

Strictly nonnegative tensors and nonnegative tensor partition

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Abstract In this paper, we introduce a new class of nonnegative tensors — strictly nonnegative tensors. A weakly irreducible nonnegative tensor is a strictly nonnegative tensor but not vice versa. We show that the spectral radius of a strictly nonnegative tensor is always positive. We give some sufficient and necessary conditions for the six well-conditional classes of nonnegative tensors, introduced in the literature, and a full relationship picture about strictly nonnegative tensors with these six classes of nonnegative tensors. We then establish global R-linear convergence of a power method for finding the spectral radius of a nonnegative tensor under the condition of weak irreducibility. We show that for a nonnegative tensor T , there always exists a partition of the index set such that every tensor induced by the partition is weakly irreducible; and the spectral radius of T can be obtained from those spectral radii of the induced tensors. In this way, we develop a convergent algorithm for finding the spectral radius of a general nonnegative tensor without any additional assumption. Some preliminary numerical results show the feasibility and effectiveness of the algorithm.

Keywords Nonnegative tensor, spectral radius, strict nonnegativity, weak irreducibility

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1 Introduction

Recently, the research topic on eigenvalues of nonnegative tensors has attracted much attention [2–8, 10–14, 16, 18–27]. Researchers studied the Perron-Frobenius theorem for nonnegative tensors and algorithms for finding the largest eigenvalue, i.e., the spectral radius, of a nonnegative tensor. Six well-conditional classes of nonnegative tensors have been introduced: irreducible nonnegative tensors [3, 12], essentially positive tensors [16], primitive tensors [4], weakly positive tensors [25], weakly irreducible nonnegative tensors [8] and weakly primitive tensors [8]. Zhang, Qi and Xu [25] concluded the relationships among the first four classes of nonnegative tensors. Friedland, Gaubert and Han [8] introduced weakly irreducible nonnegative tensors and weakly primitive tensors. These two classes, as their names suggest, are broader than the classes of irreducible nonnegative tensors and primitive tensors respectively.

In the next section, we propose a new class of nonnegative tensors, we call them *strictly nonnegative* tensors. We show that the class of strictly nonnegative tensors strictly contains the class of weakly

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irreducible nonnegative tensors mentioned above. We also prove that the spectral radius of a strictly nonnegative tensor is always positive. This further strengthens the Perron-Frobenius results for nonnegative tensors in the literature [3–5, 8, 18, 19].

In Section 3, we give sufficient and necessary conditions for the six well-conditional classes of nonnegative tensors, introduced in the literature, and a full relationship picture about strictly nonnegative tensors with these six classes of nonnegative tensors.

Friedland, Gaubert and Han [8] proposed a power method for finding the largest eigenvalue of a weakly irreducible nonnegative tensor, and established its R-linear convergence under the condition of weak primitivity. In Section 4, we modify that method and establish its global R-linear convergence for weakly irreducible nonnegative tensors.

It is well known for nonnegative matrices that: for a general nonnegative matrix, we can place it into an upper block triangular form with irreducible blocks through simultaneous row/column permutations, and the spectral radius is equal to the largest of the spectral radii of the block sub-matrices. Can this result be generalized for nonnegative tensors? The answer is not so obviously: there are two concepts of “irreducibility”, namely irreducibility [3, 12] and weak irreducibility [8]. Can the above result be generalized based on either, both or neither? We will give an affirmative answer to these questions. We show that the result can be generalized based on weak reducibility but not reducibility in Section 5. We show that for a nonnegative tensor T , always there exists a partition of the index set $\{1, \dots, n\}$ such that every tensor induced by the partition is weakly irreducible, and the largest eigenvalue of T can be obtained from those largest eigenvalues of the induced tensors.

In Section 6, based on the power method for weakly irreducible nonnegative tensors proposed in Section 4, we develop a convergent algorithm for finding the spectral radius of a general nonnegative tensor without any additional assumption. We report some preliminary numerical results of the proposed method for general nonnegative tensors. These numerical results demonstrate the feasibility and efficiency of the proposed algorithm. Conclusions and remarks are given in Section 7.

Here is some notation in this paper. A tensor T in the field \mathfrak{R} of real numbers of order m and dimension n with $m, n \geq 2$ is an m -way array which can be denoted by $(T_{i_1 \dots i_m})$ with $T_{i_1 \dots i_m} \in \mathfrak{R}$ for all $i_j \in \{1, \dots, n\}$ and $j \in \{1, \dots, m\}$. For a tensor T of order $m \geq 2$ and dimension $n \geq 2$, if there exist $\lambda \in \mathcal{C}$ and $x \in \mathcal{C}^n \setminus \{0\}$ such that

$$(Tx^{m-1})_i := \sum_{i_2, \dots, i_m=1}^n T_{ii_2 \dots i_m} x_{i_2} \cdots x_{i_m} = \lambda x_i^{m-1}, \quad \forall i \in \{1, \dots, n\} \quad (1.1)$$

holds, then λ is called an eigenvalue of T , x is called a corresponding eigenvector of T with respect to λ , and (λ, x) is called an eigenpair of T . This definition was introduced by Qi [17] when m is even and T is symmetric (i.e., $T_{j_1 \dots j_m} = T_{i_1 \dots i_m}$ among all the permutations (j_1, \dots, j_m) of (i_1, \dots, i_m)). Independently, Lim [12] gave such a definition but restricted x to be a real vector and λ to be a real number. Let $\mathfrak{R}_+^n := \{x \in \mathfrak{R}^n \mid x \geq 0\}$ and $\mathfrak{R}_{++}^n := \{x \in \mathfrak{R}^n \mid x > 0\}$. Suppose that T is a nonnegative tensor, i.e., that all of its entries are nonnegative. It can be seen that if we define function $F_T : \mathfrak{R}_+^n \rightarrow \mathfrak{R}_+^n$ associated the nonnegative tensor T as

$$(F_T)_i(x) := \left(\sum_{i_2, \dots, i_m=1}^n T_{ii_2 \dots i_m} x_{i_2} \cdots x_{i_m} \right)^{\frac{1}{m-1}} \quad (1.2)$$

for all $i \in \{1, \dots, n\}$ and $x \in \mathfrak{R}_+^n$, then (1.1) is strongly related to the eigenvalue problem for the nonlinear map F_T discussed in [15]. Denote by $\rho(T) := \max\{|\lambda| \mid \lambda \in \sigma(T)\}$ where $\sigma(T)$ is the set of all the eigenvalues of T . We call $\rho(T)$ and $\sigma(T)$ the spectral radius and the spectrum of the tensor T , respectively.

2 Strictly nonnegative tensors

In this section, we propose and analyze a new class of nonnegative tensors, namely *strictly nonnegative tensors*. To this end, we first recall several concepts related to nonnegative tensors in the literature [3, 4, 8, 12, 16, 25].

Definition 2.1. Suppose that T is a nonnegative tensor of order m and dimension n .

- T is called *reducible* if there exists a nonempty proper index subset $I \subset \{1, \dots, n\}$ such that

$$T_{i_1 i_2 \dots i_m} = 0, \quad \forall i_1 \in I, \quad \forall i_2, \dots, i_m \notin I. \tag{2.1}$$

If T is not reducible, then T is called *irreducible*.

- T is called *essentially positive* if $Tx^{m-1} \in \mathbb{R}_{++}^n$ for any nonzero $x \in \mathbb{R}_+^n$.
- T is called *primitive* if for some positive integer k , $F_T^k(x) \in \mathbb{R}_{++}^n$ for any nonzero $x \in \mathbb{R}_+^n$, here $F_T^k := F_T(F_T^{k-1})$.
- A nonnegative matrix $M(T)$ is called the *majorization* associated to the nonnegative tensor T , if the (i, j) -th element of $M(T)$ is defined to be $T_{ij \dots j}$ for any $i, j \in \{1, \dots, n\}$. T is called *weakly positive* if $[M(T)]_{ij} > 0$ for all $i \neq j$.

In Definition 2.1, the concepts of *reducibility* and *irreducibility* were proposed by Chang, Pearson and Zhang [3, Definition 2.1] and Lim [12, Page 131], the concept of *essential positivity* was given by Pearson [16, Definition 3.1], and the concept of *primitivity* was given by Chang, Pearson and Zhang [4, Definition 2.6] while we used its equivalent definition [4, Theorem 2.7]. In addition, the concept of *majorization* was given by Chang, Pearson and Zhang [3, Definition 3.5] and earned this name in [16, Definition 2.1]; and the concept of *weak positivity* was given by Zhang, Qi and Xu [25, Definition 3.1].

Friedland, Gaubert and Han [8] defined *weakly irreducible* polynomial maps and *weakly primitive* polynomial maps by using the strong connectivity of a graph associated to a polynomial map [9]. Their concepts for homogeneous polynomials gave the following two classes of nonnegative tensors.

Definition 2.2. Suppose that T is a nonnegative tensor of order m and dimension n .

- We call a nonnegative matrix $G(T)$ the *representation* associated to the nonnegative tensor T , if the (i, j) -th element of $G(T)$ is defined to be the summation of $T_{ii_2 \dots i_m}$ with indices $\{i_2, \dots, i_m\} \ni j$.
- We call the tensor T *weakly reducible* if its representation $G(T)$ is a reducible matrix, and *weakly primitive* if $G(T)$ is a primitive matrix. If T is not *weakly reducible*, then it is called *weakly irreducible*.

Now, we introduce *strictly nonnegative tensors*.

Definition 2.3. Suppose that T is a nonnegative tensor of order m and dimension n . Then, it is called *strictly nonnegative* if $F_T(x) > 0$ for any $x > 0$.

Lemma 2.1. An m -th order n dimensional nonnegative tensor T is strictly nonnegative if and only if the vector $R(T)$ with its i -th element being $\sum_{i_2, \dots, i_m=1}^n T_{ii_2 \dots i_m}$ is positive.

Proof. By Definition 2.3, $Te^{m-1} > 0$ with e being the vector of all ones. So,

$$(Te^{m-1})_i = \sum_{i_2, \dots, i_m=1}^n T_{ii_2 \dots i_m} > 0$$

for all $i \in \{1, \dots, n\}$. The “only if” part follows.

Now, suppose $R(T) > 0$. Then, for every $i \in \{1, \dots, n\}$, we could find j_2, \dots, j_{m_i} such that $T_{ij_2 \dots j_{m_i}} > 0$. So, for any $x > 0$, we have

$$(Tx^{m-1})_i = \sum_{j_2, \dots, j_m=1}^n T_{ij_2 \dots j_m} x_{j_2} \dots x_{j_m} \geq T_{ij_2 \dots j_{m_i}} x_{j_2} \dots x_{j_{m_i}} > 0$$

for all $i \in \{1, \dots, n\}$. Hence, the “if” part follows. The proof is complete. \square

Corollary 2.1. An m -th order n dimensional nonnegative tensor T is strictly nonnegative if it is weakly irreducible.

Proof. Suppose that T is weakly irreducible. By the definition, we have that the signs of vectors $G(T)e$ and $R(T)$ are the same. Moreover, $G(T)e$ is positive since T is weakly irreducible. Because, otherwise, we would have a zero row of the matrix $G(T)$, which further implies that $G(T)$ is reducible, a contradiction to Definition 2.2. So, by Lemma 2.1, T is strictly nonnegative. \square

A result similar to this corollary was given in (3.2) of [8]. On the other hand, the converse of Corollary 2.1 is not true in general.

Example 2.1. Let third order two dimensional nonnegative tensor T be defined as:

$$T_{122} = T_{222} = 1, \text{ and } T_{ijk} = 0 \text{ for other } i, j, k \in \{1, 2\},$$

then $R(T) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} > 0$. So T is strictly nonnegative by Lemma 2.1. While, $G(T) = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ is a reducible nonnegative matrix, so T is weakly reducible.

Proposition 2.1. An m -th order n dimensional nonnegative tensor T is strictly nonnegative if and only if F_T is strictly increasing [4], i.e., $F_T(x) > F_T(y)$ for any $x > y \geq 0$.

Proof. If F_T is strictly increasing, then, for any $x > 0$, we have $Tx^{m-1} > T0^{m-1} = 0$. So, T is strictly nonnegative. The “if” part follows.

Now, suppose that T is strictly nonnegative. Then, $R(T) > 0$ by Lemma 2.1. So, for every $i \in \{1, \dots, n\}$, we could find j_{2i}, \dots, j_{mi} such that $T_{ij_{2i} \dots j_{mi}} > 0$. If $0 \leq x < y$, then $x_{j_{2i}} \cdots x_{j_{mi}} < y_{j_{2i}} \cdots y_{j_{mi}}$ for any $i \in \{1, \dots, n\}$. Hence,

$$\begin{aligned} (Ty^{m-1})_i - (Tx^{m-1})_i &= \sum_{j_2, \dots, j_m=1}^n T_{ij_2 \dots j_m} (y_{j_2} \cdots y_{j_m} - x_{j_2} \cdots x_{j_m}) \\ &\geq T_{ij_{2i} \dots j_{mi}} (y_{j_{2i}} \cdots y_{j_{mi}} - x_{j_{2i}} \cdots x_{j_{mi}}) \\ &> 0 \end{aligned}$$

for any $i \in \{1, \dots, n\}$. So, by (1.2), the “only if” part follows. The proof is complete. \square

We now show that the spectral radius of a strictly nonnegative tensor is always positive. At first, we present some notation. For any nonnegative tensor T of order m and dimension n and a nonempty subset I of $\{1, \dots, n\}$, the induced tensor denoted by T_I of I is defined as the m -th order $|I|$ dimensional tensor $\{T_{i_1 \dots i_m} \mid i_1, \dots, i_m \in I\}$. Here $|I|$ denotes the cardinality of the set I .

Lemma 2.2. For any m -th order n dimensional nonnegative tensor T and nonempty subset $I \subseteq \{1, \dots, n\}$, $\rho(T) \geq \rho(T_I)$.

Proof. Let K be a nonnegative tensor of the same size of T with $K_I = T_I$ and zero anywhere else. Then, obviously, $T \geq K \geq 0$ in the sense of componentwise and $\rho(K) = \rho(T_I)$. Now, by [19, Lemma 3.4], $\rho(K) \leq \rho(T)$. So, the result follows immediately. \square

Theorem 2.1. If nonnegative tensor T is strictly nonnegative, then $\rho(T) > 0$.

Proof. By Lemma 2.1, $R(T) > 0$, so $G(T)e > 0$. We have two cases.

- (I) If $G(T)$ is an irreducible matrix, then T is weakly irreducible by Definition 2.2. By Perron-Frobenius Theorem [8], there is $x > 0$ such that $Tx^{m-1} = \rho(T)x^{[m-1]}$. Thus, by Lemma 2.1, the strict positivity of T implies $\rho(T) > 0$.
- (II) If $G(T)$ is a reducible matrix, then T is weakly reducible by Definition 2.2. T may not have a positive eigenvector. While, we can find a nonempty $I \subseteq \{1, \dots, n\}$ such that $[G(T)]_{ij} = 0$ for all $i \in I$ and $j \notin I$. Denote by K the principal submatrix of $G(T)$ indexed by I , and T' the tensor

induced by I . Since $G(T)e > 0$ and $[G(T)]_{ij} = 0$ for all $i \in I$ and $j \notin I$, we still have $Ke > 0$. Hence, T' is also strictly nonnegative since $G(T') = K$ and $Ke > 0$. By Lemma 2.2, we have that $\rho(T) \geq \rho(T')$.

So, inductively, we can finally get a tensor sequence T, T', \dots, T^* (since n is finite) with

$$\rho(T) \geq \rho(T') \geq \dots \geq \rho(T^*).$$

Moreover, T^* is either a weakly irreducible tensor when the dimension of T^* is higher than 1, or a positive scalar since T^* is strictly positive. In both cases, $\rho(T^*) > 0$ by (I). The proof is complete. \square

Example 2.2. Let third order 2 dimensional nonnegative tensor T be defined as:

$$T_{122} = 1, \text{ and } T_{ijk} = 0 \text{ for other } i, j, k \in \{1, 2\}.$$

The eigenvalue equation of the tensor T becomes

$$\begin{cases} x_2^2 &= \lambda x_1^2, \\ 0 &= \lambda x_2^2. \end{cases}$$

Obviously, $\rho(T) = 0$. Hence, a nonzero nonnegative tensor may have zero spectral radius. So, Theorem 2.1 is not vacuous in general.

At the end of this section, we summarize the Perron-Frobenius theorem for nonnegative tensors as follows.

Theorem 2.2. Let T be an m -th order n dimensional nonnegative tensor, then

- (Yang and Yang [19]) $\rho(T)$ is an eigenvalue of T with a nonnegative eigenvector;
- (Theorem 2.1) if furthermore T is strictly nonnegative, then $\rho(T) > 0$;
- (Friedland, Gaubert and Han [8]) if furthermore T is weakly irreducible, then $\rho(T)$ has a unique positive eigenvector;
- (Chang, Pearson and Zhang [3]) if furthermore T is irreducible and if λ is an eigenvalue with a nonnegative eigenvector, then $\lambda = \rho(T)$;
- (Yang and Yang [19]) if T is irreducible, and T has k distinct eigenvalues of modulus $\rho(T)$, then the eigenvalues are $\rho(T) \exp(i2\pi j/k)$ with $i^2 = -1$ and $j = 0, \dots, k - 1$;
- (Chang, Pearson and Zhang [4]) if furthermore T is primitive, then $k = 1$; and
- (Pearson [16]) if T is further essentially positive and m is even, $\rho(T)$ is real geometrically simple.

3 Relationships of the seven classes of nonnegative tensors

In this section, we make a clear diagram of the relationships among the concepts of nonnegative tensors mentioned above. Let T be a nonnegative tensor of order m and dimension n throughout this section. Let E be the identity tensor of order m and dimension n with its diagonal elements being 1 and off-diagonal elements being 0 (when $m = 2$, E is the usual identity matrix).

The relationships among irreducibility, primitivity, weak positivity and essential positivity were characterized in [25, Section 3], which can be summarized as follows:

- If T is essentially positive, then T is both weakly positive and primitive. But the converse is not true. Moreover, there exists a tensor which is weakly positive and primitive simultaneously, but not essentially positive.

- There is no inclusion relation between the class of weakly positive tensors and the class of primitive tensors.
- If T is weakly positive or primitive, then T is irreducible. But the converse is not true.

Actually, there are close relationships between (weakly) irreducibility and (weakly) primitivity, and between weak positivity and essential positivity.

Theorem 3.3. A nonnegative tensor T is weakly irreducible / irreducible / weakly positive if and only if $T + E$ is weakly primitive / primitive / essentially positive. A nonnegative tensor T is essentially positive if and only if it is weakly positive, and all of its diagonal elements are positive.

Proof.

- (Weakly irreducible / Weakly primitive) We have that nonnegative representation matrix $G(T)$ is irreducible if and only if matrix $G(T + E)$ is primitive [1, Theorem 2.1.3 and Corollary 2.4.8]. So, by Definition 2.2, tensor T is weakly irreducible if and only if tensor $T + E$ is weakly primitive.
- (Weakly positive / Essentially positive) The claims follow directly from Definition 2.1 and the nonnegativity of T .
- (Irreducible / Primitive) First, it follows from [19, Theorem 6.6] immediately that T is irreducible if and only if $F_{T+E}^{n-1}(x) > 0$ for any nonzero $x \in \mathfrak{R}_+^n$.

Second, if $T + E$ is primitive, then $T + E$ is irreducible. We can prove that $F_{T+E}^{n-1}(x) > 0$ for any nonzero $x \in \mathfrak{R}_+^n$, which implies that T is irreducible by the above result. Actually, let $K := T + \frac{1}{2}E$, then, $2K$ is irreducible and nonnegative. We have $2(T + E) = 2(K + \frac{1}{2}E) = 2K + E$ and $F_{2(T+E)}^{n-1}(x) = F_{2K+E}^{n-1}(x) > 0$ for any nonzero $x \in \mathfrak{R}_+^n$ by the above result. It is straightforward to check that $F_{2(T+E)}^{n-1}(x) = 2^{\frac{n-1}{m-1}} F_{T+E}^{n-1}(x)$ for any nonzero $x \in \mathfrak{R}_+^n$. So, the result follows. \square

We now discuss some other relations among these six classes of nonnegative tensors. By Definitions 2.1 and 2.2, if T is irreducible, then it is weakly irreducible. Nevertheless, the converse is not true in general, which can be seen from the following example.

Example 3.3. Let T be a third order three dimensional tensor which is defined by $T_{123} = T_{221} = T_{223} = T_{312} = T_{332} = 1$ and $T_{ijk} = 0$ for other $i, j, k \in \{1, 2, 3\}$. Then, $G(T) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{pmatrix}$ is irreducible but

$T_{2ij} = 0$ for all $i, j \in \{1, 3\}$, which says that T is reducible.

By Definition 2.2, if T is weakly primitive, then T is weakly irreducible. The following example demonstrates that the converse is not true.

Example 3.4. Let T be a third order three dimensional tensor which is defined by $T_{122} = T_{233} = T_{311} = 1$ and $T_{ijk} = 0$ for other $i, j, k \in \{1, 2, 3\}$. Then, T is not weakly primitive, since its representation matrix

$G(T) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ is not primitive. But it is an irreducible matrix, and hence, T is weakly irreducible by Definition 2.2.

We now discuss the relations between primitivity and weak primitivity. The following result is a complementary to that in Chang, Pearson and Zhang [4].

Lemma 3.3. For a nonnegative tensor T of order m and dimension n , if $M(T)$ is primitive, then T is primitive, and if T is primitive, then $G(T)$ is primitive.

Proof. If $M(T)$ is primitive, then let $K := [M(T)]^k$ with $k := n^2 - 2n + 2$, we have $K_{ij} > 0$ for any $i, j \in \{1, \dots, n\}$ [1, Theorem 2.4.14]. Now, for any nonzero $x \in \mathfrak{R}_+^n$, suppose $x_j > 0$. Then, for any $i \in \{1, \dots, n\}$, there exist i_2, \dots, i_k such that $M(T)_{ii_2}, M(T)_{i_2i_3}, \dots, M(T)_{i_kj} > 0$. So, $T_{ii_2\dots i_2}, \dots, T_{i_kj\dots j} >$

0. Thus, we have $[F_T^k(x)]_i > 0$ for any $i \in \{1, \dots, n\}$. As the above inequalities hold for any nonzero $x \in \mathfrak{R}_+^n$, we obtain that T is primitive.

If T is primitive, then for some integer $k > 0$, $F_T^k(x) > 0$ for any nonzero $x \in \mathfrak{R}_+^n$. For any $i \in \{1, \dots, n\}$, let e_j denote the j -th column of the $n \times n$ identity matrix for any $j \in \{1, \dots, n\}$. We thus have $[F_T^k(e_j)]_i > 0$. So, we must have indices $\{i_2^2, \dots, i_m^2\}, \{i_1^3, \dots, i_m^3\}, \dots, \{i_1^{k-1}, \dots, i_m^{k-1}\}, i_k$ such that $T_{i_2^2 \dots i_m^2}, T_{i_1^3 \dots i_m^3}, \dots, T_{i_1^{k-1} \dots i_m^{k-1}}, T_{i_k j \dots j} > 0$ and $i^{l+1} \in \{i_2^l, \dots, i_m^l\}$ for $l \in \{1, \dots, k-1\}$. Thus, if we let $L := [G(T)]^k$, we should have $L_{ij} > 0$ for all $i, j \in \{1, \dots, n\}$. Hence, $G(T)$ is primitive. \square

By Lemma 3.3 and Definition 2.2, if T is primitive, then T is weakly primitive.

We list some further relationships among the concepts mentioned above as follows:

- By Example 3.3, we see that $G(T)$ is primitive but T is reducible since $T_{2ij} = 0$ for all $i, j \in \{1, 3\}$. Hence, there exists a nonnegative tensor which is weakly primitive but not irreducible.
- Let T be a third order two dimensional tensor which is defined by $T_{122} = T_{211} = 1$ and $T_{ijk} = 0$ for other $i, j, k \in \{1, 2\}$. Then, T is weakly positive, and hence, irreducible, but not weakly primitive.
- Let T be a third order two dimensional tensor which is defined by $T_{122} = T_{211} = T_{212} = T_{121} = 1$ and $T_{ijk} = 0$ for other $i, j, k \in \{1, 2\}$. Then, T is weakly positive but not primitive, since $Te_1^2 = e_2$ and $Te_2^2 = e_1$. However, it is weakly primitive, since $G(T) = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ is obviously primitive.

Using Corollary 2.1 and Example 2.1, we can summarize the relationships obtained so far in Figure 1.

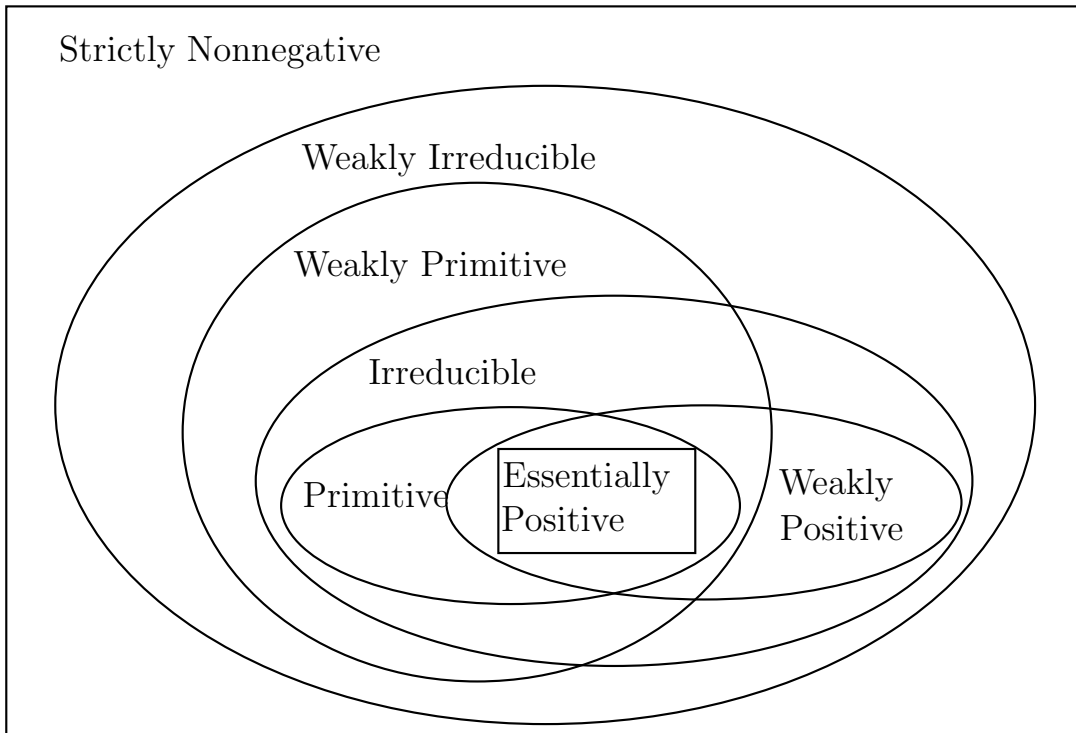


Figure 1 Relationships of the seven classes of nonnegative tensors

4 Global R-linear convergence of a power method for weakly irreducible nonnegative tensors

We present here a modification of the power method proposed in [8].

Algorithm 4.1. (A Higher Order Power Method (HOPM))

Step 0 Initialization: choose $x^{(0)} \in \mathfrak{R}_{++}^n$. Let $k := 0$.

Step 1 Compute

$$\bar{x}^{(k+1)} := T(x^{(k)})^{m-1}, \quad x^{(k+1)} := \frac{(\bar{x}^{(k+1)})^{\lfloor \frac{1}{m-1} \rfloor}}{e^T \left[(\bar{x}^{(k+1)})^{\lfloor \frac{1}{m-1} \rfloor} \right]},$$

$$\alpha(x^{(k+1)}) := \max_{1 \leq i \leq n} \frac{(T(x^{(k)})^{m-1})_i}{(x^{(k)})_i^{m-1}} \quad \text{and} \quad \beta(x^{(k+1)}) := \min_{1 \leq i \leq n} \frac{(T(x^{(k)})^{m-1})_i}{(x^{(k)})_i^{m-1}}.$$

Step 2 If $\alpha(x^{(k+1)}) = \beta(x^{(k+1)})$, stop. Otherwise, let $k := k + 1$, go to Step 1.

Algorithm 4.1 is well-defined if the underlying tensor T is a strictly nonnegative tensor, as in this case, $Tx^{m-1} > 0$ for any $x > 0$. Hence, Algorithm 4.1 is also well-defined for weakly irreducible nonnegative tensors. The following theorem establishes the convergence of Algorithm 4.1 if the underlying tensor T is weakly primitive, where we need to use the concept of Hilbert's projective metric [15]. We first recall such a concept. For any $x, y \in \mathfrak{R}_+^n \setminus \{0\}$, if there are $\alpha, \beta > 0$ such that $\alpha x \leq y \leq \beta x$, then x and y are called *comparable*. If x and y are comparable, and define

$$m(y/x) := \sup\{\alpha > 0 \mid \alpha x \leq y\} \quad \text{and} \quad M(y/x) := \inf\{\beta > 0 \mid y \leq \beta x\},$$

then, the Hilbert's projective metric d can be defined by

$$d(x, y) := \begin{cases} \log\left(\frac{M(y/x)}{m(y/x)}\right), & \text{if } x \text{ and } y \text{ are comparable,} \\ +\infty, & \text{otherwise} \end{cases}$$

for $x, y \in \mathfrak{R}_+^n \setminus \{0\}$. Note that if $x, y \in \Delta_n := \{z \in \mathfrak{R}_{++}^n \mid e^T z = 1\}$, then $d(x, y) = 0$ if and only if $x = y$. Actually, it is easy to check that d is a metric on Δ_n .

Theorem 4.4. Suppose that T is a weakly irreducible nonnegative tensor of order m and dimension n . Then, the following results hold.

- (i) T has a positive eigenpair (λ, x) , and x is unique up to a multiplicative constant.
- (ii) Let (λ_*, x^*) be the unique positive eigenpair of T with $\sum_{i=1}^n (x^*)_i = 1$. Then,

$$\min_{x \in \mathfrak{R}_{++}^n} \max_{1 \leq i \leq n} \frac{(Tx^{m-1})_i}{x_i^{m-1}} = \lambda_* = \max_{x \in \mathfrak{R}_{++}^n} \min_{1 \leq i \leq n} \frac{(Tx^{m-1})_i}{x_i^{m-1}}.$$

- (iii) If (ν, v) is another eigenpair of T , then $|\nu| \leq \lambda_*$.

- (iv) Suppose that T is weakly primitive and the sequence $\{x^{(k)}\}$ is generated by Algorithm 4.1. Then, $\{x^{(k)}\}$ converges to the unique vector $x^* \in \mathfrak{R}_{++}^n$ satisfying $T(x^*)^{m-1} = \lambda_*(x^*)^{[m-1]}$ and $\sum_{i=1}^n x_i^* = 1$, and there exist constant $\theta \in (0, 1)$ and positive integer M such that

$$d(x^{(k)}, x^*) \leq \theta^{\frac{k}{M}} \frac{d(x^{(0)}, x^*)}{\theta} \tag{4.1}$$

holds for all $k \geq 1$.

Proof. Except the result in (4.1), all the other results in this theorem can be easily obtained from [8, Theorem 4.1, Corollaries 4.2, 4.3 and 5.1]. So, we only give the proof of (4.1) here. We have the following observations first:

- \mathfrak{R}_+^n is a *normal cone* in Banach space \mathfrak{R}^n [15], since $y \geq x \geq 0$ implies $\|y\| \geq \|x\|$;

- \mathfrak{R}_+^n has nonempty interior \mathfrak{R}_{++}^n which is an open cone, and $F_T : \mathfrak{R}_{++}^n \rightarrow \mathfrak{R}_{++}^n$ is continuous and *order-preserving* by Corollary 2.1 and the nonnegativity of tensor T ;
- F_T is homogeneous of degree 1 in \mathfrak{R}_{++}^n ;
- the set Δ_n is connected and T has an eigenvector x^* in Δ_n by Theorem 4.4 (i);
- by (1.2), F_T is continuously differentiable in an open neighborhood of x^* , since $x^* > 0$;
- by Definition 2.2, $G(T)$ is primitive, hence there exists an integer N such that $[G(T)]^N > 0$. So, $[G(T)]^N x$ is comparable with x^* for any nonzero $x \in \mathfrak{R}_+^n$;
- $G(T) : \mathfrak{R}^n \rightarrow \mathfrak{R}^n$ is a compact linear map, hence its essential spectrum radius is zero [15, Page 38], while its spectral radius is positive since it is a primitive matrix [1].

Hence, by [15, Corollary 2.5 and Theorem 2.7], we have that there exist a constant $\theta \in (0, 1)$ and a positive integer M such that

$$d(x^{(Mj)}, x^*) \leq \theta^j d(x^{(0)}, x^*), \tag{4.2}$$

where d denotes the Hilbert’s projective metric on $\mathfrak{R}_+^n \setminus \{0\}$.

By [15, Proposition 1.5], we also have that

$$d(F_T(x), F_T(y)) \leq d(x, y) \tag{4.3}$$

for any $x, y \in \mathfrak{R}_+^n$. Since $\lambda_* > 0$, by the property of Hilbert’s projective metric d [15, Page 13] we have that

$$\begin{aligned} d(x^{(k+1)}, x^*) &= d\left(\frac{F_T(x^{(k)})}{e^T F_T(x^{(k)})}, x^*\right) = d\left(\frac{F_T(x^{(k)})}{e^T F_T(x^{(k)})}, \frac{1}{(\lambda_*)^{\frac{1}{m-1}}} F_T(x^*)\right) \\ &= d(F_T(x^{(k)}), F_T(x^*)) \leq d(x^{(k)}, x^*) \end{aligned}$$

holds for any k . So, for any $k \geq M$, we can find the largest j such that $k \geq Mj$ and $M(j + 1) \geq k$. Hence,

$$d(x^{(k)}, x^*) \leq d(x^{(Mj)}, x^*) \leq \theta^j d(x^{(0)}, x^*) \leq \theta^{\frac{k}{M}-1} d(x^{(0)}, x^*)$$

which implies (4.1) for all $k \geq M$. When $1 \leq k < M$, we have $\theta^{\frac{k}{M}} > \theta$, since $\theta \in (0, 1)$. Therefore, (4.1) is true for all $k \geq 1$. □

We denote by $x^{[p]}$ a vector with its i -th element being x_i^p .

By Theorems 3.3 and 4.4, the following result holds obviously.

Theorem 4.5. Suppose that T is a weakly irreducible nonnegative tensor of order m and dimension n , and the sequence $\{x^{(k)}\}$ is generated by Algorithm 4.1 with T being replaced by $T + E$. Then, $\{x^{(k)}\}$ converges to the unique vector $x^* \in \mathfrak{R}_{++}^n$ satisfying $T(x^*)^{m-1} = \lambda_*(x^*)^{[m-1]}$ and $\sum_{i=1}^n x_i^* = 1$, and there exist a constant $\theta \in (0, 1)$ and a positive integer M such that (4.1) holds for all $k \geq 1$.

Remark 4.1. (i) Compared with [8, Corollaries 5.1 and 5.2], a main advantage of our results is that (4.1) in Theorem 4.4(iv) gives the **global** R -linear convergence of Algorithm 4.1; while the geometric convergence given in [8, Corollary 5.2] is essentially a result of **local** R -linear convergence. (ii) Compared with the results in [25], a main advantage of our results is that the problem we considered in Theorem 4.5 is broader than that in [25], i.e., our results are obtained for the weakly irreducible nonnegative tensors; while the results in [25] hold for the irreducible nonnegative tensors. There are also other differences between Theorem 4.5 and those in [25], such as, Theorem 4.5 gives the global R -linear convergence of the **iterated sequence** $\{x^{(k)}\}$ of Algorithm 4.1 for weakly irreducible nonnegative tensors; while the Q -linear convergence of the corresponding **eigenvalue sequence** was proved in [25] for weakly positive nonnegative tensors.

Theorem 4.5 is one of the corner stones for us to develop a method for finding the spectral radius of a *general nonnegative tensor*.

5 Partition a general nonnegative tensor to weakly irreducible nonnegative tensors

If a nonnegative tensor T of order m and dimension n is weakly irreducible, then from Theorem 4.5, we can find the spectral radius and the corresponding positive eigenvector of T by using Algorithm 4.1. If T is not weakly irreducible, there is no guarantee for the convergence as that in Theorem 4.5.

In this section, we show that if a nonnegative tensor T is not weakly irreducible, then there exists a partition of the index set $\{1, \dots, n\}$ such that every tensor induced by the set in the partition is weakly irreducible, and the largest eigenvalue of T can be obtained from these induced tensors. Thus, we can find the spectral radius of a general nonnegative tensor by using Algorithm 4.1 for these induced weakly irreducible tensors. At the end of this section, we show that, if weakly reducibility is replaced by reducibility, a similar method does not work.

The following result is useful.

Theorem 5.6. [19, Theorem 2.3] For any nonnegative tensor T of order m and dimension n , $\rho(T)$ is an eigenvalue with a nonnegative eigenvector $x \in \mathfrak{R}_+^n$ corresponding to it.

To develop an algorithm for general nonnegative tensors, we prove the following theorem which is an extension of the corresponding result for nonnegative matrices [1]. For the convenience of the subsequent analysis, a one dimensional tensor is always considered as irreducible, hence weakly irreducible. Similarly, one dimensional positive tensors are considered as primitive. Note that Algorithm 4.1 works for one dimensional primitive tensor as well. However, weakly irreducible nonnegative tensors with dimension one may have zero spectral radius, but they are always positive when the dimension $n \geq 2$ by Theorem 2.1. Note that, n is assumed to be no smaller than two throughout this paper, while the case of one dimensional tensors is needed in the presentation of the partition results in this section.

Theorem 5.7. Suppose that T is a nonnegative tensor of order m and dimension n . If T is weakly reducible, then there is a partition $\{I_1, \dots, I_k\}$ of $\{1, \dots, n\}$ such that every tensor in $\{T_{I_j} \mid j \in \{1, \dots, k\}\}$ is weakly irreducible.

Proof. Since T is weakly reducible, by Definition 2.2 we can obtain that the matrix $G(T)$ is reducible. Thus, we can find a partition $\{J_1, \dots, J_l\}$ of $\{1, \dots, n\}$ such that

- (\star) every matrix (a second order tensor) in $\{[G(T)]_{J_i} \mid i \in \{1, \dots, l\}\}$ is irreducible and $[G(T)]_{st} = 0$ for any $s \in J_p$ and $t \in J_q$ such that $p > q$.

Actually, by the definition of reducibility of a matrix, we can find a partition $\{J_1, J_2\}$ of $\{1, \dots, n\}$ such that $[G(T)]_{st} = 0$ for any $s \in J_2$ and $t \in J_1$. If both $[G(T)]_{J_1}$ and $[G(T)]_{J_2}$ are irreducible, then we are done. Otherwise, we can repeat the above analysis to any reducible block(s) obtained above. In this way, since $\{1, \dots, n\}$ is a finite set, we can arrive at the desired result (\star).

Now, if every tensor in $\{T_{J_i} \mid i \in \{1, \dots, l\}\}$ is weakly irreducible, then we are done. Otherwise, we repeat the above procedure to generate a partition of T to these induced tensors which are not weakly irreducible. Since $\{1, \dots, n\}$ is finite, this process will stop in finite steps. Hence, the theorem follows.

□

By Theorems 5.7 and 3.3, we have the following corollary.

Corollary 5.2. Suppose that T is a nonnegative tensor of order m and dimension n . If T is weakly irreducible, then $T + E$ is weakly primitive; otherwise, there is a partition $\{I_1, \dots, I_k\}$ of $\{1, \dots, n\}$ such that every tensor in $\{(T + E)_{I_j} \mid j \in \{1, \dots, k\}\}$ is weakly primitive.

Given a nonempty subset I of $\{1, \dots, n\}$ and an n vector x , we will denote by x_I an n vector with its i -th element being x_i if $i \in I$ and zero otherwise; and $x(I)$ a $|I|$ vector after deleting x_j for $j \notin I$ from x .

Theorem 5.8. Suppose that T is a weakly reducible nonnegative tensor of order m and dimension n , and $\{I_1, \dots, I_k\}$ is the partition of $\{1, \dots, n\}$ determined by Theorem 5.7. Then, $\rho(T) = \rho(T_{I_p})$ for some $p \in \{1, \dots, k\}$.

Proof. By the proof of Theorem 5.7, for the nonnegative matrix $G(T)$, we can find a partition $\{J_1, \dots, J_l\}$ of $\{1, \dots, n\}$ such that

- every matrix in $\{[G(T)]_{J_i} \mid i \in \{1, \dots, l\}\}$ is irreducible and $[G(T)]_{st} = 0$ for any $s \in J_p$ and $t \in J_q$ such that $p > q$.

First, we have that $\rho(T_{J_i}) \leq \rho(T)$ for all $i \in \{1, \dots, l\}$ by Lemma 2.2.

Then, denote by $(\rho(T), x)$ a nonnegative eigenpair of T which is guaranteed by Theorem 5.6. Since $[G(T)]_{ij} = 0$ for all $i \in J_l$ and $j \in \cup_{s=1}^{l-1} J_s$. We must have

$$T_{i_2 \dots i_m} = 0 \quad \forall i \in J_l, \quad \forall \{i_2, \dots, i_m\} \not\subseteq J_l. \tag{5.1}$$

Hence, for all $i \in J_l$, we have

$$\begin{aligned} \rho(T)x_i^{m-1} &= (Tx^{m-1})_i \\ &= \sum_{i_2, \dots, i_m=1}^n T_{i i_2 \dots i_m} x_{i_2} \cdots x_{i_m} \\ &= \sum_{\{i_2, \dots, i_m\} \subseteq J_l} T_{i i_2 \dots i_m} x_{i_2} \cdots x_{i_m} \\ &= \left\{ T_{J_l} (x(J_l))^{m-1} \right\}_i, \end{aligned}$$

where the third equality follows from (5.1). If $x(J_l) \neq 0$, then $(\rho(T), x(J_l))$ is a nonnegative eigenpair of tensor T_{J_l} , and if $x(J_l) = 0$, then we have

$$T_{\cup_{j=1}^{l-1} J_j} (x(\cup_{j=1}^{l-1} J_j))^{m-1} = \rho(T) [x(\cup_{j=1}^{l-1} J_j)]^{[m-1]}.$$

In the later case, repeat the above analysis with T being replaced by $T_{\cup_{j=1}^{l-1} J_j}$. Since $x \neq 0$ and l is finite, we must find some $t \in \{1, \dots, l\}$ such that $x(J_t) \neq 0$ and $(\rho(T), x(J_t))$ is a nonnegative eigenpair of the tensor T_{J_t} .

Now, if T_{J_t} is weakly irreducible, we are done since $J_t = I_p$ for some $p \in \{1, \dots, k\}$ by the proof of Theorem 5.7. Otherwise, repeat the above analysis with T and x being replaced by T_{J_t} and $x(J_t)$, respectively. Such a process is finite, since n is finite. Thus, we always obtain a weakly irreducible nonnegative tensor T_{I_p} with $I_p \subseteq \{1, \dots, n\}$ for some p such that $(\rho(T), x(I_p))$ is a nonnegative eigenpair of the tensor T_{I_p} . Furthermore, $(\rho(T), x_S)$ with $S := \cup_{i=1}^p I_p$ is a nonnegative eigenpair of tensor T .

The proof is complete. □

Note that if T is furthermore symmetric, then we can get a diagonal block representation of T with diagonal blocks T_{I_i} (after some permutation, if necessary). Then, by Corollary 5.2 and Theorems 4.5, 5.7 and 5.8, we can get the following theorem.

Theorem 5.9. Suppose that T is a nonnegative tensor of order m and dimension n .

- If T is weakly irreducible, then $T + E$ is weakly primitive by Theorem 3.3, and hence, Algorithm 4.1 with T being replaced by $T + E$ converges to the unique positive eigenpair $(\rho(T + E), x)$ of $T + E$. Moreover, $(\rho(T + E) - 1, x)$ is the unique positive eigenpair of T .
- If T is not weakly irreducible, then, we can get a set of weakly irreducible tensors $\{T_{I_j} \mid j = 1, \dots, k\}$ with $k > 1$ by Theorem 5.7. For each $j \in \{1, \dots, k\}$, we use item (a) to find the unique positive eigenpair $(\rho(T_{I_j}), x^j)$ of T_{I_j} which is guaranteed by Corollary 5.2 when $|I_j| \geq 2$ or eigenpair $(T_{I_j}, 1)$ when $|I_j| = 1$. Then, $\rho(T) = \max_{j=1, \dots, k} \rho(T_{I_j})$ by Theorem 5.8. If T is furthermore symmetric, then, x with $x(I_{j^*}) = x^{j^*}$ is a nonnegative eigenvector of T where $j^* \in \operatorname{argmax}_{j=1, \dots, k} \rho(T_{I_j})$

Remark 5.2. By Theorem 5.9 and Algorithm 4.1, if nonnegative tensor T is weakly irreducible, then, the spectral radius of T can be found directly by Algorithm 4.1 with T being replaced by $T + E$. If

T is not weakly irreducible, then, we have to find the partition of $\{1, \dots, n\}$ determined by Theorem 5.7. Fortunately, we can find such a partition through the corresponding partition of the nonnegative representation matrix of T and its induced tensors according to Theorem 5.8. The specific method of finding such a partition is given in the next section.

Most of the known papers, which established the Perron-Frobenius theorem and showed the convergence of the power method for nonnegative tensors, concentrated on the class of irreducible nonnegative tensors [6]. A natural question is that, *for any given reducible nonnegative tensor T , whether a partition of T similar to the result given in Theorem 5.7 can be derived or not.* If so, *whether the spectral radius of T can be obtained by using the power method for the induced irreducible nonnegative tensors or not.* At the end of this section, we answer these two questions. The answer to the first question is positive, which is given as follows.

Theorem 5.10. Suppose that T is a nonnegative tensor of order m and dimension n . If T is reducible, then there is a partition $\{I_1, \dots, I_k\}$ of $\{1, 2, \dots, n\}$ such that any one of the tensors $\{T_{I_j} \mid j \in \{1, \dots, k\}\}$ is irreducible and

$$T_{st_2 \dots t_m} = 0, \quad \forall s \in I_p, \quad \forall \{t_2, \dots, t_m\} \subset I_q, \quad \forall p > q.$$

Proof. Since T is reducible, by the definition of reducibility, there exists a nonempty proper subset I_2 of $\{1, \dots, n\}$ such that

$$T_{ii_2 \dots i_m} = 0, \quad \forall i \in I_2, \quad \forall i_2, \dots, i_m \in I_1 := \{1, \dots, n\} \setminus I_2.$$

If both T_{I_1} and T_{I_2} are irreducible, then we are done. Without loss of generality, we assume that T_{I_1} is irreducible and T_{I_2} is reducible. Then, by the reducibility of T_{I_2} , we can get a partition $\{J_2, J_3\}$ of I_2 such that

$$T_{ii_2 \dots i_m} = 0, \quad \forall i \in J_3, \quad \forall i_2, \dots, i_m \in J_2 := I_2 \setminus J_3.$$

If both T_{J_2} and T_{J_3} are irreducible, then we are done, since $\{I_1, J_2, J_3\}$ is the desired partition of $\{1, \dots, n\}$. Otherwise, repeating the above procedure, we can get the desired results, since n is finite. \square

However, the answer to the second question is negative, which can be seen by the following example.

Example 5.5. Let T be a third order two dimensional tensor which is defined by

$$T_{111} = 1, \quad T_{112} = T_{121} = T_{211} = 4, \quad T_{122} = T_{212} = T_{221} = 0, \quad \text{and} \quad T_{222} = 1.$$

Since $T_{122} = 0$, the tensor T is reducible. And $T_{111} = T_{222} = 1$ is the largest eigenvalue of both induced tensors by Theorem 5.10. While the nonnegative eigenpairs of T are

$$(1, (0, 1)^T) \quad \text{and} \quad (7.3496, (0.5575, 0.4425)^T).$$

This example prevents us to use $\rho(T_{I_i})$'s to get $\rho(T)$ under the framework of irreducibility. In addition, it is easy to see that checking weak irreducibility of a tensor T is much easier than checking irreducibility of the tensor T , since the former is based on a nonnegative matrix which has both sophisticated theory and algorithms [1].

6 A specific algorithm and numerical experiments

In this section, based on Algorithm 4.1, Theorem 4.5, and the theory established in Section 5, we develop a specific algorithm for finding the spectral radius of a general nonnegative tensor.

6.1 A specific algorithm

In this subsection, we give an algorithm for finding the spectral radius of a nonnegative tensor. We need an algorithm to find the irreducible blocks of a nonnegative matrix M . There are sophisticated algorithms for it. For example, the following naive approach is based on the fact that a nonnegative matrix M is irreducible if and only if $(M + E)^{n-1} > 0$ [1].

Algorithm 6.2. (Irreducible blocks of nonnegative matrices)

Step 0 Given a nonnegative matrix M , let $k = 1$ and $C^1 := M + E$.

Step 1 Until $k = n - 1$, repeat $C^k := C^{k-1}(M + E)$ and $k := k + 1$.

Step 2 Sort the numbers of nonzero elements of columns of C^{n-1} in ascend order, then perform symmetric permutation to C^{n-1} according to the sorting order into a matrix K ; sort the numbers of nonzero elements of rows of K in descend order, then perform symmetric permutation to K according to the sorting order into a matrix L . Record the two sorting orders.

Step 3 Let $i = 1, j = 1$ and $s = 1$, create an index set I_j and an vector ind , put i into I_j and index it to be the s -th element in I_j , and set $ind(i)$ to be 1.

Step 4 If $L(d, I_j(s)) > 0$ and $L(I_j(s), d) > 0$ for some $d \in \{1, \dots, n\}$ with $ind(d)$ being not 1, where $I_j(s)$ is the s -th element in I_j , set $s := s + 1$ and put d into I_j and index it to be the s -th element in I_j . Set $ind(d)$ to be 1, and $i := i + 1$. If there is no such d or $i = n$, go to Step 6; if there is no such d but $i < n$, go to Step 5.

Step 5 Let $j := j + 1$ and $s := 1$. Create index set I_j , and find a d with $ind(d)$ being not 1, put d into I_j and index it to be the s -th element in I_j . Set $ind(d)$ to be 1, go back to Step 4.

Step 6 Using sorting orders in Step 2 and partition $\{I_1, \dots, I_k\}$ found by Steps 3-5, we can find the partition for the matrix M easily.

Now, we propose a specific algorithm for finding the spectral radius of a general nonnegative tensor.

Algorithm 6.3. (Spectral radius of a nonnegative tensor)

Step 0 Let v be an n vector with its elements being zeros, and $i = 1$.

Step 1 Given a nonnegative tensor T of order m and dimension n , compute its representation matrix $G(T)$ as Definition 2.2.

Step 2 Finding out the partition $\{I_1, \dots, I_k\}$ of $\{1, \dots, n\}$ using Algorithm 6.2 with M being replaced by $G(T)$.

Step 3 If $k = 1$, using Algorithm 4.1 to finding out the spectral radius ρ of T , set $v(i) = \rho$, and $i = i + 1$. Otherwise, go to Step 4.

Step 4 For $j = 1, \dots, k$, computing the induced tensor T_{I_j} and its corresponding representation matrix G_j , set T as T_{I_j} , $G(T)$ as G_j and n as $|I_j|$, run subroutine Steps 2-4.

Step 5 Out put the spectral radius of T as $\max_{i=1}^n v(i)$.

6.2 Numerical experiments

In this subsection, we report some preliminary numerical results for computing the spectral radius of a general nonnegative tensor using Algorithm 6.3 (with Algorithm 4.1). All experiments are done on a PC with CPU of 3.4 GHz and RAM of 2.0 GB, and all codes are written in MATLAB.

To demonstrate that Algorithm 6.3 works for general nonnegative tensors, we randomly generate third order nonnegative tensors with dimension n which is specialized in Table 1. We generate the testing

tensors by randomly generating their every element uniformly in $[0, 1]$ with a density **Den** which is specialized in Table 1. We use Algorithm 6.3 to find the spectral radii of the generated tensors for every case, i.e., with different dimensions n and element density **Den**. The algorithm is terminated if $|\alpha(x^{(k)}) - \beta(x^{(k)})| \leq 10^{-6}$. For every case, we simulate 50 times to get the average spectral radius $\rho := \frac{\alpha(x^{(k)}) + \beta(x^{(k)})}{2}$, the average number of iterations **Ite** performed by Algorithm 4.1, the average weakly irreducible blocks of the generated tensors **Blks** (which is the number of the sets in the partition), and the average residual of $Tx^2 - \rho x^{[2]}$ with the found spectral radius ρ and its corresponding eigenvector x in 2-norm **Res**. We also use **Per** to denote the percentage of weakly irreducible tensors generated among the 50 simulations, and **TolCpu** to denote the total cputime spent for the simulation in every case. All results are listed in Table 1.

Table 1 Numerical results

n	Den	ρ	Per	Ite	Blks	Res	TolCpu
3	0.10	0.148	0.00	6.88	2.88	2.0516e-008	0.30
3	0.20	0.409	14.00	17.88	2.52	1.3521e-007	0.38
3	0.30	0.671	36.00	25.32	2.06	6.2038e-008	0.44
3	0.40	0.787	44.00	27.6	1.9	1.4773e-006	0.41
3	0.50	1.113	66.00	31.06	1.38	8.1021e-008	0.42
3	0.60	1.032	72.00	33.76	1.44	9.8837e-008	0.44
3	0.70	1.196	78.00	31.4	1.34	8.6830e-008	0.47
3	0.80	1.244	80.00	37.82	1.28	7.3281e-007	0.50
3	0.90	1.528	86.00	33.26	1.22	2.1994e-006	0.44
4	0.10	0.368	14.00	19.56	3.26	3.5644e-008	0.47
4	0.20	0.554	24.00	27.04	2.74	7.1762e-008	0.48
4	0.40	1.060	62.00	42.28	1.64	1.6366e-006	0.50
4	0.80	1.945	90.00	35.76	1.1	6.5548e-008	0.47
10	0.05	0.466	14.00	51.6	5.72	1.8168e-006	0.94
10	0.10	1.214	56.00	49.44	2.34	5.5063e-007	1.05
10	0.15	2.363	78.00	40.96	1.26	1.5255e-008	0.98
10	0.20	3.124	88.00	31.44	1.2	1.2771e-008	0.95
20	0.05	2.537	56.00	39.54	2.08	4.8818e-009	3.58
20	0.10	5.606	86.00	31.34	1.14	4.7892e-009	3.28
30	0.05	6.103	80.00	31.84	1.28	2.9270e-009	9.42
30	0.10	12.173	84.00	27.06	1.16	2.6974e-009	8.02
40	0.05	10.698	82.00	28.18	1.26	1.8279e-009	18.02
50	0.05	16.740	94.00	27.2	1.06	1.3807e-009	36.80

In addition, we also test the following example and the related numerical results are listed in Table 2, where **Blk** denotes the block number of the computation, **Ite** denotes the iteration number, and the other items are clear from Algorithm 6.3. The initial points for both tests in Tables 1 and 2 are randomly generated with their every element uniformly in $(0, 1)$. All the simulated tensors are strictly nonnegative

by the above simulation strategy, since every component of $R(T)$ is the summation of so many terms, it is never zero in our simulation.

Example 6.6. Let third order three dimensional tensor T be defined by $T_{111} = T_{222} = 1$, $T_{122} = 3$, $T_{211} = 5$, $T_{333} = 4$ and $T_{ijk} = 0$ for other $i, j, k \in \{1, 2, 3\}$. Then, the eigenvalue problem (1.1) reduces to:

$$\begin{cases} x_1^2 + 3x_2^2 &= \lambda x_1^2, \\ 5x_1^2 + x_2^2 &= \lambda x_2^2, \\ 4x_3^2 &= \lambda x_3^2. \end{cases}$$

It is easy to see that the tensor T is not weakly irreducible and the nonnegative eigenpairs of T are

$$(4, (0, 0, 1)^T) \quad \text{and} \quad (4.8730, (0.4365, 0.5635, 0)^T).$$

Hence, the spectral radius of T is 4.8730 which agrees the numerical results in Table 2. The algorithm finds the partition as $\{I_1 = \{1, 2\}, I_2 = \{3\}\}$, which is the expected one.

Table 2 Numerical results for Example 6.6

Blk	Ite	$\alpha(x^{(k)})$	$\beta(x^{(k)})$	$\alpha(x^{(k)}) - \beta(x^{(k)})$	$\ T(x^{(k)})^2 - \frac{\alpha(x^{(k)}) + \beta(x^{(k)})}{2}(x^{(k)})[2]\ $
1	1	6.000	4.000	2.000e+000	1.414e+000
1	2	5.200	4.571	6.286e-001	1.134e-001
1	3	4.974	4.774	2.002e-001	3.573e-002
1	4	4.905	4.841	6.383e-002	1.143e-002
1	5	4.883	4.863	2.035e-002	3.641e-003
1	6	4.876	4.870	6.491e-003	1.162e-003
1	7	4.874	4.872	2.070e-003	3.704e-004
1	8	4.873	4.873	6.602e-004	1.181e-004
1	9	4.873	4.873	2.106e-004	3.767e-005
1	10	4.873	4.873	6.715e-005	1.201e-005
1	11	4.873	4.873	2.141e-005	3.832e-006
1	12	4.873	4.873	6.830e-006	1.222e-006
1	13	4.873	4.873	2.178e-006	3.897e-007
1	14	4.873	4.873	6.946e-007	1.243e-007
2	1	4.000	4.000	0.000e+000	0.000e+000

From Tables 1 and 2, we have some preliminary observations:

- Algorithm 6.3 can find the spectral radius of a general nonnegative tensor efficiently.
- As expected, the more dense of nonzero elements of the underlying tensor, the higher the probability of it being weakly irreducible. We note that function *sprand* which is used in our test in MATLAB does not return a full-dense matrix even with the density parameter being 1, so the percentage **Per** for $n = 3$ is not close enough to 100 even when **Den** = 0.9.
- From Definition 2.2, we note that elements of $G(T)$ are the summation of so many elements of T . So, it is possible that $G(T)$ is an irreducible matrix even when T is a very sparse tensor. This can be noticed from the last several rows in Table 1.

7 Conclusions and remarks

In this paper, we proposed a new class of nonnegative tensors — strictly nonnegative tensors, and proved that the spectral radii for strictly nonnegative tensors are always positive. We discussed the relationships among the seven well-conditional classes of nonnegative tensors, and showed that the class of strictly nonnegative tensors strictly contains the other six classes of nonnegative tensors. We showed that a modification of the power method in [8] for finding the spectral radius of a nonnegative tensor is globally R-linearly convergent for weakly irreducible nonnegative tensors. We also showed that the spectral radius of a general nonnegative tensor is equal to the spectral radii of some weakly irreducible sub-tensors of it. Based on these two results, we proposed an algorithm for finding the spectral radius of a general nonnegative tensor. The preliminary numerical results addressed the efficiency of the method.

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