$$S^2 = \frac{\sum_{i=1}^{n} (X_i - \overline{X})^2}{n-1}$$ **is an unbiased estimator of** σ^2.

That is,

mean of $S^2 = E\left[\frac{\sum_{i=1}^{n}(X_i - \overline{X})^2}{n-1}\right] = \sigma^2$

PROOF

$$(n-1)E[S^2] = E[\sum_{i=1}^{n}(X_i - \overline{X})^2]$$

$$= E[\sum_{i=1}^{n}(X_i^2 - 2X_i\overline{X} + \overline{X}^2)]$$

$$= E[\sum_{i=1}^{n} X_i^2 - 2\overline{X}\sum_{i=1}^{n} X_i + \sum_{i=1}^{n} \overline{X}^2]$$

$$= E[\sum_{i=1}^{n} X_i^2 - 2\overline{X}(n\overline{X}) + n\overline{X}^2]$$

$$= E[\sum_{i=1}^{n} X_i^2 - n\overline{X}^2]$$

$$= \sum_{i=1}^{n} E[X_1]^2 - nE[\overline{X}^2]$$

$$= nE[X_1^2] - nE[\overline{X}^2]$$

Recall that:

$$Var(Y) = E[Y^2] - (E[Y])^2$$

$$\Rightarrow E[Y^2] = Var(Y) + (E[Y])^2$$

$$E[X_1^2] = Var(X_1) + (E[X_1])^2$$

$$= \sigma^2 + \mu^2$$

$$E[\overline{X}^2] = Var(\overline{X}) + (E[\overline{X}])^2$$

$$= \frac{\sigma^2}{n} + \mu^2$$

Hence,

$$(n-1)E[S^2]$$

$$= n(\sigma^2 + \mu^2) - n(\frac{\sigma^2}{n} + \mu^2)$$

$$= (n-1)\sigma^2$$

That is, $E[S^2] = \sigma^2$