

CHAPTER 2 PROBABILITY

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Objectives: After working through this chapter, you should be able to:

- (i) understand the basic concepts of probability;
- (ii) compute the probability of an event;
- (iii) understand and apply the addition and multiplication rules in solving business application problems;
- (iv) understand and apply the theorem of total probability and Bayes' theorem.

2.1 Basic Concepts of Probability

In statistics, an experiment is a process leading to at least two possible outcomes with uncertainty as to which will occur.

The set of all possible outcomes of an experiment is called the sample space (S). Each outcome in S is called a sample point.

Example 1

Three items are selected at random from a manufacturing process. Each item is inspected and classified defective (D) or non-defective (N).

$$S =$$

An event is a subset of a sample space, it consists of one or more outcomes with a common characteristic.

Example 2

The event that the number of defectives in above example is greater than 1.

$$A =$$

The null space or empty space is a subset of the sample space that contains no outcomes (\emptyset).

The intersection of two events A and B denoted by $(A \cap B)$ is the event containing all outcomes that are common to A and B.

Events are mutually exclusive if they have no elements in common.

The union of two events A and B denoted by $(A \cup B)$ is the event containing all the elements that belong to A or to B or to both.

Events are collectively exhaustive if no other outcome is possible for a given experiment.

The complement of an event A with respect to S is the set of outcomes of S that are not in A denoted by $(A'$ or A^c or $\bar{A})$.

2.2 Probability of an Event

Notation : $P(A)$ The probability of an event A

Probability postulates :

1. $P(S) = 1$
2. $P(\emptyset) = 0$
3. $0 \leq P(A) \leq 1$

2.3 Methods of Assigning Probabilities

1. The classical approach

If an experiment can result in any one of N different equally likely outcomes, and if exactly n of these outcomes correspond to event A, then

$$P(A) = \frac{n}{N}$$

2. The relative frequency approach

If some number N of experiments are conducted and the event A occurs in N_A of them, then

$$P(A) = \lim_{N \rightarrow \infty} \frac{N_A}{N}$$

3. The subjective approach

Subjective probability is a personal assessment of the likelihood of an event.

2.4 Principle of Counting - Permutation and Combination

2.4.1 Counting Sample Points

Fundamental principle of counting:

If an operation can be performed in N_1 ways, a second operation can be performed in N_2 ways, and so forth, then the sequence of k operations can be performed in

$$N_1 N_2 N_3 \dots N_k \text{ ways}$$

Example 3

Suppose a licence plates containing two letters following by three digits with the first digit not zero. How many different licence plates can be printed?

A permutation is an arrangement of all or part of a set of objects.

Example 4

The possible permutations from 3 letters A, B, C

The number of permutations of n distinct objects is n!.

The number of permutations of n distinct objects taken r at a time is

$${}^n P_r = \frac{n!}{(n-r)!}$$

Example 5

In how many ways can 10 people be seated on a bench if only 4 seats are available?

The combination is a collection of n objects taken r at a time in any selections of r objects where order does not count. The number of combinations of n objects taken r at a time is

$${}^n C_r = \frac{n!}{r!(n-r)!}$$

Example 6

The combinations of letters A, B, C, D taken 3 at a time.

Example 7

A box contains 8 eggs, 3 of which are rotten. Three eggs are picked at random. Find the probabilities of the following events.

- a) Exactly two eggs are rotten.
- b) All eggs are rotten.
- c) No egg is rotten.

2.5 Addition Rule and Complementary Rule

2.5.1 Addition Rule

1. For events that are not mutually exclusive

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Example 8

A card is drawn from a complete deck of playing cards. What is the probability that the card is a heart or an ace?

2. For mutually exclusive events

$$P(A \cup B) = P(A) + P(B)$$

2.5.2 Complementary Rule

If A and A' are complementary events then

$$P(A') = 1 - P(A)$$

Example 9

A coin is tossed six times in succession. What is the probability that at least one head occurs?

2.6 Conditional Probability, Statistically Independence and Multiplication Rule**2.6.1 Conditional Probability**

Let A and B be two events. The conditional probability of event A, given event B, denoted by $P(A|B)$ is defined as

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \quad \text{provided that } P(B) > 0.$$

Similarly, the conditional probability of B given A is defined as

$$P(B|A) = \frac{P(A \cap B)}{P(A)} \quad \text{provided that } P(A) > 0.$$

Example 10

A hamburger chain found that 75% of all customers use mustard, 80% use ketchup, and 65% use both, when ordering a hamburger. What are the probabilities that

- a) a ketchup-user uses mustard?
- b) a mustard-user uses ketchup?

2.6.2 Statistically Independence

Two events are independent when the occurrence or non-occurrence of one event has no effect on the probability of occurrence of the other event.

Definition : Two events A and B are independent if and only if
$$P(A \cap B) = P(A)P(B)$$

2.6.3 Multiplication Rule

1. For dependent events

$$P(A \cap B) = P(A)P(B|A) \quad \text{or} \\ = P(B)P(A|B)$$

2. For independent events

$$P(A \cap B) = P(A)P(B)$$

Example 11

A pair of fair dice are thrown twice. What is the probability of getting totals of 7 and 11?

2.7 Theorem of Total Probability

If the events B_1, B_2, \dots, B_k constitute a partition of the sample space S such that $P(B_i) \neq 0$ ($i = 1, \dots, k$) then for any event A of S

$$P(A) = P(B_1)P(A|B_1) + P(B_2)P(A|B_2) + \dots + P(B_k)P(A|B_k)$$

Example 12

Suppose 50% of the cars are manufactured in the United States and 15% of these are compact; 30% of the cars are manufactured in Europe and 40% of these are compact; and finally, 20% are manufactured in Japan and 60% of these are compact. If a car is picked at random from the lot, find the probability that it is a compact.

2.8 Bayes' Theorem

If B_1, B_2, \dots, B_k are mutually exclusive events such that $B_1 \cup B_2 \cup \dots \cup B_k$ contains all sample points of S , then for any event A of S with $P(A) \neq 0$,

$$P(B_i|A) = \frac{P(B_i)P(A|B_i)}{P(B_1)P(A|B_1) + P(B_2)P(A|B_2) + \dots + P(B_k)P(A|B_k)}$$

for $i = 1, 2, 3, \dots, k$

Example 13

Suppose a box contains 2 red balls and 1 white ball and a second box contains 2 red balls and 2 white balls. One of the boxes is selected by chance and a ball drawn from it. If the drawn ball is red, what is the probability that it came from the 1st box?

Solution: A : the event of drawing a red ball

B : the event of choosing the 1st box

EXERCISE : PROBABILITY

1. If a multiple-choice test consists of 5 questions each with 4 possible answers of which only one is correct,
 - (a) How many different ways can a student check off one answer to each question?
 - (b) How many ways can a student check off one answer to each question and get all the questions wrong?
2.
 - (a) How many ways can 6 people be lined up to get on a bus?
 - (b) If a certain 3 persons insist on following each other, how many ways are possible?
 - (c) If a certain 2 persons insist on following each other, how many ways are possible?
3. From 5 statisticians and 6 economists a committee consisting of 3 statisticians and 2 economists is to be formed. How many different committees can be formed if (a) no restriction is imposed; (b) two particular statisticians must be on the committee; (c) one particular economist cannot be on the committee?
4. From 4 red, 5 green and 6 yellow apples, how many selections of 9 apples are possible if 3 of each color are to be selected?
5. If a permutation of the word “white” is selected at random, find the probability that the permutation (a) begins with a consonant; (b) ends with a vowel; (c) has the consonants and vowels alternating.
6. If 7 cards are dealt from an ordinary deck of 52 playing cards, what is the probability that (a) exactly 2 of them will be face cards? (b) at least 1 of them will be queen?
7. A group of 10 boys and 10 girls is divided into two equal groups. Find the probability that each group has equal number of boys and girls.
8. A ball is drawn at random from a box containing 6 red balls, 4 white balls and 5 blue balls. Determine the probability that it is (a) red, (b) white, (c) blue, (d) not red, (e) white or red.
9. A fair coin is tossed 6 times in succession. What is the probability that at least 1 head occurs?
10. In a lot of 1200 golf balls, 40 have imperfect covers, 32 cannot bounce, and 12 have both defects. If a ball is selected at random from the lot, what is the probability that the ball is defective?

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11. The probability that a student passes mathematics is $\frac{2}{3}$, and the probability that he passes English is $\frac{4}{9}$. If the probability of passing at least one course is $\frac{4}{5}$, what is the probability that he will pass both courses?
12. Two cards are drawn from a well-shuffled ordinary deck of 52 cards. Find the probability that they are both aces if the first card is (a) replaced, (b) not replaced.
13. Three balls are drawn successively from the box of Problem 8. Find the probability that they are drawn in the order red, white and blue if each ball is (a) replaced, (b) not replaced.
14. Find the probability of a 4 turning up at least once in two tosses of a fair die.
15. One bag contains 4 white balls and 2 black balls; another contains 3 white balls and 5 black balls. If one ball is drawn from each bag, find the probability that (a) both are white, (b) both are black, (c) one is white and one is black.
16. An urn contains four white balls, numbered 1, 2, 3 and 4, and six black balls, numbered 5, 6, 7, 8, 9 and 10. A ball, selected at random from the urn, is white, what is the probability that the ball chosen is numbered 1?
17. A and B play 12 games of chess of which 6 are won by A, 4 are won by B, and 2 end in a tie. They agree to play a tournament consisting of 3 games. Find the probability that (a) A wins all three games, (b) two games end in a tie, (c) A and B win alternately, (d) B wins at least one game.
18. One bag contains 4 white balls and 3 black balls, and a second bag contains 3 white balls and 5 black balls. One ball is drawn from the 1st bag and is placed unseen in the second bag. What is the probability that a ball now drawn from the 2nd bag is black?
19. The results of 100 students who sit two examinations X and Y, are given below.

| | |
|-------------------|-------------|
| Pass X and pass Y | 43 students |
| Pass X and fail Y | 19 students |
| Fail X and pass Y | 15 students |
| Fail X and fail Y | 23 students |

Required:

- (a) What is the probability that a student chosen at random from the 100 has passed examination Y?
 - (b) What is the probability that a student who has passed X will also have passed Y.
 - (c) Comment on the difference, if any, between the answers to parts (a) and (b).
20. A bag contains a hundred coins. One of the coins has head on both sides. The other 99 coins are loaded so that the probability of each of them coming up with a head is 0.6. A coin is drawn from the bag at random. It is tossed and it turns up with heads ten times consecutively. What is the probability that it is the coin with head on both sides?
21. Suppose that a test has been discovered for a disease that affects 2% of the population. In clinical trials, the test was found to have the following error rates:
- (1) 8% of the people free of the disease had a positive reaction (“false alarm”).
 - (2) 1% of the people having the disease had a negative reaction (“missed alarm”).

A mass testing program is being proposed for the whole population; anyone with a positive reaction will be suspected having the disease, and will be brought to the hospital for further observation.

- (a) Of all the people with positive reaction, what proportion actually will be free of the disease?
- (b) Of all the people with negative reaction, what proportion actually will be free of the disease?
- (c) Of the whole population, what proportion will be misdiagnosed one way or another?

22. Suppose that coloured balls are distributed in three indistinguishable boxes as follows:

| | Box 1 | Box 2 | Box 3 |
|-------|-------|-------|-------|
| RED | 2 | 4 | 3 |
| WHITE | 3 | 1 | 4 |
| BLUE | 5 | 3 | 3 |

- A box is selected at random from which a ball is selected at random and it is observed to be red. What is the probability that box 3 was selected?
23. A businessman has two secretaries. The probability that the one he hired most recently will be absent on any given day is 0.08, the probability that the other secretary will be absent on any given day is 0.07, and the probability that they will both be absent on any given day is 0.02. What is the probability that:
- (a) either or both secretaries will be absent on any given day;
 - (b) at least one secretary comes to work on any given day;
 - (c) only one secretary comes to work on any given day?