\mathrm{d}t'} z(t)$, we infer from the above estimate that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w(t)\|_{H^s} + \frac{\beta^2(t)}{2}\|w(t)\|_{H^s} \leq C|\chi|\psi(t)\|w(t)\|_{H^{s-2}}\|w(t)\|_{H^s}, \tag{5.14}$$

where $\psi(t) := \mathrm{e}^{\int_0^t \beta(t')\,\mathrm{d}W_{t'}}$. For any $R > 1$, we fix R, let $\lambda_1 > 2$ and then define

$$\tau_1 := \inf\left\{t > 0 : \psi(t)\|w(t)\|_{H^{s-2}} = \|u(t)\|_{H^{s-2}} > \frac{\beta^2(t)}{C\lambda_1|\chi|}\right\}. \tag{5.15}$$

Thanks to (2.6) and $R > 1$, we have $\mathbb{P}\{\tau_1 > 0\} = 1$, and

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w(t)\|_{H^s} + \frac{(\lambda_1 - 2)\beta^2(t)}{2\lambda_1}\|w(t)\|_{H^s} \leq 0, \quad t \in [0, \tau_1] \quad \mathbb{P}\text{-a.s.}$$

Let $\lambda_2 > \frac{2\lambda_1}{\lambda_1 - 2}$. The above inequality and $w = \mathrm{e}^{-\int_0^t \beta(t')\,\mathrm{d}W_{t'}} u$ imply that for a.e. $\omega \in \Omega$ for $t \in [0, \tau_1]$,

$$\begin{aligned}
\|u(t)\|_{H^s} &= \|w(t)\|_{H^s}\mathrm{e}^{\int_0^t \beta(t')\,\mathrm{d}W_{t'}} \\
&\leq \|w_0\|_{H^s}\mathrm{e}^{\int_0^t \beta(t')\,\mathrm{d}W_{t'} - \int_0^t \frac{(\lambda_1 - 2)\beta^2(t')}{2\lambda_1}\,\mathrm{d}t'} \\
&= \|u_0\|_{H^s}\mathrm{e}^{\int_0^t \beta(t')\mathrm{d}W_{t'} - \int_0^t \frac{\beta^2(t')}{\lambda_2}\,\mathrm{d}t'}\mathrm{e}^{-\frac{((\lambda_1 - 2)\lambda_2 - 2\lambda_1)}{2\lambda_1\lambda_2}\int_0^t \beta^2(t')\,\mathrm{d}t'}.
\end{aligned} \tag{5.16}$$

Define the stopping time

$$\tau_2 := \inf\left\{t > 0 : \mathrm{e}^{\int_0^t \beta(t')\,\mathrm{d}W_{t'} - \int_0^t \frac{\beta^2(t')}{\lambda_2}\,\mathrm{d}t'} > R\right\}.$$

Notice that $\mathbb{P}\{\tau_2 > 0\} = 1$. From (5.16), we have

$$\begin{aligned}
\|u(t)\|_{H^s} &\leq R\frac{\beta_*}{C\lambda_1|\chi|R}\mathrm{e}^{-\frac{((\lambda_1 - 2)\lambda_2 - 2\lambda_1)}{2\lambda_1\lambda_2}\int_0^t \beta^2(t')\,\mathrm{d}t'} \\
&\leq \frac{\beta_*}{C\lambda_1|\chi|}\mathrm{e}^{-\frac{((\lambda_1 - 2)\lambda_2 - 2\lambda_1)}{2\lambda_1\lambda_2}\int_0^t \beta^2(t')\,\mathrm{d}t'} \\
&\leq \frac{\beta_*}{C\lambda_1|\chi|}, \quad t \in [0, \tau_1 \wedge \tau_2) \quad \mathbb{P}\text{-a.s.}
\end{aligned} \tag{5.17}$$

Combining (5.17) and (5.15), we find that

$$\mathbb{P}\{\tau_1 \geq \tau_2\} = 1.$$

Therefore, it follows from (5.17) that

$$\mathbb{P}\left\{\|u(t)\|_{H^s} \leq \frac{\beta_*}{C\lambda_1|\chi|}\mathrm{e}^{-\frac{((\lambda_1 - 2)\lambda_2 - 2\lambda_1)}{2\lambda_1\lambda_2}\int_0^t \beta^2(t')\,\mathrm{d}t'} \quad \text{for all } t > 0\right\} \geq \mathbb{P}\{\tau_2 = +\infty\}.$$

Then we apply (2) in Lemma 3.6 with $\vartheta(t) = (\frac{1}{2} - \frac{1}{\lambda_2})\beta^2(t)$ to find that

$$\mathbb{P}\{\tau_2 = +\infty\} > 1 - \left(\frac{1}{R}\right)^{2/\lambda_2},$$

which completes the proof. □

5.2. *Nonlinear case*

Proof of Theorem 2.4. Let $s > \frac{d}{2} + 3$ and $r \in (\frac{d}{2} + 1, s - 2)$. Let

$$h(u) := \alpha(1 + \|u\|_{H^r})^{\varrho} u.$$

By mean value theorem for $(1 + \cdot)^{\varrho}$ with $\varrho \geq \frac{1}{2}$, we have that for any $u, v \in H^s$,

$$\|h(u) - h(v)\|_{H^s} \leq |\alpha| g(\|u\|_{H^s} + \|v\|_{H^s}) \|u - v\|_{H^s}$$

for some increasing function $g : [0, \infty) \mapsto [0, \infty)$. By using the above locally Lipschitz property of $h(\cdot)$ and following the steps as in Sec. 4 with slight modification, one can obtain a unique maximal solution (u, τ^*) to (1.6) in H^s such that

$$\mathbf{1}_{\{\limsup_{t\to\tau^*} \|u(t)\|_{H^s} = \infty\}} = \mathbf{1}_{\{\limsup_{t\to\tau^*} \|u(t)\|_{H^q} = \infty\}}, \quad q \in [r, s] \ \mathbb{P}\text{-a.s.} \tag{5.18}$$

Then we apply Itô's formula to $\|u(t)\|_{H^{s-2}}^2$ to find

$$\begin{aligned} \mathrm{d}\|u\|_{H^{s-2}}^2 &= 2\alpha(1 + \|u\|_{H^r})^{\varrho} \|u\|_{H^{s-2}}^2 \,\mathrm{d}W - 2(\nabla \mathcal{D}^{s-2} u, \nabla \mathcal{D}^{s-2} u)_{L^2} \,\mathrm{d}t \\ &\quad - 2\chi(\mathcal{D}^{s-2} F(u), \mathcal{D}^{s-2} u)_{L^2} \,\mathrm{d}t + \alpha^2 (1 + \|u\|_{H^r})^{2\varrho} \|u\|_{H^{s-2}}^2 \,\mathrm{d}t. \end{aligned}$$

Following [9, 51, 54], we will use the Lyapunov function $\log(1 + x^2)$ to establish global existence. Applying Itô's formula to $\log(1 + \|u\|_{H^{s-2}}^2)$ and neglecting the positive term $\frac{\|\nabla u\|_{H^{s-2}}^2}{1+\|u\|_{H^{s-2}}^2}$ on the left hand side of the equation, we find

$$\begin{aligned} &\mathrm{d}\log(1 + \|u\|_{H^{s-2}}^2) \\ &\leq \frac{2\alpha(1 + \|u\|_{H^r})^{\varrho}}{1 + \|u\|_{H^{s-2}}^2} \|u\|_{H^{s-2}}^2 \,\mathrm{d}W - \frac{1}{1 + \|u\|_{H^{s-2}}^2} 2\chi(F(u), u)_{H^{s-2}} \,\mathrm{d}t \\ &\quad + \frac{1}{1 + \|u\|_{H^{s-2}}^2} \alpha^2 (1 + \|u\|_{H^r})^{2\varrho} \|u\|_{H^{s-2}}^2 \,\mathrm{d}t \\ &\quad - 2\frac{1}{(1 + \|u\|_{H^{s-2}}^2)^2} \alpha^2 (1 + \|u\|_{H^r})^{2\varrho} \|u\|_{H^{s-2}}^4 \,\mathrm{d}t. \end{aligned}$$

Let

$$\tau_m := \inf\{t \geq 0 : \|u(t)\|_{H^{s-2}} \geq m\}.$$

Using Lemma 3.4, we find that there is a $D > 0$ such that for any $t > 0$,

$$\begin{aligned} &\mathbb{E}\log(1 + \|u(t \wedge \tau_m)\|_{H^{s-2}}^2) \\ &\leq \mathbb{E}\log(1 + \|u_0\|_{H^{s-2}}^2) + \mathbb{E}\int_0^{t\wedge\tau_m} \mathcal{M}(\|u\|_{H^r}, \|u\|_{H^s}) \,\mathrm{d}t', \end{aligned}$$

where

$$\mathcal{M}(x,y) := \frac{2D|\chi|xy^2 + \alpha^2(1+x)^{2\varrho}y^2}{1+y^2} - \frac{2\alpha^2(1+x)^{2\varrho}y^4}{(1+y^2)^2}.$$

Since ϱ and α satisfy (2.7), it is straightforward to see that there are constants $K_1, K_2 > 0$ such that for all $0 < x \le y < \infty$,

$$\mathcal{M}(x,y) + \frac{\alpha^2 K_2(1+x)^{2\varrho}y^4}{(1+y^2)^2(1+\log(1+y^2))} \le K_1. \tag{5.19}$$

Indeed, this is because

$$\begin{aligned}
&\frac{2|\chi|Dxy^2 + \alpha^2(1+x)^{2\varrho}y^2}{1+y^2} - \frac{2\alpha^2(1+x)^{2\varrho}y^4}{(1+y^2)^2} + \frac{\alpha^2 K_2(1+x)^{2\varrho}y^4}{(1+y^2)^2(1+\log(1+y^2))} \\
&\le \alpha^2(1+x)^{2\varrho}\left(\frac{2|\chi|Dx}{\alpha^2(1+x)^{2\varrho}} + 1 - \frac{2x^4}{(1+x^2)^2} + \frac{K_2}{(1+\log(1+x^2))}\right).
\end{aligned}$$

When (2.7) is satisfied,

$$\limsup_{x\to+\infty}\left(\frac{2|\chi|Dx}{\alpha^2(1+x)^{2\varrho}} + 1 - \frac{2x^4}{(1+x^2)^2} + \frac{K_2}{(1+\log(1+x^2))}\right) < 0,$$

which means (5.19) holds true. Consequently, we arrive at

$$\begin{aligned}
&\mathbb{E}\log(1+\|u(t\wedge\tau_m)\|^2_{H^{s-2}}) - \mathbb{E}\log(1+\|u_0\|^2_{H^{s-2}}) \\
&\le K_1 t - \mathbb{E}\int_0^{t\wedge\tau_m} K_2 \frac{\alpha^2(1+\|u\|_{H^r})^{2\varrho}\|u\|^4_{H^{s-2}}}{(1+\|u\|^2_{H^{s-2}})^2(1+\log(1+\|u\|^2_{H^{s-2}}))}\mathrm{d}t',
\end{aligned}$$

which means for some constant $C = C(u_0, K_1, K_2, t) > 0$,

$$\mathbb{E}\int_0^{t\wedge\tau_m} \frac{\alpha^2(1+\|u\|_{H^r})^{2\varrho}\|u\|^4_{H^{s-2}}}{(1+\|u\|^2_{H^{s-2}})^2\left(1+\log(1+\|u\|^2_{H^{s-2}})\right)}\,\mathrm{d}t' \le C(u_0, K_1, K_2, t). \tag{5.20}$$

Therefore, for any $T > 0$, it follows from the BDG inequality that

$$\begin{aligned}
&\mathbb{E}\sup_{t\in[0,T\wedge\tau_m]}\log(1+\|u\|^2_{H^{s-2}}) - \mathbb{E}\log(1+\|u_0\|^2_{H^{s-2}}) \\
&\le C\mathbb{E}\left(\int_0^{T\wedge\tau_m} \frac{\alpha^2(1+\|u\|_{H^r})^{2\varrho}\|u\|^4_{H^{s-2}}}{\left(1+\|u\|^2_{H^{s-2}}\right)^2}\,\mathrm{d}t\right)^{\frac{1}{2}} \\
&\quad + \mathbb{E}\int_0^{T\wedge\tau_m}\left|K_1 - K_2\frac{\alpha^2(1+\|u\|_{H^r})^{2\varrho}\|u\|^4_{H^{s-2}}}{(1+\|u\|^2_{H^{s-2}})^2\left(1+\log(1+\|u\|^2_{H^{s-2}})\right)}\right|\mathrm{d}t \\
&\le \frac{1}{2}\mathbb{E}\sup_{t\in[0,T\wedge\tau_m]}(1+\log(1+\|u\|^2_{H^{s-2}})) + K_1 T \\
&\quad + C(K_2)\mathbb{E}\int_0^{T\wedge\tau_m}\frac{\alpha^2(1+\|u\|_{H^r})^{2\varrho}\|u\|^4_{H^{s-2}}}{(1+\|u\|^2_{H^{s-2}})^2(1+\log(1+\|u\|^2_{H^{s-2}}))}\mathrm{d}t.
\end{aligned}$$

Thus we use (5.20) to obtain

$$\begin{aligned}
&\mathbb{E}\sup_{t\in[0,T\wedge\tau_m]}\log(1+\|u\|^2_{H^{s-2}})\\
&\le 2\mathbb{E}(1+\log(1+\|u_0\|^2_{H^{s-2}}))+K_1T\\
&\quad+C(K_2)\mathbb{E}\int_0^{T\wedge\tau_m}\frac{\alpha^2(1+\|u\|_{H^r})^{2\varrho}\|u\|^4_{H^{s-2}}}{(1+\|u\|^2_{H^{s-2}})^2(1+\log(1+\|u\|^2_{H^{s-2}}))}\,\mathrm{d}t\\
&\le C(u_0,K_1,K_2,T).
\end{aligned}$$

Let $\tau_* := \lim_{m\to\infty}\tau_m$. From the above estimate, one can follow the non-explosion text (cf. [9, page 125]. See also [37, 54]) that $\mathbb{P}\{\tau_*=\infty\}=1$. Hence $\|u\|_{H^{s-2}}$ exists globally. By (5.18), u exists globally in H^s. □

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