Contents

1. Basic Algebra

1.1 Law of Indices

A power, or an index (plural: indices), is used to write a product of numbers very compactly. Basically, the index of a number says how many times to use the number in a multiplication. For example:

$$8^4 = 8\times 8\times 8\times 8$$

With this definition, rules to manipulate indices have been developed. These rules are called the Law of Indices.

1.1.1 Basic Rules

Rule of One:

Any number raised to the power one equals itself. i.e.

$$x^1 = x$$

If 8^4 means multiplying 8 four times by itself, 8^1 would definitely means multiplying 8 one time, i.e. 8.

Product Rule:

When we multiply two numbers with the same base but different indices, we can simply add the indices up. i.e.

$$x^m \cdot x^n = x^{m+n}$$

This can be shown as follows:

$$\begin{aligned} & x^m \cdot x^n \\ &= \underbrace{(x\cdot x\cdot \ldots \cdot x)}_{m \text{ times}}\underbrace{(x\cdot x\cdot \ldots \cdot x)}_{n \text{ times}} \\ &= \underbrace{(x\cdot x\cdot \ldots \cdot x)}_{m+n \text{ times}} \end{aligned}$$

For example,

$$5^3\times 5^4 = (5\times 5\times 5)(5\times 5\times 5\times 5) = 5^7$$

Power Rule:

When we raise a power to a power, we just need to multiply the powers.

$$(x^m)^n = x^{m+n}$$

For example,

$$(5^2)^3 = (5\times5)(5\times5)(5\times5) = 5^6$$

or simply,

$$(5^2)^3 = 5^{2\times3} = 5^6$$

Quotient Rule:

The quotient rule tells us the contrary of the product rule. i.e.

$$x^m \div x^n = m^{m-n}$$

This can be shown as follows:

$$x^m \div x^n = \frac{\underbrace{(x\cdot x\cdot\ldots\cdot x)}_{m\text{ times}}}{\underbrace{(x\cdot x\cdot\ldots\cdot x)}_{n\text{ times}}} = x^{m-n}$$

For example,

$$5^6 \div 5^4 = \frac{(5\times5\times5\times5\times5\times5)}{(5\times5\times5\times5)} = 5^2$$

Rule of Zero:

Any non-zero number raised to the power of zero equals 1.

$$x^0 = 1 \quad \text{if } x \neq 0$$

This is the result of Product Rule. Note that x^n can be considered as x^{n+0}. Therefore,

$$x^n = x^{n+0} = x^n \cdot x^0 \quad \text{(by Product Rule)}$$
$$x^0 = x^n \div x^n = 1 \quad \text{if } x \neq 0$$

Rule of Negative Index:

Any non-zero number raised to a negative power equals its reciprocal raised to the corresponding positive power. i.e.

$$x^{-m} = \frac{1}{x^m} \quad \text{if } x \neq 0$$

This can be shown as follows:

$$x^m \cdot x^{-m} = x^{m+(-m)} = x^0 = 1$$

$$\therefore \quad x^{-m} = \frac{1}{x^m} \quad \text{if } x \neq 0$$

For example,

$$8^{-2} = \frac{1}{8^2}$$

(These rules are also applicable to non-integral indices.)

Example: Simplify the following expressions:

(a) $\dfrac{3^6 \times 2^4}{3^4}$

(b) $3^2 \times 3^{-5}$

(c) $\dfrac{9(x^2)^3}{3xy^2}$

(d) $a^{-1}\sqrt{a}$

Answer:

(a) $\dfrac{3^6}{3^4} \times 2^4 = 3^2 \times 2^4 = 144$

(b) $3^{-3} = \dfrac{1}{3^3} = \dfrac{1}{27}$

(c) $\dfrac{9}{3}\dfrac{x^6}{x} \times \dfrac{1}{y^2} = \dfrac{3x^5}{y^2}$

(d) $a^{-1} \times a^{\frac{1}{2}} = a^{\frac{-1}{2}} = \dfrac{1}{a^{\frac{1}{2}}}$

Please attempt Interactive Exercise Question 1.1.

1.2 Summation $\sum$

The sum of a series of numbers, such as the first n positive integers, may be written as

$$1+2+3+\ldots+n$$

However, such expressions are not convenient for manipulations. We can express the sum of a series of numbers more conveniently by using the sigma notation.

1.2.1 Notation

The **Sigma notation** $\sum$ is used to represent summation. For example, the sum of the first n positive integers can be written as $\sum_{k=1}^{n} k$, that is,

$$\sum_{k=1}^{n} k = 1+2+3+\ldots+n$$

This expression means "sum over k, from 1 to n".

In the above expression, k is called the index of summation. The choice of notation for the index is arbitrary. For example:

$$\sum_{k=1}^{n} k = \sum_{i=1}^{n} i = \sum_{j=1}^{n} j = 1+2+3+\ldots+n = \frac{n(n+1)}{2}$$

Let's have more examples using the sigma notation.

Example 1

(a) $\sum_{i=1}^{n} c = \underbrace{c+c+\ldots+c}_{n \text{ times}} = nc$

(b) $\sum_{i=1}^{n} i^2 = 1^2+2^2+\ldots+n^2$

(c) $\sum_{i=1}^{n} (2+i^2) = (2+1^2)+(2+2^2)+\ldots+(2+n^2) = 2n+1^2+2^2+\ldots+n^2$

Example 2

Evaluate the following expressions:

(a) $\sum_{i=1}^{100}(4+3i)$

(b) $\sum_{i=15}^{150}(4i+1)$

Answer

(a) $\sum_{i=1}^{100}(4+3i)$

$$=(4+3\times1)+(4+3\times2)+\ldots+(4+3\times100)$$
$$=4(100)+3(1+2+\ldots+100)$$
$$=400+3\frac{100(100+1)}{2}=15550$$

(b) $\sum_{i=15}^{150}(4i+1)$

$$=(4\times15+1)+(4\times16+1)+\ldots+(4\times150+1)$$
$$=4(15+16+\ldots+150)+1\times136$$
$$=4\frac{(15+150)136}{2}+136=45016$$

Please attempt Interactive Exercise Question 1.2.

1.2.2 Rules of Manipulation

In this section, we are going to investigate methods to manipulate sums. The given expression is transformed to another form that can be evaluated more easily than the given form.

The following basic properties in the arithmetic of numbers are applicable when we are manipulating sums with the sigma notation:

I Moving a constant inside or outside the summation does not change its value.

$$\sum_{k=1}^{n} ca_k = c\sum_{k=1}^{n} a_k$$

Example:

$$\sum_{k=1}^{3} ca_k = ca_1 + ca_2 + ca_3 = c\sum_{k=1}^{3} a_k$$

II Reordering terms within the summation does not change the value, for example,

$$a_1 + a_2 + a_3 + \ldots = a_2 + a_3 + a_1 + \ldots = a_3 + a_2 + a_1 + \ldots = ...$$

There may be more than one way to express a given sum with the sigma notation. For example,

$$\sum_{k=1}^{4} k = 1 + 2 + 3 + 4 = \sum_{k=0}^{3} (k+1) = \sum_{k=2}^{5} (k-1)$$

III Breaking a summation into two parts or combining two parts into a whole does not change its value.

$$\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$$

Example:

$$\sum_{k=1}^{3}(a_k+b_k)=(a_1+b_1)+(a_2+b_2)+(a_3+b_3)$$
$$=(a_1+a_2+a_3)+(b_1+b_2+b_3)$$
$$=\sum_{k=1}^{3}a_k+\sum_{k=1}^{3}b_k$$

Some further examples:

Example 1

$$\sum_{k=1}^{10}(3k+1)$$
$$=\sum_{k=1}^{10}3k+\sum_{k=1}^{10}(1)$$
$$=3\sum_{k=1}^{10}k+\sum_{k=1}^{10}(1)$$
$$=3\frac{10(10+1)}{2}+1(10)=175$$

Example 2

$$\sum_{k=1}^{n}(k+1)-\sum_{k=1}^{n}k$$
$$=\sum_{k=1}^{n}(k+1-k)$$
$$=\sum_{k=1}^{n}(1)=n$$

Example 3

$$
\begin{aligned}
&\sum_{k=1}^{50}[\ln(k+3)-\ln(k+2)] \\
&=\sum_{k=1}^{50}\ln(k+3)-\sum_{k=1}^{50}\ln(k+2) \\
&=\sum_{k=1}^{50}\ln(k+3)-\sum_{k=0}^{49}\ln(k+3) \\
&=\Big(\sum_{k=1}^{49}\ln(k+3)+\ln(50+3)\Big)-\Big(\ln(0+3)+\sum_{k=1}^{49}\ln(k+3)\Big) \\
&=\ln 53-\ln 3=\ln(\frac{53}{3})
\end{aligned}
$$

Please attempt Interactive Exercise Question 1.3.

1.2.3 Some Summation Formulae

Some useful summation formulae are shown below:

(1) $\sum_{k=1}^{n} k = 1+2+3+...+n = \frac{n(n+1)}{2}$

(2) $\sum_{k=1}^{n} k^2 = 1^2+2^2+3^2+...+n^2 = \frac{n(n+1)(2n+1)}{6}$

(3) $\sum_{k=1}^{n} k^3 = 1^3+2^3+3^3+...+n^3 = \frac{n^2(n+1)^2}{4}$

(4) $\sum_{k=1}^{n} \left(a+(k-1)d\right) = \frac{n\left(2a+(n-1)d\right)}{2}$ (formula for Arithmetic Series)

(5) $\sum_{k=1}^{n} ar^{(k-1)} = \frac{a(r^n-1)}{r-1} = \frac{a(1-r^n)}{1-r}$ for $r \neq 1$ (formula for Geometric Series)

(6) $\sum_{k=1}^{\infty} ar^{(k-1)} = \frac{a}{1-r}$ for $-1 < r < 1$ (formula for Infinite Geometric Series)

Example 1

$$\begin{aligned}
&\sum_{k=1}^{50}(k+1)^2 \\
&= \sum_{k=1}^{50}(k^2+2k+1) \\
&= \sum_{k=1}^{50}k^2 + 2\sum_{k=1}^{50}k + \sum_{k=1}^{50}(1) \\
&= \frac{50(51)(101)}{6} + 2\frac{50(51)}{2} + 50 = 45525
\end{aligned}$$

Alternatively, the required sum can be found as follows:

$$\sum_{k=1}^{50}(k+1)^2 = \sum_{k=2}^{51}k^2 = \sum_{k=1}^{51}k^2 - 1^2 = \frac{51(52)(103)}{6} - 1 = 45526 - 1 = 45525$$

Example 2

$$\sum_{i=1}^{50}(i^3+i^2)$$
$$=\sum_{i=1}^{50}i^3+\sum_{i=1}^{50}i^2$$
$$=\frac{50^2(51)^2}{4}+\frac{50(51)(101)}{6}=1668500$$

Example 3

$$\sum_{k=4}^{7}k^2$$
$$=\sum_{k=1}^{7}k^2-\sum_{k=1}^{3}k^2$$
$$=\frac{7(8)(15)}{6}-\frac{3(4)(7)}{6}=126$$

Please attempt Interactive Exercise Question 1.4.

1.3 Exponential, Logarithm and *e*

In this chapter, we are going to review the properties of exponential and logarithmic functions. We will also explore the interesting Euler's number, ***e***.

1.3.1 Meaning of Logarithm

Logarithm is a function of the form:

$$f(x) = \log_a x$$

where ***a*** is called the **base**. The value of logarithm can be interpreted as the power of a in order to get x. In other words,

$$y = \log_a x \ \text{ means } \ a^y = x$$

Example 1

If $\log_4 x = 2$, then $x = 4^2 = 16$.

Example 2

If $\log_9 x = \frac{1}{2}$, then $x = 9^{\frac{1}{2}} = 3$.

Example 3

If $\log_2 \frac{y}{3} = 4$, then $\frac{y}{3} = 2^4 = 16$, hence, $y = 48$.

Please attempt Interactive Exercise Question 1.5.

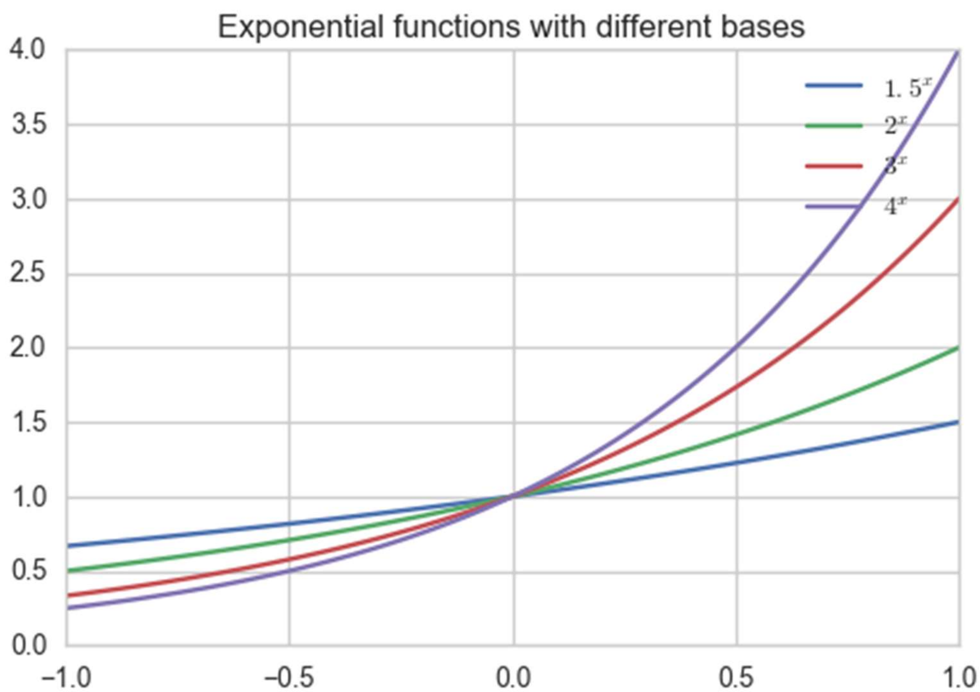
1.3.2 Exponential and Logarithmic Functions

Exponential function is a function of the form:

$$f(x) = a^x$$

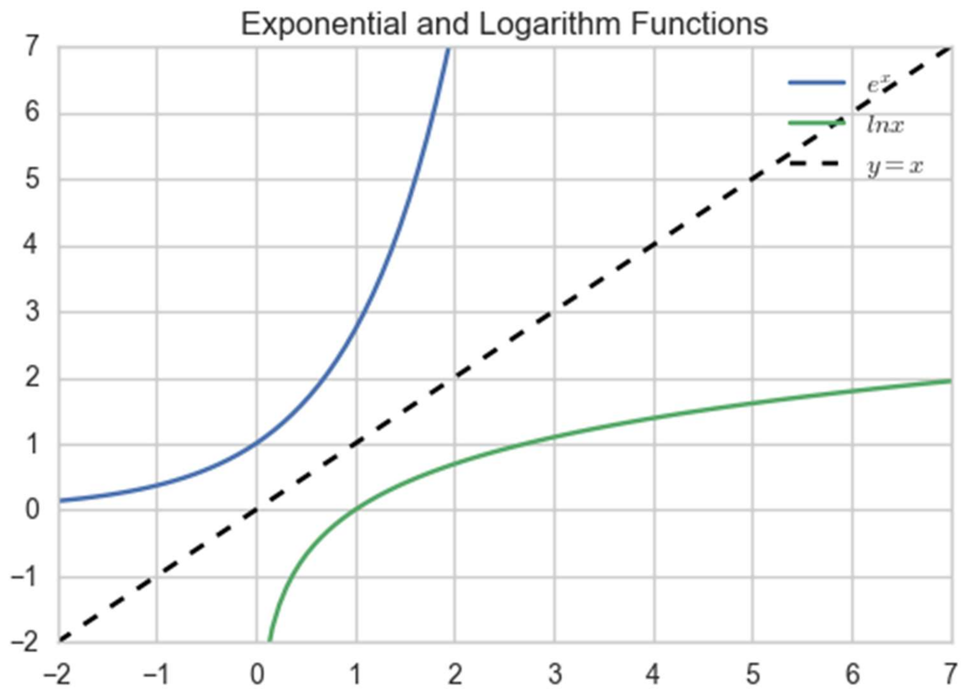
where the constant ***a*** is called **base**.

We have already done a review on law of indices in Section 1.1, which is very useful in manipulating exponential functions. Using Rule of Zero, we know that for any $a \neq 0$, $f(0) = 1$, which means that every exponential function passes through the point (0,1).



Also, by Rule of One and Product Rule in Section 1.1.1., we could evaluate any exponential functions with integral powers.

The logarithmic function is the **inverse function** of the exponential function. In the previous section, we have already seen that solving an equation involving logarithms often gives an equation with exponentials. We could grasp this relation by plotting the line $y = x$.



The figure illustrates the relation between exponential and logarithmic functions. The graph of any one of them can be obtained by reflecting the graph of the other function about the line $y = x$.

1.3.3 Manipulation of Logarithms

In this section we are going to review the rules to manipulate logarithms.

Rule of One

$$\log_a a = 1$$

Rule of Change of Base

$$\log_a x = \frac{\log_b x}{\log_b a}$$

Product Rule

$$\log_a xy = \log_a x + \log_b y$$

Quotient Rule

$$\log_a \frac{x}{y} = \log_a x - \log_a y$$

Reciprocal Rule

$$\log_a \frac{1}{y} = -\log_a y$$

Power Rule

$$\log_a x^y = y \log_a x$$

Some examples illustrating the rules are shown below:

Example 1

$$\begin{aligned}\log_b \frac{xy}{z} &= \log_b xy - \log_b z \\ &= \log_b x + \log_b y - \log_b z\end{aligned}$$

Example 2

$$\begin{aligned}\log_5 5^p &= p\log_5 5 \\ &= p(1) = p\end{aligned}$$

Example 3

$$\begin{aligned}\log_2 (8x)^{\frac{1}{3}} &= \frac{1}{3}\log_2 (8x) \\ &= \frac{1}{3}[\log_2 (8) + \log_2 x] \\ &= \frac{1}{3}(3 + \log_2 x) \\ &= 1 + \frac{1}{3}\log_2 x\end{aligned}$$

Example 4

Find x if $2\log_b 5 + \frac{1}{2}\log_b 9 - \log_b 3 = \log_b x$.

$$2\log_b 5 + \frac{1}{2}\log_b 9 - \log_b 3 = \log_b x$$

$$\begin{aligned}\log_b 5^2 + \log_b 9^{\frac{1}{2}} - \log_b 3 &= \log_b x \\ \log_b 5^2 + \log_b 3 - \log_b 3 &= \log_b x \\ \log_b 25 &= \log_b x \\ x &= 25\end{aligned}$$

Example 5

$$\begin{aligned}\log_2 \frac{8x^3}{2y} &= \log_2 8x^3 - \log_2 2y \\ &= \log_2 8 + \log_2 x^3 - (\log_2 2 + \log_2 y) \\ &= 3 + 3\log_2 x - (1 + \log_2 y) \\ &= 3 + 3\log_2 x - 1 - \log_2 y \\ &= 2 + 3\log_2 x - \log_2 y\end{aligned}$$

1.3.4 Euler's Number and Natural Logarithm

In this section we are going to introduce a constant called Euler's Number, denoted by ***e***.

Imagine that you get a loan from a bank and the annual interest is 100%! If the interest is compounded monthly instead of annually, we could calculate the value of the loan compounded monthly after one year by:

$$P(1+\frac{100\%}{12})^{12}$$

where P is the starting principal. Assume that the starting principal is \$10000, the value after one year would be approximately \$26130.

Suppose now the bank wants to receive more interest and offers a lending plan which is compounded daily. Similarly, the value after one year can be calculated by

$$10000(1+\frac{100\%}{365})^{365} \approx 27145.67$$

This amount is surely more than the previous one. However, the difference may be less than your expectation. There is merely an increase of about \$1000. The bank seems to be disappointed by this result so it offers an extraordinarily aggressive plan: compounded every second:

$$10000(1+\frac{100\%}{31536000})^{31536000} \approx 27182.818$$

The increase is again small. It seems that there is a certain value that cannot be exceeded no matter how frequently the loan is compounded. Actually, that value exists even the loan is compounded infinitely:

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = 2.71828182846\ldots$$

This value is called the Euler's Number $\boldsymbol{e}$. Therefore, even the loan is compounded infinitely, the amount would only be about \$27183.

We have already seen the relation between exponential and logarithmic functions in the previous sections. The corresponding logarithm with base $\boldsymbol{e}$ is known as the natural logarithm, and is usually denoted as ln. The rules discussed in Section 1.3.4 are also applicable to the natural logarithm.

Please attempt interactive Exercise Question 1.6.

1.4 Set Theory

Many probability rules and theorems involve extensive use of set operations. In this section we are going to introduce the basic notations and terminology of set theory.

A set is a collection of objects, which are called the **elements** of the set. If S is a set and x is an element of S, we write $x \in S$. If x is not in S, we write $x \notin S$. If a set has no element, we call it **empty set**, denoted by $\emptyset$.

We can also specify the content of a set. The number of elements in a set may be finite or infinite. If a set has a finite number of elements, we can write it as a list of elements in braces:

$$S = \{x_1, x_2, \ldots, x_n\}$$

For example, we could specify the set of outcomes of a rolling a die as $\{1, 2, 3, 4, 5, 6\}$.

Another way to specify a set is to state the property of its elements (this is applicable when the number of elements in a set is finite or infinite) as follows:

$S = \{x : x \text{ satisfies P}\}$ where P is the general property of the elements in S.

For example, the set of even integers can be specified as $\{k : \frac{k}{2} \text{ is an integer}\}$.

1.4.1 Set Operations

If every element of a set S is also an element of a set T, we say that S is a **subset** of T, and we write $S \subset T$. Two sets are said to be equal, denoted by $S = T$, if the elements of them are the same, i.e. $S \subset T$ and $T \subset S$.

A **universal set**, denoted by Ω, is the set containing all the elements involved in the problem under consideration. For example, if we only consider positive integers not more than 100, the universal set is $\{1,2,3,...,100\}$; if we toss a coin once, the elements (called "outcomes" in the terminology of probability) are "head" and "tail", so the universal set is $\{H,T\}$.

The **complement** of a set S is the set $\{x : x \in \Omega \text{ and } x \notin S\}$, which contains all elements of Ω that do not belong to S, and is denoted by S^c, S' or $\overline{S}$.

The **union** of two sets S and T is the set of all elements that belong to S or T (or both), and is denoted by $S \cup T$. The **intersection** of two sets S and T is the set of all elements that belongs to both S and T, and is denoted by $S \cap T$. In other words,

$$S \cup T = \{x : x \in S \text{ or } x \in T\}$$

$$S \cap T = \{x : x \in S \text{ and } x \in T\}$$

Two sets are said to be **disjoint** if their intersection is empty. Alternatively, two sets are said to be disjoint if they do not have any common element.

1.4.2 Properties of Set Operations

Some basic properties of set operations are:

Identity Law

$$A \cup \varnothing = A$$

$$A \cap \Omega = A$$

Remark:

An empty set contains no element. Therefore, the union of set A and the empty set is still A. Also, as every element of set A belongs to the universal set, their intersection contains all elements of A, which is A.

Domination Law

$$A \cup \Omega = \Omega$$

$$A \cap \varnothing = \varnothing$$

Remark:

Union of universal set and any other set A is still universal set because every element contained in universal set contributes to forming the union.

In contrast, the intersection of an empty set and any other set A is an empty set because no element can belong to both A and an empty set.

Commutative Law

$$A \cup B = B \cup A$$

$$A \cap B = B \cap A$$

Associative Law

$$(A \cup B) \cup C = A \cup (B \cup C)$$

$$(A \cap B) \cap C = A \cap (B \cap C)$$

Remark:

The order of a sequence of union and intersection operations does not matter.

Distributive Law

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

De Morgan's Law

$$(A \cup B)^c = A^c \cap B^c$$

$$(A \cap B)^c = A^c \cup B^c$$

Example 1

Prove that $B \cup (\varnothing \cap A) = B$.

Proof:

$$\begin{aligned} B \cup (\varnothing \cap A) &= B \cup \varnothing && \text{(by Domination Law)} \\ &= B && \text{(by Identity Law)} \end{aligned}$$

Example 2

Prove that $(A \cap B) \cup (A \cup B^C)^C = B$.

Proof:

$$\begin{aligned} (A \cap B) \cup (A \cup B^c)^c &= (A \cap B) \cup (A^c \cap B) && \text{(by De Morgan's Law)} \\ &= (A \cup A^c) \cap B && \text{(by Distributive Law)} \\ &= \Omega \cap B \\ &= B && \text{(by Identity Law)} \end{aligned}$$

Please attempt Interactive Exercise 1.7.