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LOTKA-VOLTERRA DIFFUSION-ADVECTION COMPETITION SYSTEM WITH DYNAMICAL RESOURCES

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Abstract. Competition systems describing the competition between species for resources have been widely studied in the literature and wealthy results have been developed. Most of them (if not all) have essentially assumed that the resources are spatially varying without temporal dynamics. This is an idealized assumption since the most ecological environments and/or biospecies are dynamically changing. Hence the effect of temporal dynamics of resource ought to be taken into account to predict/interpret the competition outcomes more precisely. This constitutes the main motivation of this work and we consider a Lotka-Volterra reaction-diffusion-advection competition system with a dynamical resource whose dynamics is determined by an evolution equation, where the competing species have biased movement (advection) up the resource gradient. We first establish the global existence of classical solutions via Moser iteration and global stability of spatially homogeneous steady states for the constant resource growth rate by method of Lyapunov functionals. When the resource growth rate is spatially varying, we use numerical simulations to demonstrate the possible competition outcomes and find that the asymptotic dynamics of the competition system with dynamical resources is quite different from the case that resources have no dynamics. Moreover, we numerically observe that the advective strategy and/or the relative strength of advection sensory responses are key factors determining the competition outcomes and the asymptotic profiles of the solution.

1. Introduction.

The evolution of dispersal (either random or biased) is an important topic in theoretical studies of population dynamics, for which numerous mathematical models have been proposed to understand the effect of dispersal strategies on the population dynamics and their evolutions (e.g., see the survey papers [15, 32] or book [9]). Among other things, the present work is motivated by the following

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two-species reaction-diffusion-advection competition model

$$\begin{cases} u_t = d_1\Delta u - \nabla\cdot(\lambda_1 u\nabla\chi_1(m)) + u(m(x) - b_1 u - c_1 v), & \text{in } \Omega\times\mathbb{R}^+,\\ v_t = d_2\Delta v - \nabla\cdot(\lambda_2 v\nabla\chi_2(m)) + v(m(x) - b_2 u - c_2 v), & \text{in } \Omega\times\mathbb{R}^+,\\ d_1\partial_\nu u - \lambda_1 u\partial_\nu\chi_1(m) = d_2\partial_\nu v - \lambda_2 u\partial_\nu\chi_2(m) = 0, & \text{on } \partial\Omega\times\mathbb{R}^+,\\ (u,v)(x,0) = (u_0,v_0)(x), & \text{in } \Omega, \end{cases} \tag{1}$$

where $u(x,t)$ and $v(x,t)$ represent the population densities of two competing species at location $x \in \Omega$ and time $t > 0$, and the habitat Ω is a bounded smooth domain in $\mathbb{R}^N(N \geq 2)$; $m(x)$ represents the environmental resource and $\chi_1, \chi_2 \in C^2(\bar{\Omega})$ are functions accounting for the sensory response mechanisms in response to the resource gradient, λ_1 and λ_2 are positive constants measuring the strength of sensory responses; $d_1, d_2 > 0$ are the dispersal rates of u and v, respectively; $b_i, c_i (i = 1, 2)$ are positive constants and $\partial_\nu = \frac{\partial}{\partial\nu}$, where ν denotes the outward unit normal vector on $\partial\Omega$. The zero-flux boundary conditions are prescribed to warrant that no individual crosses the boundary of the habitat. The initial data u_0 and v_0 are nonnegative and not identically zero.

When the resource is spatially homogeneous, namely $m(x) = a > 0$ is a constant, the model (1) reduces to the classical Lotka-Volterra diffusion-competition model

$$\begin{cases} u_t = d_1\Delta u + u(a - b_1 u - c_1 v), & \text{in } \Omega\times\mathbb{R}^+,\\ v_t = d_2\Delta v + v(a - b_2 u - c_2 v), & \text{in } \Omega\times\mathbb{R}^+,\\ \partial_\nu u = \partial_\nu v = 0, & \text{on } \partial\Omega\times\mathbb{R}^+,\\ (u,v)(x,0) = (u_0,v_0)(x), & \text{in } \Omega, \end{cases} \tag{2}$$

which has been extensively studied in the literature (cf. [34, 8, 26] and references therein). The global dynamics of solutions to (2) crucially depends on the ecological reaction coefficients. Set $B = b_1/b_2, C = c_1/c_2$. Then the (positive) co-existence steady state is globally asymptotically stable if $C < 1 < B$ (weak competition) while competitive exclusion is globally asymptotically achieved if $1 < \min\{B, C\}$ or $1 > \max\{B, C\}$. If $B < 1 < C$ (strong competition), the dynamics will be more complicate and is yet to be completely understood, where the two exclusion steady states are locally stable (cf. [25, 34]).

When the resource is spatially heterogeneous (i.e. $m(x)$ is non-constant) and no biased advection takes place (i.e. $\lambda_1 = \lambda_2 = 0$), (1) reduces to the following one

$$\begin{cases} u_t = d_1\Delta u + u(m(x) - b_1 u - c_1 v), & \text{in } \Omega\times\mathbb{R}^+,\\ v_t = d_2\Delta v + v(m(x) - b_2 u - c_2 v), & \text{in } \Omega\times\mathbb{R}^+,\\ \partial_\nu u = \partial_\nu v = 0, & \text{on } \partial\Omega\times\mathbb{R}^+,\\ (u,v)(x,0) = (u_0,v_0)(x), & \text{in } \Omega. \end{cases} \tag{3}$$

The most prominent dynamics of (3), in contrast to (2), is perhaps the so-called "slower diffuser prevails" phenomenon saying that *the slower diffuser wipes out its fast competitor regardless of the initial value*. This was first found in [16] for the case $b_1 = c_1 = b_2 = c_2 = 1$ and was further extended in [31] to the weak competition case $0 < c_1, b_2 < 1$ with $b_1 = c_2 = 1$. The global dynamics of (3) in more complete parameter regimes have been carried out in a series of important works [21, 20, 22].

Compared to the above two reduced models (2) and (3), the global dynamics of (1) with spatially heterogenous resources and advection (i.e. $\lambda_1 > 0$ or $\lambda_2 > 0$) are more complicated. Indeed if dispersal strategies include the advection (biased movement), the prominent phenomenon "slower diffuser prevails" may not happen, instead both co-existence and competitive exclusion are possible even if two species have different diffusion rates, see [10, 11, 14, 3] for $\chi_1 = \chi_2 = m$ and [4, 12, 17] for $\chi_1 = \chi_2 = \ln m$ for $b_1 = b_2 = c_1 = c_2 = 1$. In particular, the advective strategy $\chi_1 = \ln m$ is evolutionarily stable and ideal free distribution can be achieved if $\chi_2 - \ln m$ is not constant (cf. [4, 12]). In advective environments like the river or stream, it was proved in [33, 35, 18] that slower diffusion could be disadvantageous while the fast diffuser can evolve. We refer to [43, 44] for the study of global dynamics of more general diffusion-advection competition model than (1) with inhomogeneous coefficients or different resources. Nevertheless the global dynamics of (1) with non-constant χ_1 and χ_2 has not been completely understood and many interesting questions still remain open (cf. [15, 32]). A competition model with nonlinear repulsive advection between competitions was considered in [38, 39] and interesting dynamics were found.

A manifest assumption made in the existing models is that the environmental resource is held to be spatially varying but temporarily constant. However, ecological systems consist of dynamically interacting organisms, where many resources are consumable and hence may vary in time, such as the nutrients for plants and food for animals. Therefore it would be biologically meaningful to consider competition systems with resources having both temporal and spatial dynamics. It appears that not many analytical studies are pursued in this direction. The earliest work was perhaps [24] where the time-periodic resource $m(x,t)$ was considered and quite different results on the stability of coexistence and exclusion steady states are found. Later on, traveling wave solutions (see [42, 5]) and the free boundary problem (e.g., see [13, 37] and references therein) of (3) with time-periodic environmental resources have been investigated. In these attempts, the resources are given functions of time and/or space but lacks of temporal dynamics. To consider this realistic factor, a consumer-resource model without competition was proposed in [41] and its global dynamics of solutions was studied in [19].

The purpose of this paper is to explore the effect of temporal dynamics of resources on the competition outcomes. To this end, we consider the following mathematical model of two competing species sharing a common prey resource

$$\begin{cases} u_t = d_1\Delta u - \nabla\cdot(\lambda_1 u\nabla\chi_1(w)) + u(a_1F_1(w) - h_1(u) - c_1 v), & x\in\Omega,\ t>0,\\ v_t = d_2\Delta v - \nabla\cdot(\lambda_2 v\nabla\chi_2(w)) + v(a_2F_2(w) - b_2 u - h_2(v)), & x\in\Omega,\ t>0,\\ w_t = \Delta w - uF_1(w) - vF_2(w) + \mu w(r(x) - w), & x\in\Omega,\ t>0,\\ \partial_\nu u = \partial_\nu v = \partial_\nu w = 0, & x\in\partial\Omega,\ t>0,\\ (u,v,w)(x,0) = (u_0,v_0,w_0)(x), & x\in\Omega, \end{cases} \tag{4}$$

where $u(x,t)$ and $v(x,t)$ denote the densities of two competing species, respectively, and $w(x,t)$ represents the prey resource. The parameters d_i, λ_i, a_i, b_2, c_1 $(i=1,2)$ and μ are all positive constants, and $r(x)$ is the nutrient available to the prey resource w (i.e. the intrinsic growth rate of w). The advection in the system (4) is modelled by the biased movement of species up the resource gradient with sensory response functions $\chi_i(w)$ $(i=1,2)$ which are assumed to satisfy the following hypothesis:

(H1) $\chi_i(w)\in C^{2,\alpha}([0,\infty))$, $\alpha\in(0,1)$ and $\frac{d}{dw}\chi_i(w)\geq 0$ for all $w\geq 0$.

The functional response functions $F_i(w)$, intra-specific interaction functions h_i and the nutrient $r(x)$ are assumed to satisfy

(H2) $F_i(w) \in C^0([0,\infty))$ and $F_i(w) \gneqq 0$ for $w \geq 0$ $(i=1,2)$, $h_1(u) = \theta_1 + b_1 u$ and $h_2(v) = \theta_2 + c_2 v$ with $\theta_i \in \mathbb{R}$ and $b_1, c_2 > 0$. Moreover, $r(x) \in C(\bar{\Omega})$.

The main results of this paper are summarized below. First we establish the global existence of classical solutions to (4) with (H1) and (H2) for any $r \in C(\bar{\Omega})$ in two dimensions (see Theorem 2.1). Then we establish the global stability of solutions to (4) for constant $r(x)$ under certain conditions based on the method of Lyapunov functionals (see Theorem 3.1). Finally, we numerically explore the global dynamics of (4) with non-constant $r(x)$ and discuss the underlying biological implications in section 4.

2. Global boundedness of solutions.

In this section, we are devoted to establishing the global existence of classical solutions to (4), as given in the following theorem.

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary and the hypotheses (H1) and (H2) hold. Assume $(u_0, v_0, w_0) \in [W^{1,p}(\Omega)]^3$ with $p > 2$, $u_0, v_0, w_0 \geq 0 (\not\equiv 0)$. Then (4) admits a unique global classical solution $(u, v, w) \in [C^0([0,\infty) \times \bar{\Omega}) \cap C^{2,1}((0,\infty) \times \bar{\Omega})]^3$ satisfying $u, v, w > 0$ for all $t > 0$ and*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{L^\infty(\Omega)} + \|w(\cdot,t)\|_{W^{1,\infty}(\Omega)} \leq C,$$

where $C > 0$ is a constant independent of t. In particular, we have $0 < w \leq K$, where

$$K := \max\{\|r\|_{L^\infty}, \|w_0\|_{L^\infty}\}. \tag{5}$$

We remark that when $F_1(w) = w$ and $r(x) = 1$, the global existence of classical solutions in two dimensions and the existence of stationary and time-periodic nontrivial solutions bifurcating from the positive constant equilibrium have been established in [36]. The results in Theorem 2.1 extended the global existence results of [36]. The proof of Theorem 2.1 consists of two steps: local existence and the *a priori* estimates of solutions. We first introduce some frequently used notations.

Notation. For simplicity, we abbreviate $\int_0^t \int_\Omega f(\cdot, s) dx ds$ and $\int_\Omega f(\cdot, t) dx$ as $\int_0^t \int_\Omega f$ and $\int_\Omega f$, respectively. In addition, we denote $\|\cdot\|_{L^p(\Omega)} = \|\cdot\|_{L^p}$ for short, and use C_i $(i = 1, 2, 3, \cdots)$ to denote generic constants which may vary in the context.

2.1. Local existence and some preliminary results.

First, we establish the local existence of solutions to system (4) by the abstract theory of quasilinear parabolic systems by Amann in [1, 2].

Lemma 2.2 (Local existence)**.** *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary. Assume that the parameters μ, d_i, λ_i, a_i, b_i, c_i $(i = 1, 2)$ are positive constants, $\theta_i \in \mathbb{R}$, and the hypotheses (H1) and (H2) hold. Suppose that $(u_0, v_0, w_0) \in [W^{1,p}(\Omega)]^3$ with $u_0, v_0, w_0 \geq 0 (\not\equiv 0)$ and $p > 2$. Then there exists a constant $T_{max} \in (0, \infty]$ such that system (4) has a unique classical solution (u, v, w) fulfilling $u, v, w > 0$ for all $t > 0$ and*

$$(u, v, w) \in [C^0(\bar{\Omega} \times [0, T_{max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{max}))]^3.$$

Moreover if $T_{max} = \infty$, then

$$\|u(\cdot,t)\|_{L^\infty} + \|v(\cdot,t)\|_{L^\infty} + \|w(\cdot,t)\|_{W^{1,\infty}} \to \infty \quad \text{as} \quad t \nearrow T_{max}.$$

Proof. Denote $z = (u, v, w)$. Then the system (4) can be written as

$$\begin{cases} z_t = \nabla \cdot (P(z)\nabla z) + Q(z), & x \in \Omega, t > 0, \\ \frac{\partial z}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ z(\cdot, 0) = (u_0, v_0, w_0), & x \in \Omega, \end{cases} \tag{6}$$

where

$$P(z) = \begin{pmatrix} d_1 & 0 & -\lambda_1 u \chi_1'(w) \\ 0 & d_2 & -\lambda_2 v \chi_2'(w) \\ 0 & 0 & 1 \end{pmatrix},$$

and

$$Q(z) = \begin{pmatrix} u(a_1 F_1(w) - h_1(u) - c_1 v) \\ v(a_2 F_2(w) - b_2 u - h_2(v)) \\ -uF_1(w) - vF_2(w) + \mu w(r(x) - w) \end{pmatrix}.$$

Since $d_i > 0$ $(i = 1, 2)$, the matrix $P(z)$ is positive definite for the given initial data, which means system (6) is normally parabolic. Then the application of [2, Theorem 7.3] yields a $T_{max} > 0$ such that system (6) possesses a unique solution $(u, v, w) \in [C^0(\bar{\Omega}\times[0, T_{max}))\cap C^{2,1}(\bar{\Omega}\times(0, T_{max}))]^3$. Now we show the nonnegativity of (u, v, w) by the maximum principle. To this end, we rewrite equations of system (4) as

$$\begin{cases} u_t - d_1\Delta u + \Psi_1(x, t)\nabla w \cdot \nabla u + \Psi_2(x, t)u = 0, & x \in \Omega, t \in (0, T_{\max}), \\ \frac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega, t \in (0, T_{\max}), \\ u(x, 0) = u_0 \geq 0, & x \in \Omega, \end{cases} \tag{7}$$

where $\Psi_1(x, t) = \lambda_1 \chi_1'(w)$ and $\Psi_2(x, t) = \lambda_1 \chi_1''(w)|\nabla w|^2 + \lambda_1 \chi_1'(w)\Delta w - (a_1 F_1(w) - h_1(u) - c_1 v)$. Then one applies the maximum principle to system (7) and gets that $u(x, t) \geq 0$ for all $(x, t) \in \Omega \times (0, T_{max})$. Since $u_0 \not\equiv 0$, then $u > 0$ holds by the strong maximum principle. Similarly, we can derive that $v, w > 0$ for all $(x, t) \in \Omega \times (0, T_{max})$. Moreover, we see that the matrix $P(z)$ is an upper triangular matrix, which allows us to obtain the blowup criterion by [1, Theorem 5.2]. This completes the proof. ☐

By a comparison principle, we can prove the following result.

Lemma 2.3. *Let the assumptions in Lemma 2.2 hold. Then the solution (u, v, w) of system (4) satisfies that*

$$\|w(\cdot, t)\|_{L^\infty} \leq K \tag{8}$$

for all $t > 0$, where K is defined by (5). Moreover, we have

$$\limsup_{t\to\infty} w(\cdot, t) \leq \|r\|_{L^\infty} \text{ for all } x \in \bar{\Omega}. \tag{9}$$

Proof. Using the facts that u, v, w are non-negative and $r(x) \in C^1(\bar{\Omega})$, we have

$$\begin{cases} w_t - \Delta w \leq \mu w(\|r\|_{L^\infty} - w), & x \in \Omega,\ t > 0, \\ \frac{\partial w}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ w(x, 0) = w_0(x), & x \in \Omega. \end{cases} \tag{10}$$

Let $w^*(t)$ be the solution of the following ODE problem

$$\begin{cases} \frac{dw^*(t)}{dt} = \mu w^*(\|r\|_{L^\infty} - w^*), & t > 0, \\ w^*(0) = \|w_0\|_{L^\infty}. \end{cases} \tag{11}$$

Then $w^*(t) \le \max\{\|w_0\|_{L^\infty}, \|r\|_{L^\infty}\} = K$. It is clear that $w^*(t)$ is a super-solution of the following PDE problem

$$\begin{cases} W_t - \Delta W = \mu W(\|r\|_{L^\infty} - W), & x \in \Omega,\ t > 0, \\ \frac{\partial W}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ W(x,0) = w_0(x), & x \in \Omega, \end{cases} \tag{12}$$

and hence it holds that

$$0 < W(x,t) \le w^*(t) \text{ for all } (x,t) \in \bar{\Omega} \times (0,\infty), \tag{13}$$

where $W > 0$ results from the strong maximum principle. Combining (10), (12) and (13), and using the comparison principle, one has

$$0 < w(x,t) \le W(x,t) \le w^*(t) \le K \text{ for all } (x,t) \in \bar{\Omega} \times (0,\infty), \tag{14}$$

which gives (8). We further have from (11) that $\limsup_{t\to\infty} w^*(t) \le \|r\|_{L^\infty}$, which along with (14) gives (9). Therefore, we complete the proof of Lemma 2.3. □

Lemma 2.4. *Let assumptions in Lemma 2.2 hold and (u, v, w) be the solution of system (4). Then it holds that*

$$\int_\Omega |\nabla w(\cdot,t)|^2 \le C \tag{15}$$

and

$$\int_t^{t+\tau} \int_\Omega (u^2 + v^2) \le C \quad \text{and} \quad \int_t^{t+\tau} \int_\Omega |\Delta w(\cdot,t)|^2 \le C, \tag{16}$$

where $\tau = \min\{1, \frac{T_{max}}{2}\}$ and $C > 0$ is a constant independent of t.

Proof. Integrating the first equation of system (4) over Ω and using Young's inequality, thanks to (8) as well as the positivity of u and v, we have

$$\begin{aligned} \frac{d}{dt}\int_\Omega u + \int_\Omega u &= a_1 \int_\Omega u F_1(w) - \int_\Omega u(\theta_1 + b_1 u) - c_1 \int_\Omega uv + \int_\Omega u \\ &\le -\frac{b_1}{2}\int_\Omega u^2 + \frac{\left(a_1 \max\limits_{w\in[0,K]} F_1(w) + |\theta_1| + 1\right)^2 |\Omega|}{2b_1}, \end{aligned}$$

which gives

$$\frac{d}{dt}\int_\Omega u + \int_\Omega u + \frac{b_1}{2}\int_\Omega u^2 \le C_1, \tag{17}$$

where $C_1 = \frac{\left(a_1 \max\limits_{w\in[0,K]} F_1(w) + |\theta_1| + 1\right)^2 |\Omega|}{2b_1}$. To proceed, we use the same way to the second equation of system (4) and derive that

$$\frac{d}{dt}\int_\Omega v + \int_\Omega v + \frac{c_2}{2}\int_\Omega v^2 \le C_2 \tag{18}$$

with $C_2 = \frac{\left(a_2 \max\limits_{w\in[0,K]} F_2(w) + |\theta_2| + 1\right)^2 |\Omega|}{2c_2}$.

In the sequel, we let $M = \left(\max_{w\in[0,K]}\{F_1(w), F_2(w)\}\right)^2$. Multiplying the equation of w in system (4) by $-\Delta w$ and using the fact $\|w\|_{L^\infty} \le K$, one derives that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla w|^2 + \frac{1}{2}\int_\Omega |\nabla w|^2 + \int_\Omega |\Delta w|^2 \\
&\le M\int_\Omega (u+v)|\Delta w| + \mu K(K + \|r\|_{L^\infty})\int_\Omega |\Delta w| - \frac{1}{2}\int_\Omega w\Delta w \\
&\le \frac{3}{4}\int_\Omega |\Delta w|^2 + M^2\int_\Omega (u^2+v^2) + C_3
\end{aligned}$$

with $C_3 = K^2\left(\mu(\|r\|_{L^\infty} + K) + \frac{1}{2}\right)^2|\Omega|$, that is,

$$\frac{d}{dt}\int_\Omega |\nabla w|^2 + \int_\Omega |\nabla w|^2 + \frac{1}{2}\int_\Omega |\Delta w|^2 \le 2M^2\int_\Omega (u^2+v^2) + 2C_3. \tag{19}$$

Multiplying (17) and (18) by $\frac{6M^2}{b_1}$ and $\frac{6M^2}{c_2}$, respectively, and combining them with (19), we end up with

$$\phi' + \phi + M^2\int_\Omega (u^2+v^2) + \frac{1}{2}\int_\Omega |\Delta w|^2 \le C_4, \tag{20}$$

where

$$\phi(t) = M^2\left(\frac{6}{b_1}\int_\Omega u + \frac{6}{c_2}\int_\Omega v\right) + \int_\Omega |\nabla w|^2$$

and $C_4 = \left(\frac{6C_1}{b_1} + \frac{6C_2}{c_2}\right)M^2 + 2C_3$. Then the application of Grönwall's inequality on (20) gives

$$\phi(t) \le \phi(0) + C_4, \tag{21}$$

which yields (15). Furthermore, letting $\tau = \min\{1, \frac{T_{max}}{2}\}$, integrating (20) over $(t, t+\tau)$ and using (21), one derives that

$$\begin{aligned}
&M^2\int_t^{t+\tau}\int_\Omega (u^2+v^2) + \frac{1}{2}\int_t^{t+\tau}\int_\Omega |\Delta w|^2 \\
&\le \phi(t) + C_4\tau \le \phi(0) + C_4(1+\tau),
\end{aligned}$$

which gives (16). Hence the proof of Lemma 2.4 is completed. □

2.2. **The priori estimates of solutions.** Next we will derive the boundedness of $\|u(\cdot,t)\|_{L^2}$ and $\|v(\cdot,t)\|_{L^2}$ with the help of (16). Furthermore, we derive the uniform boundedness of the solution.

Lemma 2.5. *Assume that the hypotheses of Lemma 2.2 hold. Then there exists a constant $C > 0$ independent of t such that for all $t > 0$,*

$$\|u(\cdot,t)\|_{L^2} + \|v(\cdot,t)\|_{L^2} + \|w(\cdot,t)\|_{W^{1,4}} \le C. \tag{22}$$

Proof. Multiplying the first equation of system (4) by u and applying Young's inequality, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega u^2 + \frac{d_1}{2}\int_\Omega |\nabla u|^2 \\
&\le \lambda_1^2\int_\Omega \frac{|\chi_1'(w)|^2}{2d_1}u^2|\nabla w|^2 + \left(a_1\max_{w\in[0,K]}F_1(w) + |\theta_1|\right)\int_\Omega u^2 - b_1\int_\Omega u^3 \qquad (23)\\
&\le \frac{\lambda_1^2 K_1}{2d_1}\left(\int_\Omega u^4\right)^{\frac{1}{2}}\left(\int_\Omega |\nabla w|^4\right)^{\frac{1}{2}} - \frac{b_1}{2}\int_\Omega u^3 + C_1,
\end{aligned}$$

where $C_1 = \frac{16\left(a_1 \max\limits_{w\in[0,K]} F_1(w)+|\theta_1|\right)^3|\Omega|}{27b_1^2}$. On one hand, one can use the Gagliardo-Nirenberg inequality to derive that

$$\|u\|_{L^4}^2 \le C_2(\|\nabla u\|_{L^2}\|u\|_{L^2} + \|u\|_{L^2}^2). \tag{24}$$

On the other hand, due to (15), we apply the Gagliardo-Nirenberg inequality in two dimensions along with the Neumann boundary conditions (cf. [7]) to get

$$\|\nabla w\|_{L^4}^2 \le C_3(\|\Delta w\|_{L^2}\|\nabla w\|_{L^2} + \|\nabla w\|_{L^2}^2) \le C_4(\|\Delta w\|_{L^2} + 1), \tag{25}$$

which holds only true in two dimensions. The combination of (24) and (25) gives that

$$\begin{aligned}
&\frac{\lambda_1^2 K_1}{2d_1}\left(\int_\Omega u^4\right)^{\frac{1}{2}}\left(\int_\Omega |\nabla w|^4\right)^{\frac{1}{2}} \\
&\le \frac{\lambda_1^2 K_1 C_2 C_4}{2d_1}(\|\nabla u\|_{L^2}\|u\|_{L^2}\|\Delta w\|_{L^2} + \|u\|_{L^2}^2\|\Delta w\|_{L^2} + \|\nabla u\|_{L^2}\|u\|_{L^2} + \|u\|_{L^2}^2) \\
&\le \frac{d_1}{2}\|\nabla u\|_{L^2}^2 + C_5\|u\|_{L^2}^2\|\Delta w\|_{L^2}^2 + C_5\|u\|_{L^2}^2
\end{aligned} \tag{26}$$

with $C_5 = \frac{1}{d_1}\left(\frac{\lambda_1^2 K_1 C_2 C_4}{d_1} + d_1\right)^2$. Substituting (26) into (23) and using Young's inequality give that

$$\frac{d}{dt}\|u\|_{L^2}^2 - 2C_5\|u\|_{L^2}^2\|\Delta w\|_{L^2}^2 \le C_6 \tag{27}$$

for all $t \in (0, T_{max})$, where $C_6 = 2(C_1 + \frac{64C_5^3}{27b_1^2}|\Omega|)$.

Continuously, in order to deal with the differential inequality (27) and derive the estimate for $\|u\|_{L^2}$, we need some information on $\|u\|_{L^2}$ and $\int_t^{t+\tau}\int_\Omega |\Delta(\cdot,t)|^2$ at some time $t = t_0$. To this end, we recall the result (16), then for any $t \in (0, T_{max})$, one can find a $t_0 = t_0(t) \in ((t-\tau)_+, t)$ such that

$$\|u(\cdot, t_0)\|_{L^2}^2 \le C_7 \tag{28}$$

in both cases $t \in (0, \tau)$ and $t \ge \tau$, where τ is defined in Lemma 2.4. Moreover, by (16), there exists a constant $C_8 > 0$ such that

$$\int_{t_0}^{t_0+\tau}\int_\Omega |\Delta w(\cdot, t)|^2 \le C_8. \tag{29}$$

Noting that we can derive from (27) that

$$\frac{d}{dt}\left(\|u(\cdot,t)\|_{L^2}^2 e^{-2C_5\int_0^t \|\Delta w(\cdot,s)\|_{L^2}^2 ds}\right) \le C_6 e^{-2C_5\int_0^t \|\Delta w(\cdot,s)\|_{L^2}^2 ds}.$$

Integrating the above inequality over (t_0, t) and using the fact $t_0 < t \le t_0+\tau \le t_0+1$, we derive

$$\begin{aligned}
\|u(\cdot,t)\|_{L^2}^2 \le& \|u(\cdot,t_0)\|_{L^2}^2 e^{2C_5\int_{t_0}^t \|\Delta w(\cdot,s)\|_{L^2}^2 ds} + C_6\int_{t_0}^t e^{2C_5\int_s^t \|\Delta w(\cdot,\xi)\|_{L^2}^2 d\xi} ds \\
\le& \|u(\cdot,t_0)\|_{L^2}^2 e^{2C_5\int_{t_0}^{t_0+\tau} \|\Delta w(\cdot,s)\|_{L^2}^2 ds} + C_6\int_{t_0}^{t_0+\tau} e^{2C_5\int_{t_0}^{t_0+\tau} \|\Delta w(\cdot,\xi)\|_{L^2}^2 d\xi} ds \\
\le& (C_7 + C_6\tau)e^{2C_5C_8},
\end{aligned} \tag{30}$$

for all $t \in (0, T_{max})$, where the last inequality holds due to (28) and (29).

We treat v in the same way to derive that

$$\|v(\cdot,t)\|_{L^2}^2 \le C_9 \tag{31}$$

for all $t \in (0, T_{max})$. Furthermore, we apply the classical parabolic regularity to the third equation of system (4) and obtain that $\|w(\cdot,t)\|_{W^{1,4}} \le C_9$ in two dimensions, which, together with (30)–(31), yields (22). Consequently, the proof is finished. □

With the boundedness of $\|u(\cdot,t)\|_{L^2}$ and $\|v(\cdot,t)\|_{L^2}$ in hands, we are ready to derive the uniform boundedness of the solution (u, v, w).

Lemma 2.6. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary and the assumptions in Lemma 2.2 hold. Then we have*

$$\|w(\cdot,t)\|_{W^{1,\infty}} \le C, \tag{32}$$

where $C > 0$ is a constant independent of t.

Proof. Noting $0 < w \le K$ in (8) and $|\chi_1'(w)|^2 \le K_1$. Then one multiplies the first equation of system (4) by u^2 and integrates the result over Ω to derive that

$$\begin{aligned}
&\frac{1}{3}\frac{d}{dt}\int_\Omega u^3 + 2\int_\Omega d_1 u|\nabla u|^2 + b_1\int_\Omega u^4 \\
&\le 2\lambda_1\int_\Omega \chi_1'(w)u^2\nabla u\cdot\nabla w + a_1\int_\Omega u^3F_1(w) - \theta_1\int_\Omega u^3 - c_1\int_\Omega u^3 v \\
&\le \int_\Omega d_1 u|\nabla u|^2 + \frac{4\lambda_1^2K_1}{d_1}\left(\int_\Omega u^6\right)^{\frac{1}{2}}\left(\int_\Omega|\nabla w|^4\right)^{\frac{1}{2}} \\
&\quad + \left(a_1\max_{w\in[0,K]}F_1(w) + |\theta_1|\right)\int_\Omega u^3.
\end{aligned} \tag{33}$$

From Lemma 2.5, we have $\|\nabla w\|_{L^4} \le C_1$ and $\|u^{\frac{3}{2}}\|_{L^{\frac{4}{3}}}^2 = \|u\|_{L^2}^3 \le C_2$. Then one applies the Gagliardo-Nirenberg inequality (in two dimensions) and Young's inequality to obtain that

$$\begin{aligned}
\frac{4\lambda_1^2K_1}{d_1}\left(\int_\Omega u^6\right)^{\frac{1}{2}}\left(\int_\Omega|\nabla w|^4\right)^{\frac{1}{2}} &\le \frac{4\lambda_1^2K_1C_1^2}{d_1}\|u^{\frac{3}{2}}\|_{L^4}^2 \\
&\le C_3(\|\nabla u^{\frac{3}{2}}\|_{L^2}^{\frac{4}{3}}\|u^{\frac{3}{2}}\|_{L^{\frac{4}{3}}}^{\frac{2}{3}} + \|u^{\frac{3}{2}}\|_{L^{\frac{4}{3}}}^2) \\
&\le \frac{2d_1}{9}\|\nabla u^{\frac{3}{2}}\|_{L^2}^2 + C_4,
\end{aligned} \tag{34}$$

where $C_4 = C_2C_3(1 + \frac{3C_3^2}{d_1^2})$. Since $\frac{2d_1}{9}\|\nabla u^{\frac{3}{2}}\|_{L^2}^2 = \frac{d_1}{2}\|u^{\frac{1}{2}}\nabla u\|_{L^2}^2$, we substitute (34) into (33) and employ Young's inequality again to show that

$$\frac{1}{3}\frac{d}{dt}\int_\Omega u^3 + \frac{1}{3}\int_\Omega u^3 + \frac{d_1}{2}\int_\Omega u|\nabla u|^2 + \frac{b_1}{2}\int_\Omega u^4 \le C_5$$

with $C_5 = \frac{27|\Omega|}{32b_1^3}\left(a_1\max_{w\in[0,K]}F_1(w) + |\theta_1| + \frac{1}{3}\right)^4 + C_4$, that is

$$\frac{d}{dt}\int_\Omega u^3 + \int_\Omega u^3 \le 3C_5.$$

Therefore, the application of Grönwall's inequality to the above inequality gives that

$$\int_\Omega u^3 \le \int_\Omega u_0^3 + 3C_5. \tag{35}$$

We conclude similarly that $\|v\|_{L^3} \le C_6$, which together with (35) and (8) gives (32) directly by the parabolic regularity. □

Lemma 2.7. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary and the assumptions in Lemma 2.2 hold. Assume that (u, v, w) is the unique solution of system (4). Then there exists a positive constant C independent of t such that*

$$\|u(\cdot,t)\|_{L^\infty} + \|v(\cdot,t)\|_{L^\infty} \le C \text{ for all } t \in (0, T_{max}).$$

Proof. Since $0 < w \le K$ and $F_1(w) \in C^0([0,\infty))$, multiplying the first equation of (4) by u^{p-1} with $p \ge 2$ and integrating the result by parts, we derive

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int_\Omega u^p + (p-1)\int_\Omega d_1 u^{p-2}|\nabla u|^2 \\
&\le (p-1)\lambda_1 \int_\Omega \chi_1'(w) u^{p-1}|\nabla u||\nabla w| + a_1\int_\Omega F_1(w)u^p - \theta_1\int_\Omega u^p \\
&\le \frac{(p-1)}{2}\int_\Omega d_1 u^{p-2}|\nabla u|^2 + \frac{(p-1)\lambda_1^2}{2}\int_\Omega \frac{|\chi_1'(w)|^2}{d_1} u^p |\nabla w|^2 \\
&+ \left(a_1 \max_{w\in[0,K]} F_1(w) + |\theta_1|\right)\int_\Omega u^p,
\end{aligned}$$

which combining with the fact $|\chi_1'(w)|^2 \le K_1$ and $w \in W^{1,\infty}(\Omega \times [0, T_{\max}])$, implies that

$$\frac{d}{dt}\int_\Omega u^p + p(p-1)\int_\Omega u^p + \frac{d_1 p(p-1)}{2}\int_\Omega u^{p-2}|\nabla u|^2 \le C_1 p(p-1)\int_\Omega u^p \tag{36}$$

with $C_1 = \frac{\lambda_1^2 K_1}{2d_1}\|\nabla w\|_{L^\infty}^2 + \left(a_1 \max_{w\in[0,K]} F_1(w) + |\theta_1|\right) + 1$. Then we apply the Gagliardo-Nirenberg inequality to $\int_\Omega u^p$ and get that

$$\begin{aligned}
C_1 p(p-1)\int_\Omega u^p &= C_1 p(p-1)\|u^{\frac{p}{2}}\|_{L^2}^2 \\
&\le C_2 p(p-1)(\|\nabla u^{\frac{p}{2}}\|_{L^2}^{\frac{2n}{n+2}}\|u^{\frac{p}{2}}\|_{L^1}^{\frac{4}{n+2}} + \|u^{\frac{p}{2}}\|_{L^1}^2) \\
&\le \frac{2d_1(p-1)}{p}\|\nabla u^{\frac{p}{2}}\|_{L^2}^2 + C_3 p(p-1)(p^n+1)\|u^{\frac{p}{2}}\|_{L^1}^2,
\end{aligned} \tag{37}$$

where $C_3 = C_2[(\frac{C_2}{2d_1})^{\frac{n}{2}} + 1]$. Noting that $\int_\Omega u^{p-2}|\nabla u|^2 = \frac{4}{p^2}\int_\Omega |\nabla u^{\frac{p}{2}}|^2$ and that $p^n + 1 \le (p+1)^n$, one derives that

$$\frac{d}{dt}\int_\Omega u^p + p(p-1)\int_\Omega u^p \le C_3 p(p-1)(p+1)^n \left(\int_\Omega u^{\frac{p}{2}}\right)^2 \tag{38}$$

from (36) and (37). Furthermore, it follows from (38) that

$$\int_\Omega u^p \le \int_\Omega u_0^p + C_3(p+1)^n \sup_{0\le t\le T_{max}}\left(\int_\Omega u^{\frac{p}{2}}\right)^2. \tag{39}$$

Denote

$$N(p) = \max\left\{\|u_0\|_{L^\infty}, \sup_{0\le t\le T_{max}}\left(\int_\Omega u^p\right)^{\frac{1}{p}}\right\}$$

and let $C_4 = C_3 + |\Omega|$. Then it follows from (39) that

$$N(p) \leq C_4^{\frac{1}{p}} (p+1)^{\frac{n}{p}} N\left(\frac{p}{2}\right).$$

Taking $p = 2^j$, $j = 1, 2, ...$, one has that

$$\begin{aligned} N(2^j) &\leq C_4^{2^{-j}} (1+2^j)^{n2^{-j}} N(2^{j-1}) \\ &\leq \prod_{k=1}^{j} C_4^{2^{-k}} (1+2^k)^{n2^{-k}} N(1) \\ &\leq \prod_{k=1}^{j} (1+2^{-k})^{n2^{-k}} \left(C_4^{\sum_{k=1}^{j} 2^{-k}}\right) \left(2^{\sum_{k=1}^{j} kn2^{-k}}\right) N(1) \\ &\leq 2^{3n} C_4 N(1). \end{aligned} \tag{40}$$

Since $u \in L^1(\Omega \times [0,T])$, we get $N(1) \leq C_5$. Letting $j \to \infty$ in (40), one has

$$\|u\|_{L^\infty} \leq 2^{3n} C_5 N(1) \leq C_6$$

for all $t \in (0, T_{max})$. Performing the same procedure to v, we can get a constant $C_7 > 0$ such that $\|v\|_{L^\infty} \leq C_7$ for all $t \in (0, T_{max})$. This completes the proof. □

Now we are in a position to prove Theorem 2.1.

Proof of Theorem 2.1. The extension criterion in Lemma 2.2 with Lemma 2.6 and Lemma 2.7 gives Theorem 2.1 immediately.

3. Global stability of the solution.

In this section, we will investigate the asymptotical behavior of the solution to the system (4) with constant $r(x)$ and establish the global stability of spatially homogeneous steady states. Since the spatially homogeneous steady states can not be explicitly found without specifying the general function F_i and h_i, it is very hard to derive a general stability result. In what follows, we shall focus on the following typical situation where

$$F_i(w) = w,\ h_1(u) = b_1 u \text{ and } h_2(v) = c_2 v.$$

That is we study the global stability of solution for the following diffusion-advection competition model

$$\begin{cases} u_t = d_1 \Delta u - \nabla \cdot (\lambda_1 u \nabla \chi_1(w)) + u(a_1 w - b_1 u - c_1 v), & x \in \Omega,\ t > 0, \\ v_t = d_2 \Delta v - \nabla \cdot (\lambda_2 v \nabla \chi_2(w)) + v(a_2 w - b_2 u - c_2 v), & x \in \Omega,\ t > 0, \\ w_t = \Delta w - w(u+v) + \mu w(1-w), & x \in \Omega,\ t > 0, \\ \partial_\nu u = \partial_\nu v = \partial_\nu w = 0, & x \in \partial\Omega,\ t > 0, \\ (u, v, w)(x, 0) = (u_0, v_0, w_0)(x), & x \in \Omega, \end{cases} \tag{41}$$

where we have assumed $r(x) = 1$. One of the benefits of considering (41) is we can compare the global stability results of (41) with the classical diffusion-competition model (2) or (3) to find the role of resource dynamics and/or the advection (see Remark 3.2). We remark that the model (41) is comparable with a model with dynamical resource considered in [40] where the density-dependent diffusion was discussed. In (41), we focus on the density-dependent advection and will examine the effect of advection on the global dynamics.

The spatially homogeneous steady states of (41) are determined by the following equations

$$\begin{cases} u(a_1 w - b_1 u - c_1 v) = 0, \\ v(a_2 w - b_2 u - c_2 v) = 0, \\ w(\mu w - \mu + u + v) = 0. \end{cases} \tag{42}$$

Except two trivial solutions: extinction state $(0,0,0)$ and resource-only state $(0,0,1)$, (42) has some other non-trivial solutions depending on the value of parameters a_i, b_i, c_i which can be divided into the following three categories similar to the Lotka-Volterra competition system (2):

Case 1: $\frac{c_1}{c_2} < \frac{a_1}{a_2} < \frac{b_1}{b_2}$ (weak competition);
Case 2: $\frac{a_1}{a_2} < \min\{\frac{b_1}{b_2}, \frac{c_1}{c_2}\}$ (v is superior to u in the competition);
Case 3: $\frac{a_1}{a_2} > \max\{\frac{b_1}{b_2}, \frac{c_1}{c_2}\}$ (u is superior to v in the competition).

For convenience, we denote

$$L := \mu(b_2 c_1 - b_1 c_2) + a_1(b_2 - c_2) + a_2(c_1 - b_1).$$

One can check that $L < 0$ in Case 1 ($\frac{c_1}{c_2} < \frac{a_1}{a_2} < \frac{b_1}{b_2}$). Then the corresponding homogeneous steady state (u_s, v_s, w_s) can be explicitly solved as

$$(u_s, v_s, w_s) = \begin{cases} (u_1^*, v_1^*, w_1^*) \text{ or } (0, v_2^*, w_2^*) \text{ or } (u_3^*, 0, w_3^*), & \text{in Case 1}, \\ (0, v_2^*, w_2^*) \text{ or } (u_3^*, 0, w_3^*), & \text{in Case 2 and Case 3}, \end{cases}$$

where

$$(u_1^*, v_1^*, w_1^*) := \frac{\mu}{L}(a_2 c_1 - a_1 c_2, a_1 b_2 - a_2 b_1, b_2 c_1 - b_1 c_2) \tag{43}$$

and

$$(v_2^*, w_2^*) := \left(\frac{\mu a_2}{a_2 + \mu c_2}, \frac{\mu c_2}{a_2 + \mu c_2}\right), \quad (u_3^*, w_3^*) := \left(\frac{\mu a_1}{a_1 + \mu b_1}, \frac{\mu b_1}{a_1 + \mu b_1}\right). \tag{44}$$

To state our results on the large time behavior of the solution, we further introduce some notations. Denote

$$K_i := \max_{0 \le w \le K} |\chi_i'(w)|^2, \quad i = 1, 2, \tag{45}$$

where K is defined in (5). Let

$$\begin{cases} \delta_1 = (a_1 b_2 + a_2 c_1)^2 - 4a_1 a_2 b_1 c_2, & \text{in Case 1}, \\ \delta_2 = a_1(b_2 + c_2)^2 - 4a_2 b_1 c_2, & \text{in Case 2}, \\ \delta_3 = a_2(b_1 + c_1)^2 - 4a_1 b_1 c_2, & \text{in Case 3}, \end{cases} \tag{46}$$

and

$$\begin{cases} \mu_1^* = \frac{c_1 c_2 (a_1 c_2 + a_2 b_1 - a_1 b_2)^2 + b_1 b_2 (a_1 c_2 - a_2 c_1 + a_2 b_1)^2}{4 b_2 c_1 (a_1 c_2 + a_2 b_1)(b_1 c_2 - b_2 c_1)}, & \text{in Case 1}, \\ \mu_2^* = \frac{(a_1(b_2 + c_2) - 2a_2 b_1)^2}{4 b_1 c_2 (a_2 b_1 - a_1 b_2)}, & \text{in Case 2}, \\ \mu_3^* = \frac{(a_2(b_1 + c_1) - 2a_1 c_2)^2}{4 b_1 c_2 (a_1 c_2 - a_2 c_1)}, & \text{in Case 3}. \end{cases} \tag{47}$$

Then the global stability results are stated in the following theorem.

Theorem 3.1. *Let the conditions in Theorem 2.1 hold and suppose (u, v, w) is the solution obtained in Theorem 2.1 with $r(x) = 1$. Then the following convergence results hold.*

1. *Assume the parameters satisfy* $\frac{c_1}{c_2} < \frac{a_1}{a_2} < \frac{b_1}{b_2}$ *(weak competition) and suppose* $\frac{\lambda_1^2 K_1}{4b_1 d_1} + \frac{\lambda_2^2 K_2}{4c_2 d_2} \leq 1$ *("=" holds if* $\|w_0\|_{L^\infty} \leq 1$*). Then*
$$\|(u,v,w) - (u_1^*, v_1^*, w_1^*)\|_{L^\infty(\Omega)} \to 0 \text{ as } t \to \infty$$
if $\delta_1 < 0$ *or* $\delta_1 \geq 0$ *and* $\mu > \mu_1^*$.
2. *Assume the parameters satisfy* $\frac{a_1}{a_2} < \min\{\frac{b_1}{b_2}, \frac{c_1}{c_2}\}$ *(predation capability of* v *is stronger) and* $\frac{\lambda_2^2 K_2}{4c_2 d_2} \leq 1$ *("=" holds if* $\|w_0\|_{L^\infty} \leq 1$ *). If* $\delta_2 < 0$ *or* $\delta_2 \geq 0$ *and* $\mu > \mu_2^*$*, then*
$$\|(u,v,w) - (0, v_2^*, w_2^*)\|_{L^\infty(\Omega)} \to 0 \text{ as } t \to \infty.$$
3. *Assume the parameters satisfy* $\frac{a_1}{a_2} > \max\{\frac{b_1}{b_2}, \frac{c_1}{c_2}\}$ *(predation capability of* u *is stronger) and* $\frac{\lambda_1^2 K_1}{4b_1 d_1} \leq 1$ *("=" holds if* $\|w_0\|_{L^\infty} \leq 1$ *). If* $\delta_3 < 0$ *or* $\delta_3 \geq 0$ *and* $\mu > \mu_3^*$*, then*
$$\|(u,v,w) - (u_3^*, 0, w_3^*)\|_{L^\infty(\Omega)} \to 0 \text{ as } t \to \infty.$$

Remark 3.2. *The Lyapunov method used in the paper to prove Theorem 3.1 essentially relies on that* $r(x)$ *is constant. When* $r(x)$ *is not constant, the asymptotic behavior of the solution to (41) remains open and we shall use numerical simulations to demonstrate that advective sensory response functions* $\chi_i(w)$ *are indeed important in determining the competition outcomes. We also mention that* K_i *defined in (45) is a quantity accounting for the intensity of biased movement (or advection) towards the resource. Hence the condition* $\frac{\lambda_1^2 K_1}{4b_1 d_1} + \frac{\lambda_2^2 K_2}{4c_2 d_2} \leq 1$ *can be understood that the advection of both species is weak. Similar interpretation can be applied to the condition* $\frac{\lambda_1^2 K_1}{4b_1 d_1} \leq 1$ *or* $\frac{\lambda_2^2 K_2}{4c_2 d_2} \leq 1$*. Although we obtain the global stability of the coexistence and exclusion steady states similar to the classical diffusion-competition model (2) without resource dynamics and advection, the conditions are much more complicated due to the complexity caused by the resource dynamics and advection. A further understanding of global dynamics is still very much demanded.*

We shall prove Theorem 3.1 by constructing the Lyapunov functionals alongside the following well-known result.

Lemma 3.3. *(Barălat's Lemma [6]) Suppose that* $h : [1, \infty) \to \mathbb{R}$ *is a uniformly continuous function such that* $\lim_{t\to\infty} \int_1^t h(s)ds$ *exists, then* $\lim_{t\to\infty} h(t) = 0$.

We also need higher regularity of the solution given in the following.

Lemma 3.4. *Let* (u, v, w) *be the unique global bounded classical solution of (41) given by Theorem 2.1 with* $r(x) = 1$*. Then for any given* $0 < \alpha < 1$*, there exists a constant* $C(\alpha) > 0$ *such that*
$$\|u\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{\Omega}\times[1,\infty))} + \|v\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{\Omega}\times[1,\infty))} + \|w\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{\Omega}\times[1,\infty))} \leq C(\alpha). \quad (48)$$

Proof. This proof is based on the standard regularity for parabolic equations. For readers' convenience, we sketch the proof here. Due to the boundedness of (u, v, w), applying the interior L^p estimate([30]) to (41), we derive that
$$\|u\|_{W_p^{2,1}(\Omega\times[i+\frac{1}{4},i+3])} + \|v\|_{W_p^{2,1}(\Omega\times[i+\frac{1}{4},i+3])} + \|w\|_{W_p^{2,1}(\Omega\times[i+\frac{1}{4},i+3])} \leq C_1, \ \forall i \geq 0. \quad (49)$$
Using the Sobolev embedding theorem, we derive
$$\|u\|_{C^{1+\alpha,\frac{1+\alpha}{2}}(\bar{\Omega}\times[\frac{1}{4},\infty))} + \|v\|_{C^{1+\alpha,\frac{1+\alpha}{2}}(\bar{\Omega}\times[\frac{1}{4},\infty))} + \|w\|_{C^{1+\alpha,\frac{1+\alpha}{2}}(\bar{\Omega}\times[\frac{1}{4},\infty))} \leq C_2. \quad (50)$$

Applying (50) and the Schauder estimate [28] to the third equation of (41), we obtain

$$\|w\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{\Omega}\times[i+\frac{1}{3},i+3])} \leq C_3,\ \forall i \geq 0, \tag{51}$$

which implies

$$\|w\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{\Omega}\times[\frac{1}{3},+\infty))} \leq C_4. \tag{52}$$

Rewrite the first equation in (41) as

$$u_t - d_1\Delta u + \lambda_1\chi_1'(w)\nabla w\cdot\nabla u = G(x,t),\ x\in\Omega,\ t>0, \tag{53}$$

where

$$G(x,t) = -\lambda_1 u\chi_1''(w)|\nabla w|^2 - \lambda_1 u\chi_1'(w)\Delta w + u(a_1 w - b_1 u - c_1 v).$$

Due to (50) and (51), we see that

$$\|G\|_{C^{\alpha,\frac{\alpha}{2}}(\bar{\Omega}\times[i+\frac{1}{3},i+3])} + \|\lambda_1\chi_1'(w)\nabla w\|_{C^{\alpha,\frac{\alpha}{2}}(\bar{\Omega}\times[i+\frac{1}{3},i+3])} \leq C_5,\ \forall i\geq 0.$$

Applying the Schauder estimate to (53) we have $\|u\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{\Omega}\times[i+1,i+3])} \leq C_6$ for all $i \geq 0$. Thus

$$\|u\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{\Omega}\times[1,+\infty))} \leq C_7. \tag{54}$$

Similarly, we can apply the Schauder estimate to the second equation in (41) and obtain

$$\|v\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{\Omega}\times[1,+\infty))} \leq C_8. \tag{55}$$

Then (48) follows from (52), (54) and (55). This completes the proof of Lemma 3.4. □

Next we shall prove Theorem 3.1 and split out analysis into two different cases.

Case 1: $\frac{c_1}{c_2} < \frac{a_1}{a_2} < \frac{b_1}{b_2}$. In this case, we can easily check that u_1^*, v_1^* and w_1^* defined by (43) are all positive. Then we consider the following energy functional

$$\mathcal{E}_1(t) = \xi_1 I_1(t) + \eta_1 I_2(t) + I_3(t), \tag{56}$$

where

$$I_1(t) = \int_\Omega \left(u - u_1^* - u_1^*\ln\frac{u}{u_1^*}\right), I_2(t) = \int_\Omega \left(v - v_1^* - v_1^*\ln\frac{v}{v_1^*}\right),$$
$$I_3(t) = \int_\Omega \left(w - w_1^* - w_1^*\ln\frac{w}{w_1^*}\right)$$

and $\xi_1, \eta_1 > 0$ are constants defined by

$$\xi_1 = \begin{cases} \frac{1}{a_1}, & \delta_1 < 0, \\ \frac{b_2}{a_1c_2+a_2b_1}, & \delta_1 \geq 0, \end{cases} \quad\text{and}\quad \eta_1 = \begin{cases} \frac{1}{a_2}, & \delta_1 < 0, \\ \frac{c_1}{a_1c_2+a_2b_1}, & \delta_1 \geq 0 \end{cases} \tag{57}$$

with δ_1 is defined in (46).

Lemma 3.5. *Suppose that $\frac{c_1}{c_2} < \frac{a_1}{a_2} < \frac{b_1}{b_2}$, $r(x) = 1$ and δ_1, μ_1^* are defined by (46) and (47), respectively. Let $\mathcal{E}_1(t)$ be the energy functional defined by (56). Then the following results hold.*

(1) $\mathcal{E}_1(t) \geq 0$ for all $t > 0$.

(2) Assume that

$$\frac{\lambda_1^2 K_1}{4b_1 d_1} + \frac{\lambda_2^2 K_2}{4c_2 d_2} \le 1 \quad (\text{“} = \text{” holds if } \|w_0\|_{L^\infty} \le 1), \tag{58}$$

where K_1, K_2 are defined by (45). There exists two constants $\alpha_1 > 0$ and $T_1 > 0$ such that

$$\frac{d}{dt}\mathcal{E}_1(t) \le -\alpha_1 \mathcal{F}_1(t) \tag{59}$$

holds for all $t > T_1$ either $\delta_1 < 0$ or $\delta_1 \ge 0$ and $\mu > \mu_1^*$, where

$$\mathcal{F}_1(t) = \int_\Omega (u-u_1^*)^2 + \int_\Omega (v-v_1^*)^2 + \int_\Omega (w-w_1^*)^2. \tag{60}$$

Proof. First, we show that $\mathcal{E}_1(t) \ge 0$ for all $t > 0$. To this end, we define $\varphi(z) := z - u_1^* \ln z$ for $z > 0$, which yields $\varphi'(z) = 1 - \frac{u_1^*}{z}$ and $\varphi''(z) = \frac{u_1^*}{z^2}$. By Taylor's expansion, we can find a constant $\xi > 0$ between u and u_1^* such that

$$u - u_1^* - u_1^* \ln \frac{u}{u_1^*} = \varphi(u) - \varphi(u_1^*) = \frac{\varphi''(\xi)}{2}(u-u_1^*)^2 = \frac{u_1^*}{2\xi^2}(u-u_1^*)^2 \ge 0,$$

which implies $I_1(t) \ge 0$. Similarly, we have that $I_2(t) \ge 0$ and $I_3(t) \ge 0$. Therefore, by (56), $\mathcal{E}_1(t) \ge 0$ for all $t > 0$ since $\xi_1, \eta_1 > 0$.

Next we show $\mathcal{E}_1(t)$ satisfies (59) under certain conditions. Indeed with the fact that $a_1 w_1^* - b_1 u_1^* - c_1 v_1^* = 0$, we have

$$\begin{aligned}
\frac{d}{dt} I_1(t) &= \int_\Omega \left(1 - \frac{u_1^*}{u}\right) u_t \\
&= -u_1^* \int_\Omega \frac{d_1 |\nabla u|^2}{u^2} + \lambda_1 u_1^* \int_\Omega \frac{\chi_1'(w) \nabla u \cdot \nabla w}{u} + \int_\Omega (u-u_1^*)(a_1 w - b_1 u - c_1 v) \\
&= -u_1^* \int_\Omega \frac{d_1 |\nabla u|^2}{u^2} + \lambda_1 u_1^* \int_\Omega \frac{\chi_1'(w) \nabla u \cdot \nabla w}{u} - c_1 \int_\Omega (u-u_1^*)(v-v_1^*) \\
&\quad - b_1 \int_\Omega (u-u_1^*)^2 + a_1 \int_\Omega (u-u_1^*)(w-w_1^*).
\end{aligned} \tag{61}$$

Similarly, from the second and third equations of system (41), we get

$$\begin{aligned}
\frac{d}{dt} I_2(t) = &-v_1^* \int_\Omega \frac{d_2 |\nabla v|^2}{v^2} + \lambda_2 v_1^* \int_\Omega \frac{\chi_2'(w) \nabla v \cdot \nabla w}{v} - b_2 \int_\Omega (u-u_1^*)(v-v_1^*) \\
&- c_2 \int_\Omega (v-v_1^*)^2 + a_2 \int_\Omega (v-v_1^*)(w-w_1^*)
\end{aligned} \tag{62}$$

and

$$\begin{aligned}
\frac{d}{dt} I_3(t) = &-w_1^* \int_\Omega \frac{|\nabla w|^2}{w^2} - \int_\Omega (u-u_1^*)(w-w_1^*) - \int_\Omega (v-v_1^*)(w-w_1^*) \\
&- \mu \int_\Omega (w-w_1^*)^2,
\end{aligned} \tag{63}$$

where we have used identities $a_2 w_1^* = b_2 u_1^* + c_2 v_1^*$ and $u_1^* + v_1^* = \mu(1 - w_1^*)$. Combining (61)–(63) with (56) gives that

$$\frac{d}{dt}\mathcal{E}_1(t) = -\int_\Omega X_1 A_1 X_1^T - \int_\Omega Y_1 B_1 Y_1^T, \tag{64}$$

where $X_1 = (u-u_1^*, v-v_1^*, w-w_1^*)$ and $Y_1 = \left(\frac{\nabla u}{u}, \frac{\nabla v}{v}, \nabla w\right)$ and A_1, B_1 are matrices denoted by

$$A_1 := \begin{pmatrix} b_1\xi_1 & \frac{c_1\xi_1+b_2\eta_1}{2} & \frac{1-a_1\xi_1}{2} \\ \frac{c_1\xi_1+b_2\eta_1}{2} & c_2\eta_1 & \frac{1-a_2\eta_1}{2} \\ \frac{1-a_1\xi_1}{2} & \frac{1-a_2\eta_1}{2} & \mu \end{pmatrix},$$

and

$$B_1 := \begin{pmatrix} \xi_1 u_1^* d_1 & 0 & -\frac{\lambda_1\xi_1 u_1^*\chi_1'(w)}{2} \\ 0 & \eta_1 v_1^* d_2 & -\frac{\lambda_2\eta_1 v_1^*\chi_2'(w)}{2} \\ -\frac{\lambda_1\xi_1 u_1^*\chi_1'(w)}{2} & -\frac{\lambda_2\eta_1 v_1^*\chi_2'(w)}{2} & \frac{w_1^*}{w^2} \end{pmatrix}.$$

Next, we shall show the nonnegativity of the matrices A_1 and B_1. When $\delta_1 < 0$, we let $\xi_1 = \frac{1}{a_1}$ and $\eta_1 = \frac{1}{a_2}$. Then

$$|A_{11}| := \begin{vmatrix} b_1\xi_1 & \frac{c_1\xi_1+b_2\eta_1}{2} \\ \frac{c_1\xi_1+b_2\eta_1}{2} & c_2\eta_1 \end{vmatrix} = \frac{-\delta_1}{4a_1^2a_2^2} > 0 \quad \text{and} \quad |A_1| = \mu|A_{11}| > 0.$$

When $\delta_1 \geq 0$, we choose $\xi_1 = \frac{b_2}{a_1c_2+a_2b_1}$ and $\eta_1 = \frac{c_1}{a_1c_2+a_2b_1}$. Then one can derive that

$$|A_{11}| = \frac{b_2c_1(b_1c_2-b_2c_1)}{(a_1c_2+a_2b_1)^2}$$

and

$$\begin{aligned} |A_1| &= \mu|A_{11}| + \frac{1}{4(a_1c_2+a_2b_1)^3}\big(2b_2c_1(a_1c_2+a_2b_1-a_1b_2)(a_1c_2-a_2c_1+a_2b_1) \\ &\quad - c_1c_2(a_1c_2+a_2b_1-a_1b_2)^2 - b_1b_2(a_1c_2-a_2c_1+a_2b_1)^2\big) \\ &> |A_{11}|\left(\mu - \frac{c_1c_2(a_1c_2+a_2b_1-a_1b_2)^2 + b_1b_2(a_1c_2-a_2c_1+a_2b_1)^2}{4b_2c_1(a_1c_2+a_2b_1)(b_1c_2-b_2c_1)}\right). \end{aligned}$$

Therefore under the conditions $\frac{c_1}{c_2} < \frac{a_1}{a_2} < \frac{b_1}{b_2}$ and $\mu > \mu_1^*$ defined in (46), one has that $|A_{11}| > 0$ and $|A_1| > 0$. Based on Sylvester's criterion, the matrix A_1 is positive defined and we can find a constant $\alpha_1 > 0$ such that

$$X_1A_1X_1^T \geq \alpha_1|X_1|^2, \quad \text{if } \delta_1 < 0 \text{ or } \delta_1 \geq 0 \text{ and } \mu > \mu_1^*. \tag{65}$$

For B_1, first we see $\xi_1u_1^*d_1 > 0$ and hence

$$\begin{vmatrix} \xi_1u_1^*d_1 & 0 \\ 0 & \eta_1v_1^*d_2 \end{vmatrix} = \xi_1\eta_1u_1^*v_1^*d_1d_2 > 0.$$

To proceed, we claim that

$$\frac{\xi_1u_1^*}{w_1^*} < \frac{1}{b_1} \quad \text{and} \quad \frac{\eta_1v_1^*}{w_1^*} < \frac{1}{c_2} \tag{66}$$

if $\frac{c_1}{c_2} < \frac{a_1}{a_2} < \frac{b_1}{b_2}$. In fact, since $\frac{b_1}{b_2} > \frac{a_1}{a_2}$, we have $a_2b_1 > a_1b_2$, which implies that

$$\frac{a_1c_2-a_2c_1}{a_1(b_1c_2-b_2c_1)} < \frac{1}{b_1} \quad \Leftrightarrow \quad \frac{u_1^*}{a_1w_1^*} < \frac{1}{b_1} \tag{67}$$

by recalling the definition of u_1^* and w_1^*. On the other hand, one has

$$\frac{b_2u_1^*}{(a_1c_2+a_2b_1)w_1^*} < \frac{b_2u_1^*}{a_2b_1w_1^*} < \frac{u_1^*}{a_1w_1^*} < \frac{1}{b_1} \tag{68}$$

thanks to $a_2b_1 > a_1b_2$ and (67). The combination of (67) and (68) gives that $\frac{\xi_1u_1^*}{w_1^*} < \frac{1}{b_1}$. Similarly, we can derive that $\frac{\eta_1v_1^*}{w_1^*} < \frac{1}{c_2}$. Hence, (66) holds in Case 1.

Furthermore, we can find $T_1 > 0$ such that for all $t > T_1$

$$\frac{\lambda_1^2 \xi_1 u_1^* w^2 |\chi_1'(w)|^2}{4w_1^* d_1} + \frac{\lambda_2^2 \eta_1 v_1^* w^2 |\chi_2'(w)|^2}{4w_1^* d_2} < 1 \tag{69}$$

due to (58) and (66). In fact first we see $\|w(\cdot,t)\|_{L^\infty} \le 1$ if $\|w_0\|_{L^\infty} \le 1$ by (8). Then it follows from (58) and (66) that

$$\frac{\lambda_1^2 \xi_1 u_1^* w^2 |\chi_1'(w)|^2}{4w_1^* d_1} + \frac{\lambda_2^2 \eta_1 v_1^* w^2 |\chi_2'(w)|^2}{4w_1^* d_2} < \frac{\lambda_1^2 w^2 K_1}{4b_1 d_1} + \frac{\lambda_2^2 w^2 K_2}{4c_2 d_2} \le \frac{\lambda_1^2 K_1}{4b_1 d_1} + \frac{\lambda_2^2 K_2}{4c_2 d_2} \le 1.$$

If $\|w_0\|_{L^\infty} > 1$, we suppose that $\frac{\lambda_1^2 K_1}{4b_1 d_1} + \frac{\lambda_2^2 K_2}{4c_2 d_2} < 1$ holds, then there exists a constant $\varepsilon_0 > 0$ such that

$$\frac{\lambda_1^2 K_1}{4b_1 d_1} + \frac{\lambda_2^2 K_2}{4c_2 d_2} + \varepsilon_0 \le 1. \tag{70}$$

Since $w \in C^{2,1}(\bar{\Omega} \times (0,\infty))$, it follows from (9) that

$$\limsup_{t\to\infty} \left(\frac{\lambda_1^2 w^2 K_1}{4b_1 d_1} + \frac{\lambda_2^2 w^2 K_2}{4c_2 d_2} \right) \le \frac{\lambda_1^2 K_1}{4b_1 d_1} + \frac{\lambda_2^2 K_2}{4c_2 d_2},$$

which allows us to find a constant $T_1 > 0$ such that

$$\frac{\lambda_1^2 w^2 K_1}{4b_1 d_1} + \frac{\lambda_2^2 w^2 K_2}{4c_2 d_2} \le \frac{\lambda_1^2 K_1}{4b_1 d_1} + \frac{K_2 \lambda_2^2}{4c_2 d_2} + \varepsilon_0 \tag{71}$$

for all $t > T_1$. The combination of (66) and (70)–(71) guarantees (69) in the case $\|w_0\|_{L^\infty} > 1$. From (69), we obtain directly that

$$|B_1| = \frac{\xi_1 \eta_1 u_1^* v_1^* w_1^* d_1 d_2}{w^2} \left(1 - \frac{\lambda_1^2 \xi_1 u_1^* w^2 |\chi_1'(w)|^2}{4w_1^* d_1} - \frac{\lambda_2^2 \eta_1 v_1^* w^2 |\chi_2'(w)|^2}{4w_1^* d_2} \right) > 0$$

for all $t > T_1$. The application of Sylvester's criterion enables us to get

$$Y_1 B_1 Y_1^T \ge 0$$

under the condition (58). Hence, the combination of (60), (64) and (65) yields that for all $t > T_1$,

$$\frac{d}{dt}\mathcal{E}_1(t) \le -\alpha_1 \mathcal{F}_1(t) \quad \text{either } \delta_1 < 0 \text{ or } \delta_1 \ge 0 \text{ and } \mu > \mu_1^*,$$

which yields (59). □

Lemma 3.6. *Suppose that the conditions of Lemma 3.5 hold. Then we have*

$$\|u(\cdot,t) - u_1^*\|_{L^\infty(\Omega)} + \|v(\cdot,t) - v_1^*\|_{L^\infty(\Omega)} + \|w(\cdot,t) - w_1^*\|_{L^\infty(\Omega)} \to 0 \quad \text{as } t \to \infty. \tag{72}$$

Proof. By Lemma 3.5, we have $\frac{d}{dt}\mathcal{E}_1(t) \le -\alpha_1 \mathcal{F}_1(t)$, where

$$\mathcal{E}_1(t) = \xi_1 \int_\Omega (u - u_1^* - u_1^* \ln \frac{u}{u_1^*}) + \eta_1 \int_\Omega (v - v_1^* - v_1^* \ln \frac{v}{v_1^*}) + \int_\Omega (w - w_1^* - w_1^* \ln \frac{w}{w_1^*}),$$

and

$$\mathcal{F}_1(t) = \int_\Omega (u - u_1^*)^2 + \int_\Omega (v - v_1^*)^2 + \int_\Omega (w - w_1^*)^2.$$

Since $\mathcal{E}_1(t) \ge 0$, we have

$$\int_1^\infty \mathcal{F}_1(t) \le \frac{1}{\alpha_1}\mathcal{E}_1(1) < \infty.$$

It follows from the regularity of u, v, w that $\mathcal{F}_1(t)$ is uniformly continuous in $[1, \infty)$. An application of Lemma 3.3 yields

$$\mathcal{F}_1(t) = \int_\Omega (u - u_1^*)^2 + \int_\Omega (v - v_1^*)^2 + \int_\Omega (w - w_1^*)^2 \to 0 \text{ as } t \to \infty. \tag{73}$$

By Lemma 3.4, we derive that $u(\cdot, t), v(\cdot, t)$ and $w(\cdot, t)$ are bounded for $t > 1$ in the space $W^{1,\infty}(\Omega)$. Applying the Gagliardo-Nirenberg inequality

$$\|\phi\|_\infty \le c\|\phi\|_{W^{1,\infty}(\Omega)}^{\frac{n}{n+2}} \|\phi\|_2^{\frac{2}{n+2}}, \ \forall \phi \in W^{1,\infty}(\Omega)$$

to $u - u_1^*, v - v_1^*$ and $w - w_1^*$, respectively. Thus we get (72) from (73). □

Case 2: $\frac{a_1}{a_2} < \min\{\frac{b_1}{b_2}, \frac{c_1}{c_2}\}$**.** In this case, we employ the following energy functional

$$\mathcal{E}_2(t) := \xi_2 J_1(t) + \frac{1}{a_2} J_2(t) + J_3(t) \tag{74}$$

to study the asymptotic behavior of the solution (u, v, w) solving system (41), where

$$J_1(t) = \int_\Omega u, \ J_2(t) = \int_\Omega \left(v - v_2^* - v_2^* \ln \frac{v}{v_2^*} \right), \ J_3(t) = \int_\Omega \left(w - w_2^* - w_2^* \ln \frac{w}{w_2^*} \right).$$

(v_2^*, w_2^*) is given in (44) and

$$\xi_2 = \begin{cases} \frac{1}{a_1}, & \delta_2 < 0, \\ \frac{2a_2 b_1 - a_1 b_2}{a_1^2 c_2}, & \delta_2 \ge 0. \end{cases}$$

More precisely, we have the following results.

Lemma 3.7. *Let $\mathcal{E}_2(t)$ be the functional defined by (74) and $r(x) = 1$. Then for all $t > 0$, we have $\mathcal{E}_2(t) \ge 0$. Moreover, under the condition $\frac{a_1}{a_2} < \min\{\frac{b_1}{b_2}, \frac{c_1}{c_2}\}$ and*

$$\frac{\lambda_2^2 K_2}{4c_2 d_2} \le 1 \quad (\text{“}=\text{” holds if } \|w_0\|_{L^\infty} \le 1), \tag{75}$$

there exist two constants $\alpha_2 > 0$ and $T_2 > 0$ such that if $\delta_2 < 0$ or $\delta_2 \ge 0$ and $\mu > \mu_2^*$,

$$\frac{d}{dt}\mathcal{E}_2(t) \le -\alpha_2 \mathcal{F}_2(t), \tag{76}$$

where

$$\mathcal{F}_2(t) = \int_\Omega u^2 + \int_\Omega (v - v_2^*)^2 + \int_\Omega (w - w_2^*)^2$$

and μ_2^* is denoted by (47).

Proof. By the similar arguments as in Lemma 3.5, we apply the Taylor formula to obtain that $J_2 \ge 0$ and $J_3 \ge 0$. Hence, one derives directly that $\mathcal{E}_2(t) \ge 0$ thanks to the positiveness of u.

To proceed, we show that (76) is true. Noting that the homogeneous steady state v_2^* and w_2^* satisfies $c_2 v_2^* = a_2 w_2^*$ and $v_2^* = \mu(1 - w_2^*)$, which together with the equations of system (41) gives

$$\begin{aligned} \frac{d}{dt} J_1(t) &= a_1 \int_\Omega uw - b_1 \int_\Omega u^2 - c_1 \int_\Omega uv \\ &\le a_1 \int_\Omega u(w - w_2^*) - b_1 \int_\Omega u^2 - \frac{a_1 c_2}{a_2} \int_\Omega u(v - v_2^*) \end{aligned} \tag{77}$$

due to $\frac{a_1}{a_2} < \frac{c_1}{c_2}$ and

$$\begin{aligned}\frac{d}{dt}J_2(t) = a_2\int_\Omega (v-v_2^*)(w-w_2^*) - b_2\int_\Omega u(v-v_2^*) - c_2\int_\Omega (v-v_2^*)^2 \\ - v_2^*\int_\Omega \frac{d_2|\nabla v|^2}{v^2} + v_2^*\int_\Omega \frac{\lambda_2\chi_2'(w)\nabla v\cdot\nabla w}{v}\end{aligned} \tag{78}$$

as well as

$$\frac{d}{dt}J_3(t) = -\int_\Omega u(w-w_2^*) - \int_\Omega (v-v_2^*)(w-w_2^*) - \mu\int_\Omega (w-w_2^*)^2 - w_2^*\int_\Omega \frac{|\nabla w|^2}{w^2}. \tag{79}$$

Then the combination of (77)–(79) and (74) leads to

$$\frac{d}{dt}\mathcal{E}_2(t) \le -\int_\Omega X_2A_2X_2^T - \int_\Omega Y_2B_2Y_2^T, \tag{80}$$

where $X_2 = (u, v-v_2^*, w-w_2^*)$, $Y_2 = \left(\frac{\nabla v}{v}, \nabla w\right)$, and two matrixes A_2, B_2 are defined as follows

$$A_2 := \begin{pmatrix} b_1\xi_2 & \frac{b_2+\xi_2a_1c_2}{2a_2} & \frac{1-\xi_2a_1}{2} \\ \frac{b_2+\xi_2a_1c_2}{2a_2} & \frac{c_2}{a_2} & 0 \\ \frac{1-\xi_2a_1}{2} & 0 & \mu \end{pmatrix}, \qquad B_2 := \begin{pmatrix} \frac{v_2^*d_2}{a_2} & -\frac{\lambda_2v_2^*\chi_2'(w)}{2a_2} \\ -\frac{\lambda_2v_2^*\chi_2'(w)}{2a_2} & \frac{w_2^*}{w^2} \end{pmatrix}.$$

If $\delta_2 < 0$, we have $\xi_2 = \frac{1}{a_1}$ from the definition of ξ_2, and hence

$$|A_{21}| := \begin{pmatrix} b_1\xi_2 & \frac{b_2+\xi_2a_1c_2}{2a_2} \\ \frac{b_2+\xi_2a_1c_2}{2a_2} & \frac{c_2}{a_2} \end{pmatrix} = \frac{-\delta_2}{4a_1a_2^2} > 0, \qquad |A_2| = -\frac{\mu\delta_2}{4a_1a_2^2} > 0.$$

On the other hand, if $\delta_2 \ge 0$ we select $\xi_2 = \frac{2a_2b_1-a_1b_2}{a_1^2c_2}$ and derive that

$$|A_{21}| = \frac{b_1(a_2b_1-a_1b_2)}{a_1^2a_2} > 0$$

and

$$|A_2| = \mu|A_{21}| - \frac{(a_1(b_2+c_2)-2a_2b_1)^2}{4a_1^2a_2c_2} = |A_{21}|\left(\mu - \frac{(a_1(b_2+c_2)-2a_2b_1)^2}{4b_1c_2(a_2b_1-a_1b_2)}\right) > 0$$

when $\mu > \mu_2^*$. Hence, there exists a constant $\alpha_2 > 0$ such that

$$X_2A_2X_2^T \ge \alpha_2|X_2|^2, \quad \text{if } \delta_2 < 0 \text{ or } \delta_2 \ge 0 \text{ and } \mu > \mu_2^* \tag{81}$$

based on Sylvester's criterion. Under the condition (75), we can use the similar arguments as in Lemma 3.5 to find $T_2 > 0$ such that

$$\frac{\lambda_2^2w^2|\chi_2'(w)|^2}{4c_2d_2} < \frac{\lambda_2^2K_2}{4c_2d_2} \le 1 \quad \text{for all } t > T_2$$

(" = " holds if $\|w_0\|_{L^\infty} \le 1$), which implies that $\frac{c_2d_2}{w^2} > \frac{\lambda_2^2|\chi_2'(w)|^2}{4}$. Recalling the definition of v_2^* and w_2^*, one has $\frac{v_2^*d_2}{a_2} > 0$ and

$$|B_2| = \frac{v_2^*}{a_2^2}\left(\frac{a_2w_2^*d_2}{w^2} - \frac{\lambda_2^2v_2^*|\chi_2'(w)|^2}{4}\right) = \frac{\mu^2}{(a_2+\mu c_2)^2}\left(\frac{c_2d_2}{w^2} - \frac{\lambda_2^2|\chi_2'(w)|^2}{4}\right) > 0$$

for all $t > T_2$. Therefore, the matrix B_2 is positive definite and then $Y_2B_2Y_2^T \ge 0$, which together with (80) and (81), gives that

$$\frac{d}{dt}\mathcal{E}_2(t) \le -\alpha_2\mathcal{F}_2(t), \quad \text{if } \delta_2 < 0 \text{ or } \delta_2 \ge 0 \text{ and } \mu > \mu_2^*$$

for all $t > T_2$. Consequently, we finish the proof. □

Lemma 3.8. *Let* (u, v, w) *be the solution of system* (41) *and* (v_2^*, w_2^*) *be defined by* (44). *Assume that the conditions in Lemma 3.7 hold. Then it follows that*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t) - v_2^*\|_{L^\infty(\Omega)} + \|w(\cdot,t) - w_2^*\|_{L^\infty(\Omega)} \to 0 \quad \text{as } t \to \infty. \tag{82}$$

Proof. In Lemma 3.7, we have shown that the non-negative functional $\mathcal{E}_2(t)$ satisfies $\frac{d}{dt}\mathcal{E}_2(t) \le -\alpha_2 \mathcal{F}_2(t)$. It follows that

$$\int_1^\infty \mathcal{F}_2(t) \le \frac{1}{\alpha_2}\mathcal{E}_2(1) < \infty.$$

The regularity of the solution (u, v, w) entails that $\mathcal{F}_2(t)$ is uniformly continuous in $[1, \infty)$. Then the application of Lemma 3.3 yields

$$\mathcal{F}_2(t) = \int_\Omega u^2 + \int_\Omega (v - v_2^*)^2 + \int_\Omega (w - w_2^*)^2 \to 0 \text{ as } t \to \infty. \tag{83}$$

By Lemma 3.4, we derive that $u(\cdot,t), v(\cdot,t)$ and $w(\cdot,t)$ are bounded for $t > 1$ in the space $W^{1,\infty}(\Omega)$. By the Gagliardo-Nirenberg inequality

$$\|\phi\|_\infty \le c\|\phi\|_{W^{1,\infty}(\Omega)}^{\frac{n}{n+2}} \|\phi\|_2^{\frac{2}{n+2}}, \ \forall \phi \in W^{1,\infty}(\Omega),$$

we get the limit (82) from (83). □

Now we are in a position of prove Theorem 3.1.

Proof of Theorem 3.1. The assertions (1) and (2) of Theorem 3.1 are the results of Lemma 3.6 and Lemma 3.8. The proof of assertion (3) of Theorem 3.1 is completely parallel to that of assertion (2).

4. **Effects of resource dynamics and advective strategies: simulations and discussions.**

In this paper, we consider a three-species Lotka-Volterra diffusion-advection competition system (41) with a dynamical resource determined by an evolution equation where the advection is biased up the resource gradient. This model is an extension of existing Lotka-Volterra competition systems where the resource is a given spatially varying function. We first establish the global existence of classical solutions of (41) in two dimensions (see Theorem 2.1) using the maximum principle and Moser iteration. Then we investigate the global stability of constant equilibria (see Theorem 3.1) for the constant intrinsic growth rate of the resource (i.e. $r(x) = 1$) by the method of Lyapunov functional. While if the intrinsic growth rate of the resource $r(x)$ is non-constant, the global dynamics of (41) remains unknown, which is a very challenging question due to quite a number of equations and cross-diffusion structures. Before any analytical results can be proved, it would be instructive to numerically investigate the time-asymptotic dynamics to foresee the possible results.

Note that (41) is a complex system with many parameters and various mathematical questions can be numerically explored. Below we shall focus on two questions: (i) how does the resource dynamics affect the competition outcomes? (ii) how does the advective strategy affect the global dynamics of competing species? To this end, we shall always assume that two competing species u and v have the same ecological situations, namely

$$a_1 = b_1 = c_1 = a_2 = b_2 = c_2 = 1. \tag{84}$$

Then for the above question (i), we shall compare the competition outcomes of system (41) with (1), where (41) has dynamical resource w while the resource $m(x)$

in (1) has no dynamics but is a given spatially varying function. For question (ii), we shall mainly vary the forms of $\chi_1(w)$ and $\chi_2(w)$ that represent the advective strategies employed by the competing species, along with the variations of $d_i(i = 1, 2)$, λ_1 and λ_2, to see how the competition outcomes are changed. We perform the numerical simulations in an interval $\Omega = [0, 10]$ by implementing the Matlab PDEPE solver based on the finite difference scheme. In the simulations, we fix the initial value as

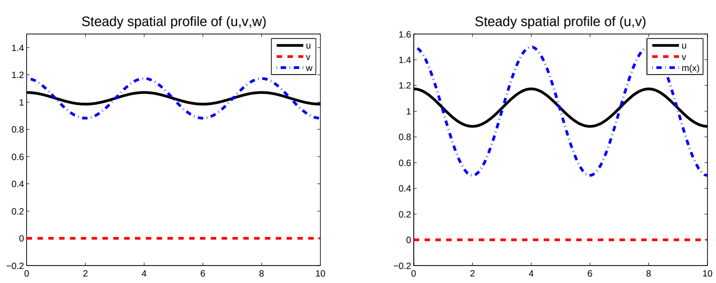
$$u_0(x) = 1 + \cos(\pi x),\ v_0(x) = 1 + \cos(\pi x),\ w_0(x) = 1 + 0.5\cos(\pi x). \tag{85}$$

The intrinsic growth rate function $r(x)$ for the dynamical resource w in (41) and the environmental resource $m(x)$ in (1) are set to be same as

$$r(x) = m(x) = 1 + 0.5\cos(\pi x/2). \tag{86}$$

When simulating systems (1) and (41) with advection (i.e. $\lambda_1, \lambda_2 > 0$), we need to specify the forms of sensory response functions $\chi_i(i = 1, 2)$. In the literature (cf. [15]), there are two major forms: $\chi_i(s) = s$ (linear response) and $\chi_i(s) = \ln s$ (logarithmic response). These two types of sensory responses have been widely used in chemotaxis system (cf. [27, 23, 29]). Therefore in our simulations, we shall employ the above two forms of sensory responses.

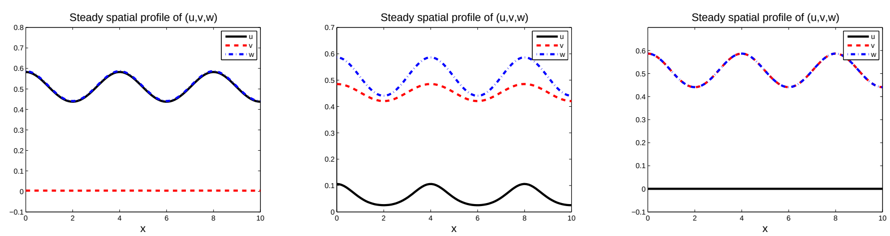
4.1. **Effects of resource dynamics.** The main effect of resource dynamics we will explore is whether the so-called "slower diffuser prevails" phenomenon occurred in the competition system (3) without advection (i.e. (1) with $\lambda_1 = \lambda_2 = 0$) still holds true for the dynamical resource. Therefore we set $d_1 = 1, d_2 = 10$ and numerically solve (41) with $\lambda_1 = \lambda_2 = 0$ for $m(x)$ and $r(x)$ given in (86). The numerical simulations for the large-time profile of the solution (i.e. steady spatial profile of the solution) are plotted in Fig. 1, from which we see that the slower species u wipes out its fast competitor v for both models (41) and (3). This implies the resource dynamics has no impact on the "slower diffuser prevails" phenomenon in the absence of advection.



(a) (b)

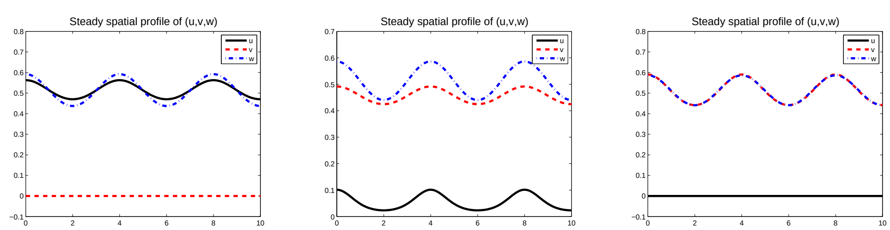
Figure 1. Numerical simulations of large-time profile of the solution to (41) shown in (a) and to (1) shown in (b), where $d_1 = 1, d_2 = 10, \lambda_1 = \lambda_2 = 0$, initial values are given in (85) and other parameter values are given in (84).

Next we proceed to explore the effect of resource dynamics on the competition outcomes in the presence of advection (biased movement) as a dispersal strategy aside from diffusion. In this case, we assume that two competing species have the same sensory responses (i.e. $\chi_1(w) = \chi_2(w)$ and $\lambda_1 = \lambda_2$), and the corresponding numerical results are plotted in Fig.2 for the logarithmic response and Fig.3 for the linear response, respectively. Remarkably we find that for both types of sensory responses, the slower diffuser u may wipe out its fast competitor (see Fig.2-(a) and Fig.3-(a)) if the advective responses are weak (i.e. $\lambda_1 = \lambda_2$ are small) or vice versa (see Fig.2-(c) and Fig.3-(c)) if the advective responses are strong (i.e. $\lambda_1 = \lambda_2$ are large). Moreover the two competing species may coexist if the advective responses are moderate (see Fig.2-(b) and Fig.3-(b)). However, for the competition system (1) without resource dynamics, our numerical simulations show that the slower diffuser always prevails as depicted in Fig.1-(b) for both linear and logarithmic sensory responses with the same parameter values given in Fig.2 and Fig.3. Hence we do not plot them here for brevity. These numerical simulations essentially indicate that as long as the advection (biased movement) towards the resource gradients) takes place, the global dynamics will be quite different between fixed and dynamical resources. In particular, if the resource has temporal dynamics, the slower diffuser may lose the competition and coexistence may also be achieved depending on the strength of advection, which is in sharp contrast to the case of fixed resource for which only the phenomenon "slower diffuser wins" happens. In addition to this, we find that when competitive exclusion occurs, the "ideal free distribution" (meaning the spatial distribution of winning species perfectly matches the resource distribution) may be achieved (see Fig.2-(a) or (c) and Fig.3-(c)).



(a) (b) (c)

FIGURE 2. Numerical simulations of large-time solution profiles of (41), where $d_1 = 1, d_2 = 10, \chi_1(w) = \chi_2(w) = \ln w$ and λ_1, λ_2 values are: (a) $\lambda_1 = \lambda_2 = 1$; (b) $\lambda_1 = \lambda_2 = 5$; (c) $\lambda_1 = 10, \lambda_2 = 10$.



(a) (b) (c)

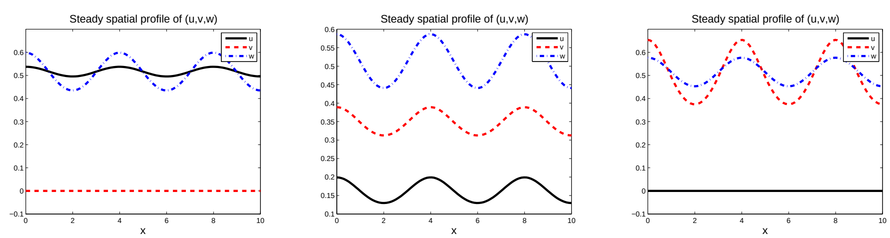
FIGURE 3. Numerical simulations of large-time solution profiles of (41), where $d_1 = 1, d_2 = 10, \chi_1(w) = \chi_2(w) = w$ and λ_1, λ_2 values are: (a) $\lambda_1 = \lambda_2 = 1$; (b) $\lambda_1 = \lambda_2 = 10$; (c) $\lambda_1 = 20, \lambda_2 = 20$.

4.2. **Effects of advective strategies.** In the preceding subsection, we numerically demonstrate the effects of resource dynamics on the competition outcomes and we find the resource dynamics may lead to more diverse competition outcomes if the advection, as a strategy of movement in contrast to diffusion, is employed. Below, we shall proceed to explore the effect of advective strategies on the competition outcomes. To this end, we let $d_1 = d_2$ and $\lambda_1 = \lambda_2$ and consider two cases:

(a) $\chi_1(w) = \ln w$ and $\chi_2(w) = w$, that is the species u uses logarithmic law and v uses linear law for the advection.
(b) $\chi_1(w) = \ln w$ (or $\chi_1(w) = w$) and $\chi_2(w) = 0$, that is the species u uses the advection while v does not.

In case (a), we can tell how the advective strategies impact the competition outcomes. In case (b), we can examine whether the advection is advantageous or not for the competition.

The numerical results for case (a) are shown in Fig.4, where we see that mutual exclusion between two competing species may occur (see Fig.4-(a) and Fig.4-(c)) if the advection strength is weak or strong, while the coexistence will be achieved if the advection strength is moderate (see Fig.4-(b)). Precisely, the species with



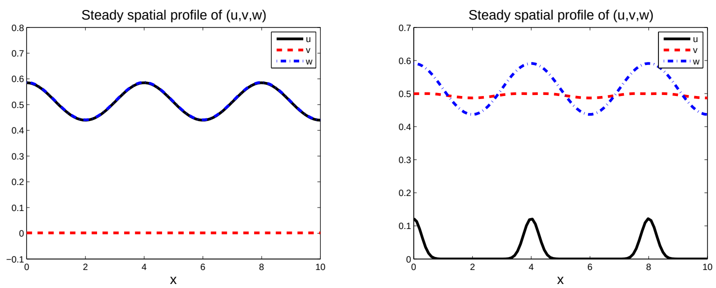
(a) (b) (c)

FIGURE 4. Numerical simulations of large-time solution profiles of (41) with $\chi_1(w) = \ln w, \chi_2(w) = w$, where $d_1 = d_2 = 1$ and λ_1, λ_2 values are: (a) $\lambda_1 = \lambda_2 = 0.1$; (b) $\lambda_1 = \lambda_2 = 1.5$; (c) $\lambda_1 = \lambda_2 = 5$.

logarithmic advection will win (resp. lose to) its competitor with linear advection if the advection is weak (resp. strong), while the two species will coexist regardless of advective strategies employed if the advection is moderate.

The numerical results for case (b) are plotted in Fig.5 for $\chi_1(w) = \ln w, \chi_2(w) = 0$. From the simulations, we find that the species with weak advection can wipe out the species without advection (see Fig.5-(a)) and achieve the ideal free distribution, while the species with strong advection can coexist with the species without advection although the supported population mass is much less than the weak advection (see Fig.5-(b)). The similar numerical results can be observed for the case $\chi_1(w) = w$ and $\chi_2(w) = 0$, where the only difference is that the species u can not achieve the ideal free distribution when the advection is weak. For brevity we do not plot the numerical results for this case here.

4.3. **Effect of advection strength.** From the previous subsection, we have observed that given the same advection strength (i.e. $\lambda_1 = \lambda_2$), the magnitude of advection strength plays an important role in determining the global dynamics of the solution (see Fig.2-Fig.4). Now we further explore the effect of advection strength when two competing species have unequal advection strength (i.e. $\lambda_1 \neq \lambda_2$) for the

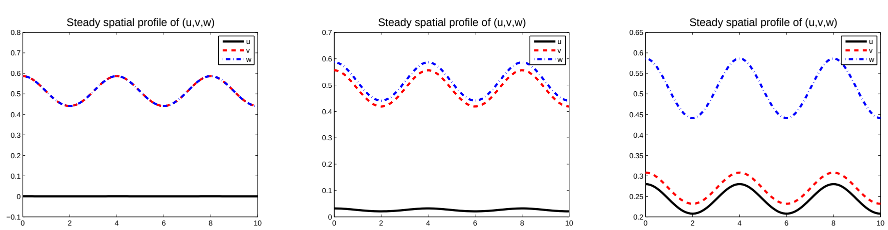


(a) $\lambda_1 = 1$ (b) $\lambda_1 = 50$

FIGURE 5. Numerical simulations of large-time solution profiles of (41) with $\chi_1(w) = \ln w, \chi_2(w) = 1$, where the value of λ_1 is indicated in the figure.

same advective sensory response mechanism (i.e. $\chi_1(w) = \chi_2(w)$). The numerical results are shown in Fig.6 for the logarithmic response and Fig.7 for the linear response.

For the logarithmic response functions $\chi_i(w) = \ln w (i = 1, 2)$, we find that if the ratio $\frac{\lambda_1}{\lambda_2}$ is large, then competitive exclusion will occur and the species with weaker advection strength wins (see Fig.6-(a)). While if $\frac{\lambda_1}{\lambda_2} \approx 1$, two competing species may coexist (see Fig.6-(b) and Fig.6-(c)) but the species with weaker advection strength have slight advantage in terms of the population number supported. For linear response functions $\chi_i(w) = w(i = 1, 2)$, we find that if $\frac{\lambda_1}{\lambda_2}$ is large, co-existence may be achieved (see Fig.7-(a) and Fig.7-(b)) which is substantially different from the case of logarithmic response functions shown in Fig.6-(a). If $\frac{\lambda_1}{\lambda_2} \approx 1$, co-existence may be achieved only if the advection strength of two species are very close (see Fig.7-(d)) and there is an intermediate value of $\frac{\lambda_1}{\lambda_2} = 1.5$ such that the competitive exclusion happens and the species with stronger advection strength wins (see Fig.7-(c)) which is distinct from the case of logarithmic responses shown in Fig.6-(b).

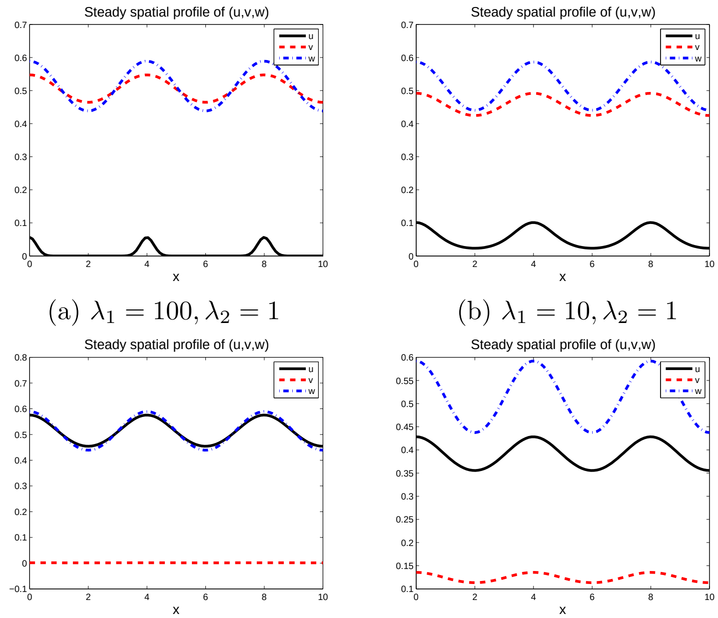


(a) $\lambda_1 = 10, \lambda_2 = 1$ (b) $\lambda_1 = 1.5, \lambda_2 = 1$ (c) $\lambda_1 = 1.05, \lambda_2 = 1$

FIGURE 6. Numerical simulations of large-time profile of the solution to (41), where $d_1 = d_2 = 1, \chi_1(w) = \chi_2(w) = \ln w$.

From numerical simulations shown in Fig.6 and Fig.7, we can make the following general conclusions.

- Given the same advective sensory response functions $\chi_i(w)$ $(i = 1, 2)$, the advective response strength ratio $\frac{\lambda_1}{\lambda_2}$ between species is a key factor determining the competition outcomes (see Fig.6 or Fig.7);



(a) $\lambda_1 = 100, \lambda_2 = 1$ (b) $\lambda_1 = 10, \lambda_2 = 1$

(c) $\lambda_1 = 1.5, \lambda_2 = 1$ (d) $\lambda_1 = 1.05, \lambda_2 = 1$

FIGURE 7. Numerical simulations of large-time profile of the solution to (41), where $d_1 = d_2 = 1, \chi_1(w) = \chi_2(w) = w$.

- Given the same ratio $\frac{\lambda_1}{\lambda_2}$, different advective sensory response functions $\chi_i(w)$ $(i = 1, 2)$ may result in different competition outcomes (compare Fig.6-(a) with Fig.7-(b) and Fig.6-(c) with Fig.7-(c)).

In summary, from the numerical simulations demonstrated above, we find that global dynamics of the competition system (41) are quite different from (1) if the advection, as a dispersal strategy different from diffusion, is taken into account. Typically the prevailing phenomenon "slower diffuser prevails" appeared in (1) may not occur in (41). This implies that the resource temporal dynamics is an indispensable factor to predict the competition outcomes more precisely. We also numerically find that different advective sensory mechanisms (linear or logarithmic) may give rise to very different competition outcomes. For the same advective sensory mechanism, the relative sensory response strength between species (i.e. the value of ratio $\frac{\lambda_1}{\lambda_2}$) is a key factor in determining which species will win the competition. The numerical simulations have partially illustrated wealthy dynamics and patterns of the competition system (41) when the resource intrinsic growth rate $r(x)$ is non-constant. However the justification of these numerical observations remains open. These are difficult questions to explore since the system (41) has three equations and cross diffusions. Our numerical simulations undoubtedly provide some useful clues for further studies.

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