Chebyshev's Theorem

The probability that any random variable X falls within k standard deviations of the mean is at least

$(1-\frac{1}{k^2})$. That is

$$\Pr(\mu - k\sigma < x < \mu + k\sigma) \geq 1 - \frac{1}{k^2}$$

Proof

$$\sigma^2 = E\left[(x-\mu)^2\right]$$

$$= \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx$$

$$= \int_{-\infty}^{\mu-k\sigma} (x-\mu)^2 f(x)dx + \int_{\mu-k\sigma}^{\mu+k\sigma} (x-\mu)^2 f(x)dx + \int_{\mu+k\sigma}^{\infty} (x-\mu)^2 f(x)dx$$

$$\geq \int_{-\infty}^{\mu-k\sigma} (x-\mu)^2 f(x)dx + \int_{\mu+k\sigma}^{\infty} (x-\mu)^2 f(x)dx$$

Since the 2nd of the three integrals is non-negative.

Now since $|x-\mu| \geq k\sigma$ wherever $x \geq \mu + k\sigma$ or $x \leq \mu - k\sigma$, we have $(x-\mu)^2 \geq k^2\sigma^2$ in both remaining integrals. It follows that

$$\sigma^2 \geq \int_{-\infty}^{\mu-k\sigma} k^2\sigma^2 f(x)dx + \int_{\mu+k\sigma}^{\infty} k^2\sigma^2 f(x)dx$$

and that

$$\frac{1}{k^2} \geq \int_{-\infty}^{\mu-k\sigma} f(x)dx + \int_{\mu+k\sigma}^{\infty} f(x)dx$$

$$\therefore 1 - \frac{1}{k^2} \leq 1 - \int_{-\infty}^{\mu-k\sigma} f(x)dx - \int_{\mu+k\sigma}^{\infty} f(x)dx$$

$$1 - \frac{1}{k^2} \leq \int_{\mu-k\sigma}^{\mu+k\sigma} f(x)dx = \Pr(\mu - k\sigma < x < \mu + k\sigma)$$