Contents

3. Probability

In this chapter, we are going to review the basic concepts of probability: basic properties, addition and multiplication rules of probability.

3.1 Basic Concepts

3.1.1 Experiment

In probability, an **experiment** is a process which has at least two possible outcomes, and it is uncertain that which outcome will occur before the experiment is done. For example, if you are going to flip a coin (which has a head and a tail) once, there are two possible outcomes: "getting a head" or "getting a tail". The result will be known only when you have flipped the coin. The outcome of the second flipping of the coin may be different from that of the first flipping.

3.1.2 Sample Space and Event

For a given experiment, the set of all possible outcomes is called the **sample space**, denoted by S. Each outcome in the sample space is called a **sample point**. An **event** is a subset of the sample space consisting of one or more sample points.

Example:

If you are going to flip a coin once, the sample space is $S = \{H, T\}$ where H denotes the outcome of "getting a head", and T denotes the outcome of "getting a tail".

If you are going to flip a coin three times, the sample space has 8 sample points as follow:

$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$.

The event of "getting 3 heads" has only one sample point "*HHH*", while the event of "getting two heads" has three sample points: "*HHT*", "*HTH*", and "*THH*".

3.1.3 Basic Properties of Probability

A measure of the likelihood that an event will occur is known as the **probability** of the event. The probability of the event *A* is denoted by $P(A)$. There are some basic properties for assigning probabilities to events.

1. $0 \le P(A) \le 1$ for any event *A*;
2. $P(S) = 1$;
3. $P(\phi) = 0$, where ϕ is called the **empty space**, the event containing no outcomes.

3.1.4 Classical Approach of Assigning Probabilities

The classical approach is based on the assumption that all the outcomes (sample points) in the sample space of the concerned experiment are equally likely to occur. Hence, by counting the total number of sample points in the sample space, say, *N*, and the number of sample points corresponding to a certain event *A*, say, *n*, the probability of event *A* is computed as

$$P(A) = \frac{n}{N} .$$

A common way of stating the classical approach of assigning probabilities is

$$P(A) = \frac{\text{number of outcomes favourable to event } A}{\text{total number of possible outcomes}} .$$

Example:

Referring to the experiment of flipping a coin three times, there are 8 sample points in the sample space and 3 of them have two heads each. Hence, based on the classical approach, the probability of "getting two heads when a coin is flipped three times" is $\frac{3}{8}$.

3.2 Counting

According to the classical approach, the probability of an event can be found by dividing the number of sample points corresponding to the event by the total number of sample points in the sample space. However, counting the number of sample points is not easy for some events.

In the coming sections, we will introduce two types of counting methods, namely, permutation and combination.

3.2.1 Permutation

Let us consider the following situation. There is a class with 30 students and we are going to pick 10 students out of them and arrange them in a line. How many ways can the line be formed?

There are 30 students available before picking the first one. After choosing the first one, the picked one is arranged in the first position in the line. Then we choose another student for the second position among the remaining 29 students. The number of arrangements for the first and the second positions is therefore 30×29. Similarly, we choose 8 more students for the remaining positions. The total number of arrangements for forming the line is

$$30\times 29\times \ldots\times 22\times 21$$

A **permutation** is an arrangement of all or part of a set of objects. For the same group of 10 students, different orders in the line are counted as different permutations.

Example

How many arrangements consisting of five distinct letters can be formed? There are 26 distinct letters and we are going to pick 5 of them and arrange them in different orders. Following the discussion above, the number of permutations is

$$26\times 25\times 24\times 23\times 22 = 7893600$$

The number of permutations of *n* distinct objects taken *r* at a time, denoted by nPr, is

$$nPr = n(n-1)...(n-r+2)(n-r+1) = \frac{n(n-1)...2(1)}{(n-r)...2(1)}$$

The number of permutations of *n* distinct object is *n*-factorial and is denoted by $n!$

$$n! = n(n-1)...2(1)$$

The formula for nPr can be expressed as

$$nPr = \frac{n!}{(n-r)!}$$

The number of ways of choosing 10 students out of 30 and arranging them in a line is ${}_{30}P_{10}$. We can form ${}_{26}P_5$ arrangements consisting of 5 distinct letters.

3.2.2 Combination

Suppose we are going to form a committee of 5 members from a class of 30 students. How many different committees can be formed?

If we denote 5 particular students by A, B, C, D and E. These 5 students can be arranged in 5! ways in a line. For example, "A,B,C,D,E" and "B,A,C,D,E" represent different arrangements when they are lining up. However, these two permutations and all other permutations of these 5 students are actually the same committee.

The example of forming a committee from a given group of people indicates that some problems require "selections" instead of "arrangements". These selections, without regard to order, are called **combinations.** The number of combinations of *n* distinct objects taken *r* at a time, denoted by nCr, is

$$nCr = \frac{n!}{r!(n-r)!}$$

The number of 5-member committees that can formed from 30 students is therefore

$$_{30}C_5 = \frac{30!}{5!25!} = 142506$$

3.2.3 Miscellaneous Examples

In this session, we would explore more examples using the techniques of counting.

Example 1

Find the probability of getting a King Full House when 5 cards are randomly drawn from a well-shuffled deck.

In order to get a King Full House, we have to get 3 Kings and a pair.

As the order does not matter, there are $_4C_3$ ways for drawing three Kings out of four.

The pair can be any one of the 12 kinds other than King. There are $_{12}C_1$ ways for selecting a kind for the pair. For the selected kind, there are $_4C_2$ ways of picking two cards out of four.

To summarize, the total number of sample points of drawing King Full House is

$$({}_4C_3)({}_{12}C_1)({}_4C_2)$$

Also, the total number of sample points in the sample space (selecting 5 cards out of 52) is ${}_{52}C_5$.

Hence, by the classical approach, the required probability is

$$\frac{({}_4C_3)({}_{12}C_1)({}_4C_2)}{{}_{52}C_5} = \frac{1}{54145}$$

Example 2

Question:

Suppose that there are six cards labelled 1,2,3,4,5,6, respectively.

(a) If you choose 3 cards out of 6 and arrange them in a row, how many 3-digit numbers can be formed?

(b) If you choose 4 cards out of 6 and arrange them in a row, how many 4-digit numbers larger than 4000 can be formed?

Answer:

(a) The question requires us to pick 3 cards out of 6 and arrange them. Clearly, order matters in this case. Therefore, the number of 3-digit numbers that can be formed is ${}_6P_3 = 120$.

(b) In order that the numbers formed will be larger than 4000, the first digit can only be 4, 5 or 6. Then, we pick any 3 numbers in the remaining 5. Also, order of digits matters in this case. Hence, the number of 4-digit numbers larger than 4000 that can be formed is

$$({}_3C_1)({}_5P_3) = 180$$

Example 3

Question:

There are 7 people and 2 of them are sick. These 7 people will be arranged in a row. Find the number of possible ways that the 2 sick people will be next to each other.

Answer:

We could consider the two sick people as one entity. First we find the number of permutations of the 6 entities (5 people and the group of sick people), which is ${}_6P_6$. Then, within the group, there are ${}_2P_2$ permutations for the 2 sick people. Therefore, the total number of permutations is

$$({}_6P_6)({}_2P_2) = 1440$$

Example 4

Question:

There are 25 members in a football team. The team will play 2 matches. In each match, 11 members are needed for the starting lineup.

(a) How many ways can the starting lineup be formed for the first match?
(b) Three members not in the starting lineup for the first match will be selected for the starting lineup for the second match. How many ways can the starting lineup be formed for the second match, after the first match has been played?

Answer:

(a) As the order does not matter, the number of possible ways of forming the starting lineup for the first match is ${}_{25}C_{11} = 4457400$.

(b) Three members in the starting lineup for the first match have to leave, and 3 members from the 14 members not in the starting lineup for the first match have to join. Again, order does not matter. Hence, after the first match has been played, the total number of ways of forming the starting lineup for the second match is $({}_{11}C_3)({}_{14}C_3) = 60060$.

Example 5

Question:

There are 26 letters (A to Z) and 10 numbers (0-9).

(a) If we want to choose 4 different letters and 4 different numbers, how many combinations are there?
(b) If each password consists of 4 different letters and 4 different numbers, how many passwords can be formed?

Answer:

(a) To choose 4 letters out of 26, there are ${}_{26}C_4$ ways. To choose 4 numbers out of 10, there are ${}_{10}C_4$ ways. The total number of combinations would be $({}_{26}C_4)({}_{10}C_4) = 3139500$.

(b) For each combination in (a), different orders or permutations give different passwords. So

for each combination in (a), ${}_8P_8$ passwords can be formed. Therefore, the total number of passwords that can be formed is $({}_{26}C_4)({}_{10}C_4)({}_8P_8) = 126,584,640,000$.

Please attempt Interactive Exercise Question 3.1.

3.3 Addition Rule of Probability

3.3.1 Addition Rule for Mutually Exclusive Events

In probability, the union of events *A* and *B*, denoted by $A \cup B$, is the event containing all the elements that belong to *A* only, to *B* only, or to both. Events are **disjoint** or **mutually exclusive** if they have no elements in common.

Addition Rule for mutually exclusive events:

If *A* and *B* are mutually exclusive events, then

$$P(A \cup B) = P(A) + P(B)$$

Example:

Suppose we are going to toss a coin twice. What is the probability of the event of "getting one head and one tail"?

For this experiment, the sample space is $S = \{HH, HT, TH, TT\}$. The event of "getting one head and one tail" has two sample points: "*HT*" and "*TH*". The required probability is therefore $\frac{2}{4}$ or $\frac{1}{2}$. This probability can also be found by using the Addition Rule of probability as follows:

Let *E* be the event of getting one head and one tail, *A* be the event of getting "*HT*", and *B* be the event of getting "*TH*".

E happens if *A* or *B* happens. Furthermore, *A* and *B* cannot happen at the same time as they have no elements in common. So *A* and *B* are mutually exclusive.

Applying the addition rule, we have

$$P(E) = P(A \cup B) = P(A) + P(B) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

We obtain the same result as expected.

The Addition Rule for mutually exclusively events can be extended to the situation involving more than two mutually exclusive events. For example, in the example of flipping a coin three times in section 3.1.2, "*HHT*", "*HTH*" and "*THH*" are mutually exclusive. Hence, the probability of getting 2 heads can also be found as:

$$\begin{aligned} P(\text{getting 2 heads}) &= P(HHT) + P(HTH) + P(THH) \\ &= \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8} \end{aligned}$$

Example:

There are ninety attendees in a party, including Chris and Winnie. The attendees will be randomly split into three groups of equal size. What is the probability that Chris and Winne will end up in the same group?

Let us label the three groups I, II and III. Define events *A*, *B* and *C* as Chris and Winnie will be in groups I, II and III, respectively. These three events are mutually exclusive. Hence,

$$\begin{aligned} &P(\text{Chris and Winnie will be in the same group}) \\ &= P(A \cup B \cup C) = P(A) + P(B) + P(C) \end{aligned}$$

For event A to occur,

- Chris and Winnie must be in group I, hence, ${}_2C_2$ way;
- there are ${}_{88}C_{28}$ ways to select 28 attendees out of 88 (as Chris and Winnie have been chosen) to fill the remaining 28 positions in group I;
- there are $({}_{60}C_{30})({}_{30}C_{30})$ ways to form groups II and III from the remaining 60 attendees.

Total number of ways to form the three groups is $({}_{90}C_{30})({}_{60}C_{30})({}_{30}C_{30})$. Hence,

$$P(A)=\frac{({}_{2}C_{2})({}_{88}C_{28})({}_{60}C_{30})({}_{30}C_{30})}{({}_{90}C_{30})({}_{60}C_{30})({}_{30}C_{30})}=\frac{{}_{88}C_{28}}{{}_{90}C_{30}}$$

As the three groups have the same size, we have $P(A)=P(B)=P(C)$.

$$P(\text{Chris and Winnie will be in the same group})$$
$$=P(A)+P(B)+P(C)=3P(A)=3\left(\frac{{}_{88}C_{28}}{{}_{90}C_{30}}\right)=\frac{29}{89}$$

3.3.2 Addition Rule for Non-mutually Exclusive Events

In probability, the **intersection** of two events *A* and *B*, denoted by $A\cap B$, is the event containing all the elements that are common to *A* and *B*.

Addition Rule for non-mutually exclusive events:

If events *A* and *B* are not mutually exclusive, then

$$P(A\cup B)=P(A)+P(B)-P(A\cap B)$$

Example

If we are going to toss a die having six faces once, the sample space for the number of dots to be obtained is

$$S=\{1,2,3,4,5,6\}$$

Let *A* be the event that an odd number of dots will be obtained, and *B* be the event that more than 3 dots will be obtained. We want to find $P(A\cup B)$. Events *A* and *B* are not mutually exclusive as they have the element "5" in common. So we have to use the addition rule for non-mutually exclusive events. *A* has 3 sample points, *B* has 3 sample points and $A\cap B$ has 1 sample point. Hence, we have the following:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$= \frac{3}{6} + \frac{3}{6} - \frac{1}{6} = \frac{5}{6}$$

3.4 Conditional Probability and Independence

Consider the experiment of tossing a fair die once. The probability of getting 5 dots is $\frac{1}{6}$. If it is known that an odd number of dots is obtained with the toss, is the probability of getting 5 dots still $\frac{1}{6}$?

If there is no additional information about the toss, the sample space consists of 6 sample points. Now with the additional information or the given condition that an odd number of dots is obtained with the toss, the number of sample points or possible outcomes is reduced from 6 to 3. So knowing that an odd number of dots is obtained with the toss, we would expect that the probability of getting 5 dots is $\frac{1}{3}$ instead of $\frac{1}{6}$.

3.4.1 Conditional Probability

The probability of event **A** occurs given that event **B** occurs is a **conditional probability**, and is denoted as $P(A \mid B)$.

The definition of conditional probability is as follows:

Let A and B be two events. The conditional probability of event A, given event B, is defined as

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

provided that $P(B) > 0$.

Referring to the experiment of tossing a fair die once. Let A be the event of getting 5 dots, and B be the event of getting an odd number of dots. Then by definition,

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1/6}{1/2} = \frac{1}{3}$$

This is the same as what we have expected in the discussion before introducing the definition of conditional probability.

3.4.2 Multiplication Rule

Rearranging the formula in the definition of conditional probability, we obtain

$$P(A \cap B) = P(A|B)P(B)$$

We can also obtain $$P(A \cap B) = P(B|A)P(A)$$

This is called the **multiplication rule**.

The multiplication rule can be extended to three or more events. In the case of three events, say $C_1, C_2\ and\ C_3$, we could derive the multiplication rule from the multiplication rule of two events:

$$\begin{aligned} P(C_1 \cap C_2 \cap C_3) &= P\big((C_1 \cap C_2) \cap C_3\big) \\ &= P\big(C_3 \mid (C_1 \cap C_2)\big)P(C_1 \cap C_2) \\ &= P\big(C_3 \mid (C_1 \cap C_2)\big)P(C_2 \mid C_1)P(C_1) \end{aligned}$$

Example

Suppose the probability that you would miss a bus is $\frac{1}{8}$ on a sunny day and is $\frac{1}{4}$ on a rainy day. Also, both the probabilities of a day being sunny and a day being rainy are $\frac{1}{2}$ each. What is the probability that tomorrow will be sunny but you will miss a bus?

Let M be the event that you miss a bus, S be the event that a given day is sunny and R be the event that a given day is rainy.

We have the following probabilities:

$$P(S) = P(R) = \frac{1}{2} \ , \ \ P(M \mid S) = \frac{1}{8} \ , \ P(M \mid R) = \frac{1}{4}$$

Hence, the required probability is found as:

$$P(S \cap M) = P(M \mid S)P(S) = \frac{1}{8}(\frac{1}{2}) = \frac{1}{16}$$

3.4.3 Independence

It happens that for some events A and B, $P(A \mid B) = P(A)$. (If this happens, we also have $P(B \mid A) = P(B)$.) This means that the occurrence (or non-occurrence) of event B does not change the probability of the occurrence of event A. We say that A and B are **independent events**. The formal definition of independence in probability is:

Independence

Two events A and B are independent if and only if

$$P(A \cap B) = P(A)P(B)$$

This is called the **Multiplication Rule for independent events**.

Example

Suppose we are throwing a fair die. Find the probability that we will get three consecutive "1".

Let C_i be the event of getting a "1" in the *i*th round. As each throwing is independent of one another, we can simply multiply the individual probabilities together, i.e.,

$$\begin{aligned} P(\text{getting three consecutive "1"}) &= P(C_1 \cap C_2 \cap C_3) \\ &= P(C_1)P(C_2)P(C_3) \\ &= \frac{1}{6}(\frac{1}{6})(\frac{1}{6}) = \frac{1}{216} \end{aligned}$$

Please attempt Interactive Exercise Question 3.2.