Central Limit Theorem

Theorem. Let $X_1, X_2, \ldots, X_n$ denote the items of a random sample from a distribution that has mean μ and positive variance σ^2. Then the random variable $Y = \dfrac{\left(\sum_1^n X_i - n\mu\right)}{\sqrt{n}\,\sigma} = \dfrac{\overline{X} - \mu}{\sigma/\sqrt{n}}$ has a limiting distribution that is normal with mean zero and variance 1.

Proof. In the modification of the proof, we assume the existence of the moment-generating function $M(t) = E\left(e^{tX}\right)$, $-h < t < h$, of the distribution. However, this proof is essentially the same one that would be given for this theorem in a more advanced course by replacing the moment-generating function by the characteristic function $\varphi(t) = E\left(e^{itX}\right)$.

The function

$$m(t) = E\left[e^{t(X-\mu)}\right] = e^{-\mu t} M(t)$$

also exists for $-h < t < h$. Since $m(t)$ is the moment-generating function for $X - \mu$, it must follow that $m(0) = 1$, $m'(0) = E(X - \mu) = 0$, and $m''(0) = E\left[(X-\mu)^2\right] = \sigma^2$. By Taylor's formula there exists a number ξ between 0 and t such that

$$\begin{aligned} m(t) &= m(0) + m'(0)t + \frac{m''(\xi)t^2}{2} \\ &= 1 + \frac{m''(\xi)t^2}{2}. \end{aligned}$$

If $\dfrac{\sigma^2 t^2}{2}$ is added and subtracted, then

$$m(t) = 1 + \frac{\sigma^2 t^2}{2} + \frac{\left[m''(\xi) - \sigma^2\right]t^2}{2}.$$

Next consider the moment generating function of Y, $M(t; n)$, where

$$
\begin{aligned}
M(t; n) &= E\left[exp\left(t\frac{\sum X_i - n\mu}{\sigma\sqrt{n}}\right)\right] \\
&= E\left[exp\left(t\frac{X_1 - \mu}{\sigma\sqrt{n}}\right) \cdot exp\left(t\frac{X_2 - \mu}{\sigma\sqrt{n}}\right) \quad \cdots \quad exp\left(t\frac{X_n - \mu}{\sigma\sqrt{n}}\right)\right] \\
&= E\left[exp\left(t\frac{X_1 - \mu}{\sigma\sqrt{n}}\right) \quad \cdots \quad exp\left(t\frac{X_n - \mu}{\sigma\sqrt{n}}\right)\right] \\
&= \left\{E\left[exp\left(t\frac{X - \mu}{\sigma\sqrt{n}}\right)\right]\right\}^n \\
&= \left[m\left(\frac{t}{\sigma\sqrt{n}}\right)\right]^n, \quad -h < \frac{t}{\sigma\sqrt{n}} < h.
\end{aligned}
$$

In $m(t)$, replace t by $\frac{t}{\sigma\sqrt{n}}$ to obtain

$$
m\left(\frac{t}{\sigma\sqrt{n}}\right) = 1 + \frac{t^2}{2n} + \frac{\left[m''(\xi) - \sigma^2\right]t^2}{2n\sigma^2},
$$

where now ξ is between 0 and $\frac{t}{\sigma\sqrt{n}}$ with $-h\sigma\sqrt{n} < t < h\sigma\sqrt{n}$.

Accordingly,

$$
M(t; n) = \left\{1 + \frac{t^2}{2n} + \frac{\left[m''(\xi) - \sigma^2\right]t^2}{2n\sigma^2}\right\}^n.
$$

Since $m''(t)$ is continuous at $t = 0$ and since $\xi \to 0$ as $n \to \infty$, we have

$$
\lim_{n\to\infty}\left[m''(\xi) - \sigma^2\right] = 0.
$$

<u>The limit proposition (refer to the next page) shows that</u>

$$
\lim_{n\to\infty} M(t; n) = e^{t^2/2}.
$$

for all real values of t. This proves that the random variable $Y = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}}$ has a limiting normal distribution with mean zero and variance 1.

We interpret this theorem as saying, with n a large, fixed positive integer, that the random variable $\frac{\bar{X} - \mu}{\sigma/\sqrt{n}}$ has an approximate normal distribution with mean zero and variance 1 ($\bar{X}$ has an approximate normal distribution with mean μ and variance $\frac{\sigma^2}{n}$); and in applications we use the approximate normal p.d.f. as though it were the exact p.d.f. of $\frac{\bar{X} - \mu}{\sigma/\sqrt{n}}$.

<u>Limit proposition (stated on the previous page)</u>

We refer to the limit of the form

$$\lim_{n \to \infty} \left[1 + \frac{b}{n} + \frac{\psi(n)}{n} \right]^{cn},$$

where b and c do not depend upon n and where $\lim_{n \to \infty} \psi(n) = \infty$. Then

$$\lim_{n \to \infty} \left[1 + \frac{b}{n} + \frac{\psi(n)}{n} \right]^{cn} = \lim_{n \to \infty} \left[1 + \frac{b}{n} \right]^{cn} = e^{bc}.$$