Mean of binomial distribution

$$\mu = E[X]$$

$$= \sum_{x=0}^{n} x \cdot \Pr(X = x)$$

$$= \sum_{x=0}^{n} x \cdot \binom{n}{x} p^x (1-p)^{n-x}$$

$$= \sum_{x=1}^{n} x \cdot \frac{n!}{x!(n-x)!} \cdot p^x (1-p)^{n-x}$$

$$= \sum_{x=1}^{n} \frac{n!}{(x-1)!(n-x)!} \cdot p^x (1-p)^{n-x}$$

$$= \sum_{x=1}^{n} \frac{n(n-1)!}{(x-1)!(n-x)!} \cdot p \cdot p^{x-1} (1-p)^{n-x}$$

$$= np \sum_{x=1}^{n} \frac{(n-1)!}{(x-1)!(n-1-x+1)!} \cdot p^{x-1} (1-p)^{(n-1)-(x-1)}$$

$$= np \sum_{x=1}^{n} \binom{n-1}{x-1} \cdot p^{x-1} (1-p)^{(n-1)-(x-1)}$$

Let $y = x - 1$, $m = n - 1$

$$= np \sum_{y=0}^{m} \binom{m}{y} \cdot p^y (1-p)^{m-y}$$

$$= \underline{\underline{np}}$$

Variance of binomial distribution

$$Var(X) = E[X^2] - (E[X])^2$$

and $E[X^2] = E[X(X-1)] + E[X]$

Now:

$$E[X(X-1)] = \sum_{x=0}^{n} (x)(x-1)\binom{n}{x} p^x (1-p)^{n-x}$$

Since the 1st and the 2nd terms =0

$$= \sum_{x=2}^{n} (x)(x-1)\frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

$$= \sum_{x=2}^{n} \frac{n!}{(x-2)!(n-x)!} p^x (1-p)^{n-x}$$

$$= n(n-1)p^2 \sum_{x=2}^{n} \frac{(n-2)!}{(x-2)!(n-x)!} p^{x-2} (1-p)^{n-x}$$

Let $y = x - 2$, $m = n - 2$

$$= n(n-1)p^2 \sum_{x=0}^{m} \binom{m}{y} p^y (1-p)^{m-y}$$

$$= n(n-1)p^2$$

Hence $Var(X) = n(n-1)p^2 + np - n^2p^2$

$$= np[(n-1)(p) + 1 - np]$$

$$= np[(np - p + 1 - np]$$

$$= np(1-p)$$

$$= \underline{\underline{npq}}$$